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A Hybrid Steepest-descent Scheme for Convex Minimization over Optimization Problems

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ABSTRACT

Within the context of Hilbert spaces, we demonstrate the robust convergence of a hybrid steepest-descent approximant towards a solution for a convex minimization problem. This problem is situated within the space of solutions for equilibrium problems and the fixed point set of a finite family of η-demimetric operators. Additionally, we present numerical results that shed light on the effectiveness of the proposed approximants, offering insights into potential applications.

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1. Introduction

Throughout this paper, we assume that $\mathcal{H}$ is a real Hilbert space and $\mathcal{D} \subseteq \mathcal{H}$ is nonempty, closed and convex. Let $S : \mathcal{D} \to \mathcal{D}$ be a nonexpansive mapping (i.e., $\|Su - Sv\| \leq \|u - v\|$) and let $\Phi : \mathcal{H} \to \mathbb{R}$ be a convex and bounded below function. The minimization problem over the fixed point set of a mapping is defined as:

$$\text{find } u \in Fix(S) \text{ such that } \Phi(u) = \inf \Phi(Fix(S)), \tag{1.1}$$

where $Fix(S) = \{u \in \mathcal{D};\ Su = u\}$ denotes the fixed point set of S.

It is remarked that the problem (1.1) is equivalent to the following variational inequality problem VIP$(\acute{\Phi}, Fix(S))$ ([15]):

$$\text{find } u \in Fix(S) \text{ such that } \langle v - u, \acute{\Phi}(u)\rangle \geq 0,\ \forall v \in Fix(S), \tag{1.2}$$

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provided that Φ is Gâteaux differentiable over an open set including $Fix(S)$ where $\acute{\Phi}$ denotes the derivative of Φ.

For a slow decreasing sequence $\alpha_k^* \subset (0,1)$, the following class of hybrid steepest-descent approximants (HSDA):

$$y_{k+1} = S(y_k) - \alpha_{k+1}^* \acute{\Phi}(S(y_k)), \tag{1.3}$$

is prominent for solving (1.2). The approximants (1.3) converges strongly to the set of solutions of (1.2), involving a (quasi-)nonexpansive mapping S, under suitable set of conditions on $\Phi, \acute{\Phi}$ and (α_k^*) [28, 29]. A robust variant of HSDA, involving (asymptotically) quasi-shrinking operators, was analyzed in [30].

In 2008, Maingé [18] studied the problem (1.1) involving a more general class of demicontractive and demiclosed mapping via the following Mann-type variant of the HSDA:

$$\begin{cases} y_k := x_k - \alpha_k^* \acute{\Phi}(x_k); \\ x_{k+1} := (1-\beta) y_k + \beta S y_k. \end{cases} \tag{1.4}$$

The following compact form of (1.4) coincides with the HSDA:

$$y_{k+1} = S_\beta y_k - \alpha_{k+1}^* \acute{\Phi}(S_\beta(y_k)), \tag{1.5}$$

where $S_\beta := (1-\beta) Id + \beta S$ and Id denotes the identity mapping.

In 1994, Blum and Oettli [13] proposed a systematic mathematical formulation of equilibrium problems to solve a diverse range of problems occurring in various branches of sciences. Note that an equilibrium problem with respect to a (monotone) bifunction $\tilde{g}$ defined on a nonempty subset C of a real Hilbert space $\mathcal{H}_1$ aims to find a point $\bar{u} \in C$ such that

$$\tilde{g}(\bar{u}, \bar{v}) \geq 0, \text{ for all } \bar{v} \in C. \tag{1.6}$$

The set of equilibrium points or the set of solutions of the problem (1.6) is denoted by $EP(\tilde{g})$. The current literature provides various classical approximants to solve the equilibrium problem. In 2006, Tada and Takahashi [23] suggested a hybrid framework for the analysis of monotone equilibrium problem and fixed point problem in Hilbert spaces. This pioneering work drives the mathematical research community to propose and analyze a combination of approximants to address two or more abstract mathematical problems. On the other hand, the approximants proposed in [23] fails for the case of pseudomonotone equilibrium problem. In order to address this issue, Anh [1] suggested a hybrid extragradient method, based on the seminal work of Korpelevich [17], to address pseudomonotone equilibrium problem together with the fixed point problem (see also [2, 3, 4, 5, 6, 9, 10, 16, 8, 7, 10, 22]).

Motivated by these advancements and ongoing research, there is a natural inclination to explore pseudomonotone EP and FPP within the realm of η-demimetric operators. Consequently, we propose several variations of the classical Mann iterative algorithm [19] within Hilbert spaces. These variants incorporate, aiming for robust strong convergence outcomes in Hilbert spaces. Thus the following natural question arises in view of the architecture of the approximants (1.4):

Can one modify the approximants (1.4) to solve the convex minimization problem (1.1) over pseudomonotone equilibrium and the fixed point set of η-demimetric mapping? Answering this question in the affirmative, we propose a HSDA for the following convex minimization problem over the solution set of pseudomonotone equilibrium and the fixed point set of a finite family of η-demimetric mapping in Hilbert spaces:

$$\text{find } \bar{u} \in (Fix(S) \cap EP(\tilde{g})) \text{ such that } \Phi(\bar{u}) = \inf \Phi((Fix(S) \cap EP(\tilde{g})). \tag{1.7}$$

Recall that if Φ is Gâteaux differentiable over an open set including $(Fix(S) \cap EP(\tilde{g}))$, with its derivative denoted by $\acute{\Phi}$, then (1.7) is equivalent to the variational inequality problem VIP$(\acute{\Phi}, (Fix(S) \cap EP(\tilde{g}))$, that is

$$\text{find } \bar{u} \in (Fix(S) \cap EP(\tilde{g})) \text{ such that } \langle u - \bar{u}, \acute{\Phi}(\bar{u})\rangle \geq 0,\ \forall u \in (Fix(S) \cap EP(\tilde{g})). \tag{1.8}$$

As far as we know, such results have not so far appeared in the literature. The rest of the paper is organized as follows: Section 2 contains some relevant preliminary concepts and results for convex minimization problem, pseudomonotone equilibrium satisfying Lipschitz-type continuity and fixed point problem. Section 3 comprises strong convergence results of the proposed a HSDA whereas Section 4 provides numerical results concerning the viability of the proposed approximants.

2. Preliminaries

For a nonempty closed and convex subset $\mathcal{D} \subseteq \mathcal{H}$, if $S : \mathcal{D} \to \mathcal{H}$ is an operator then $Fix(S) = \{\bar{\nu} \in \mathcal{H} \mid \bar{\nu} = S\bar{\nu}\}$ represents the set of fixed points of the operator S. Recall that the operator S is called η-demimetric (see [24]) where $\eta \in (-\infty, 1)$, if $Fix(S) \neq \emptyset$ and

$$\langle \bar{\mu} - \bar{\nu}, \bar{\mu} - S\bar{\mu}\rangle \geq \frac{1}{2}(1-\eta)\|\bar{\mu} - S\bar{\mu}\|^2,\ \forall \bar{\mu} \in \mathcal{H} \text{ and } \bar{\nu} \in Fix(S).$$

The above definition is equivalently represented as

$$\|S\bar{\mu} - \bar{\nu}\|^2 \leq \|\bar{\mu} - \bar{\nu}\|^2 + \eta\|\bar{\mu} - S\bar{\mu}\|^2,\ \forall \bar{\mu} \in \mathcal{H} \text{ and } \bar{\nu} \in Fix(S).$$

For every point $\bar{u} \in \mathcal{H}$, there exists a unique nearest point in $\mathcal{D}$, denote by $\mathcal{P}_{\mathcal{D}}u$, such that $\|u - \mathcal{P}_{\mathcal{D}}u\| \leq |u - v\|\ \forall u, v \in \mathcal{D}$. The mapping $\mathcal{P}_{\mathcal{D}}$ is called the metric projection of $\mathcal{H}$ onto $\mathcal{D}$. It is well known that $\mathcal{P}_{\mathcal{D}}$ is nonexpansive and satisfies $\langle u - \mathcal{P}_{\mathcal{D}}u, b - \mathcal{P}_{\mathcal{D}}u\rangle \leq 0,\ \forall b \in \mathcal{D}$.

Asumption 2.1 ([12, 13]). *Let $\tilde{g} : \mathcal{D} \times \mathcal{D} \to \mathbb{R} \cup \{+\infty\}$ be a bifunction satisfying the following assumptions:*
(L1): $\tilde{g}$ is pseudomonotone, i.e., $\tilde{g}(u, v) \leq 0 \Rightarrow g(u, v) \geq 0$, for all $u, v \in \mathcal{D}$;
(L2): $\tilde{g}$ is Lipschitz-type continuous, i.e., there exist two nonnegative constants d_1, d_2 such that

$$\tilde{g}(u, v) + \tilde{g}(v, n) \geq g(u, n) - d_1\|u - v\|^2 - d_2\|v - n\|^2,\ \textit{for all } u, v, n \in \mathcal{D};$$

(L3): $\tilde{g}$ is weakly continuous on $\mathcal{D} \times \mathcal{D}$ in the sense that, if $u, v \in \mathcal{D}$ and $(\Theta_k), (b_k)$ are two sequences in $\mathcal{D}$ converge weakly to u and v respectively, then $\tilde{g}(\Theta_k, b_k)$ converges to $\tilde{g}(u, v)$;
(L4): For each fixed $u \in \mathcal{D}$, $\tilde{g}(u, .)$ is convex and subdifferentiable on $\mathcal{D}$.

Now we introduce the architecture of the modified HSDA for computing the solution of (1.7)-(1.8):

Let $\tilde{g}_i : \mathcal{D} \times \mathcal{D} \to \mathbb{R} \cup \{+\infty\}$, $i \in \{1, 2, \cdots, M\}$ be a finite family of bifunctions satisfying Assumption 2.1 and let $S_j : \mathcal{D} \to \mathcal{H}$, $j \in \{1, 2, \cdots, N\}$ is a finite family of η-demimetric mappings. Let $\Phi : \mathcal{D} \to \mathbb{R} \cup (-\infty, +\infty]$ is a proper, convex and bounded below function. Assume that $\Pi := (\bigcap_{i=1}^{M}(EP(\tilde{g}_i)) \cap \bigcap_{j=1}^{N} Fix(S_j)) \neq \emptyset$. To this end, in a more general framework, we investigate the convergence analysis of the sequence (Θ_k) generated with an arbitrary Θ_0 in $\mathcal{H}$:

$$\begin{cases} b_k := \Theta_k - \alpha_k \acute{\Phi}(\Theta_k); \\ p_k := \arg\min\{\tau \tilde{g}_i(b_k, p) + \dfrac{1}{2}\|b_k - p\|^2 : p \in \mathcal{D}\}, \quad i = 1, 2, \cdots, M; \\ q_k := \arg\min\{\tau \tilde{g}_i(p_k, p) + \dfrac{1}{2}\|b_k - p\|^2 : p \in \mathcal{D}\}, \quad i = 1, 2, \cdots, M; \\ \Theta_{k+1} := \displaystyle\sum_{j=1}^{N} \gamma_j((1 - \beta_k)Id + \beta_k S_j)q_k, \end{cases} \tag{2.1}$$

for $j = \{1, 2, \cdots, N\}$, $\gamma_j \in (0, 1)$ such that $\sum_{j=1}^{N} \gamma_j = 1$, $0 < \tau < \min(\frac{1}{2d_1}, \frac{1}{2d_2})$, $\alpha_k \subset [0, 1)$ and $\beta_k \in (0, 1)$. The following conditions are needed throughout paper:

(A1) $0 < a^* \leq \beta_k \leq \min\{1 - \eta_1, \cdots, 1 - \eta_N\}$;

(A2) $\lim_{k \to \infty} \alpha_k = 0$;

(A3) $\sum_{k \geq 0} \alpha_k = +\infty$;

(A4) $\acute{\Phi}$ is L-Lipschitz continuous on $\mathcal{H}$ (for some $L \geq 0$); i.e.

$$\|\acute{\Phi}(u) - \acute{\Phi}(v)\| \leq L\|u - v\|, \ \forall u, v \in \mathcal{H}.$$

(A5) $\acute{\Phi}$ is Ψ-strongly monotone on $\mathcal{H}$ (for some $\Psi > 0$); i.e.

$$\langle \acute{\Phi}(u) - \acute{\Phi}(v), u - v \rangle \geq \Psi\|u - v\|^2, \ \forall u, v \in \mathcal{H}.$$

It is noted that the unique existence of the solution of (1.8) is ensured by the conditions (A5) and (A6) (see for instance [28]).

The following lemmas are helpful to prove the strong convergence results in the next section.

Lemma 2.2. *Let $u, v, n \in \mathcal{H}$ and $a \in [0, 1] \subset \mathbb{R}$, then*

1. $\|u + v\|^2 \leq \|u\|^2 + 2\langle v, u + v \rangle$;
2. $\|u - v\|^2 = \|u\|^2 - \|v\|^2 - 2\langle u - v, v \rangle$;
3. $\|au + (1 - a)v - n\|^2 = a\|u - n\|^2 + (1 - a)\|v - n\|^2 - a(1 - a)\|u - v\|^2$.

Lemma 2.3 ([24]). *Let $S : \mathcal{D} \to \mathcal{H}$ be an η-demimetric operator defined on a nonempty, closed and convex subset $\mathcal{D}$ of a Hilbert space $\mathcal{H}$ with $\eta \in (-\infty, 1)$. Then $Fix(S)$ is closed and convex.*

Lemma 2.4 ([25]). *Let $S : \mathcal{D} \to \mathcal{H}$ be an η-demimetric operator defined on a nonempty, closed and convex subset $\mathcal{D}$ of a Hilbert space $\mathcal{H}$ with $\eta \in (-\infty, 1)$. Then the operator $L = (1-\gamma)Id + \gamma S$ is quasi-nonexpansive provided that $Fix(S) \neq \emptyset$ and $0 < \gamma < 1 - \eta$.*

Lemma 2.5 ([11]). *Let $S : \mathcal{D} \to \mathcal{D}$ be a nonexpansive operator defined on a nonempty closed convex subset $\mathcal{D}$ of a real Hilbert space $\mathcal{H}$ and let (Θ_k) be a sequence in $\mathcal{D}$. If $\Theta_k \rightharpoonup x$ and if $(Id - S)\Theta_k \to 0$, then $x \in Fix(S)$.*

Lemma 2.6 ([12]). *If the bifunction g satisfies Assumption 2.1, then the solution set $EP(g)$ is weakly closed and convex.*

Lemma 2.7 ([26]). *Let $\mathcal{D}$ be a nonempty closed and convex subset of a real Hilbert space $\mathcal{H}_1$ and let $f : \mathcal{D} \to \mathbb{R}$ be a convex and subdifferentiable function on $\mathcal{D}$. Then, $\bar{u}$ is the solution of convex problem $\min\{f(u) : u \in \mathcal{D}\}$, if and only if $0 \in \partial\hbar(\bar{u}) + N_{\mathcal{D}}(\bar{u})$, where $\partial f(\cdot)$ denotes the subdifferential of f and $N_{\mathcal{D}}(\bar{u})$ is the normal cone of $\mathcal{D}$ at $\bar{u}$.*

Lemma 2.8 ([1, 21]). *Suppose that $\bar{u} \in EP(\tilde{g}_i)$, and Θ_k, p_k, q_k, $i \in \{1, 2, \cdots, M\}$ are defined in via the approximants (2.1). Then we have*

$$\|q_k - \bar{u}\|^2 \leq \|b_k - \bar{u}\|^2 - (1 - 2\tau d_1)\|p_k - b_k\|^2 - (1 - 2\tau d_2)\|p_k - q_k\|^2.$$

Lemma 2.9. *Let $\tilde{g}_i : \mathcal{D} \times \mathcal{D} \to \mathbb{R} \cup \{+\infty\}$, $i \in \{1, 2, \cdots, M\}$ be a finite family of bifunctions satisfying Assumption 2.1. Let S_j, $j \in \{1, 2, \cdots, N\}$ be a finite family of η-demimetric mapping on $\mathcal{H}$ and Φ be a convex, bounded below and Gâteaux differentiable function on $\mathcal{H}$ with derivative $\acute{\Phi}$. Further assume that the conditions $(A1) - (A2)$ and $(A5)$ hold. Then the sequence (Θ_k) given by (2.1) satisfies for all $k \geq 0$,*

$$U_{k+1} - U_k + \frac{1}{2}(1 - 2L\alpha_k)\|\Theta_{k+1} - \Theta_k\|^2 \leq -\alpha_k\langle \Theta_k - \bar{u}, \acute{\Phi}(\Theta_k)\rangle, \tag{2.2}$$

where $\bar{u} \in Fix(S_j)$ and

$$U_k := \frac{1}{2}\|\Theta_k - \bar{u}\|^2 + \alpha_k(\Phi(\Theta_k) - \inf \Phi). \tag{2.3}$$

Proof. Let $\bar{u} \in Fix(S_j)$. Now, it follows from the approximants (2.1) and Lemma 2.4 that

$$\begin{aligned} \|\Theta_{k+1} - \bar{u}\| &= \|\sum_{j=1}^{N} \gamma_j((1-\beta_k)Id + \beta_k S_j)q_k - \bar{u}\| \leq \sum_{j=1}^{N} \gamma_j\|((1-\beta_k)Id + \beta_k S_j)q_k - \bar{u}\| \\ &\leq \sum_{j=1}^{N} \gamma_j\|q_k - \bar{u}\| = \|q_k - \bar{u}\|. \end{aligned} \tag{2.4}$$

From (2.1), we have

$$\sum_{j=1}^{M} \gamma_j\|\Theta_{k+1} - q_k\| = \frac{1}{\beta_k}(\Theta_{k+1} - q_k).$$

Setting $\xi := \frac{1}{\beta_k}(1 - \eta_j - \beta_k)$, we get

$$\|\Theta_{k+1} - \bar{u}\|^2 \leq \|q_k - \bar{u}\|^2 - \xi\|\Theta_{k+1} - q_k\|^2. \tag{2.5}$$

So therefore, if $\beta_k \in (0, \frac{1-\eta_j}{2}]$ (so that $\xi \geq 1$), we obtain

$$\|\Theta_{k+1} - \bar{u}\|^2 \leq \|q_k - \bar{u}\|^2 - \|\Theta_{k+1} - q_k\|^2. \tag{2.6}$$

From the definition of (b_k) and (A2)-(A3), we have

$$\lim_{k\to\infty} \|b_k - \Theta_k\| = 0. \tag{2.7}$$

From (2.1) and (A3), we have

$$\begin{aligned} \|b_k - \bar{u}\|^2 &= \|(\Theta_k - \bar{u}) - \alpha_k \acute{\Phi}(\Theta_k)\|^2 \\ &= \|\Theta_k - \bar{u}\|^2 - 2\alpha_k\langle\Theta_k - \bar{u}, \acute{\Phi}(\Theta_k)\rangle + \alpha_k^2\|\acute{\Phi}(\Theta_k)\|^2 \\ &= \|\Theta_k - \bar{u}\|^2. \end{aligned} \tag{2.8}$$

Thus we obtain

$$\|\Theta_{k+1} - \bar{u}\| \leq \|\Theta_k - \bar{u}\|.$$

Consider the following re-arranged variant of the estimate (2.4) and by applying Lemma 2.8:

$$(1-2\tau d_1)\|p_k - b_k\|^2 - (1-2\tau d_2)\|p_k - q_k\|^2 \quad \leq \quad (\|\Theta_k - \bar{u}\| + \|b_k - \bar{u}\|)\|\Theta_k - b_k\|.$$

Letting $k \to \infty$ and utilizing (2.7), we have

$$(1-2\tau d_1)\lim_{k\to\infty} \|p_k - b_k\|^2 - (1-2\tau d_2)\lim_{k\to\infty} \|p_k - q_k\|^2 = 0. \tag{2.9}$$

This implies that

$$\lim_{k\to\infty} \|p_k - b_k\|^2 = \lim_{k\to\infty} \|p_k - q_k\|^2 = 0. \tag{2.10}$$

Further, from (2.7), (2.10) and the following triangular inequality, we have

$$\|q_k - \Theta_k\| \quad \leq \quad \|q_k - p_k\| + \|p_k - b_k\| + \|b_k - \Theta_k\| \to 0. \tag{2.11}$$

From the estimate (2.6) and the following triangle inequality, we have

$$\|\Theta_{k+1} - q_k\| \leq \|\Theta_{k+1} - b_k\| + \|b_k - \Theta_k\| + \|\Theta_k - q_k\|.$$

From the above estimate and utilizing (2.7) and (2.11), we get

$$\|\Theta_{k+1} - q_k\| \leq \|\Theta_{k+1} - b_k\|.$$

Rearranged the estimate (2.6), we have

$$\|\Theta_{k+1} - \bar{u}\|^2 \leq \|b_k - \bar{u}\|^2 - \|\Theta_{k+1} - b_k\|^2. \tag{2.12}$$

Moreover

$$\begin{aligned} \|b_k - \Theta_{k+1}\|^2 &= \|(\Theta_{k+1} - \Theta_k) + \alpha_k\acute{\Phi}(\Theta_k)\|^2 \\ &= \|\Theta_{k+1} - \Theta_k\|^2 + 2\alpha_k\langle\Theta_{k+1} - \Theta_k, \acute{\Phi}(\Theta_k)\rangle + \alpha_k^2\|\acute{\Phi}(\Theta_k)\|^2 \\ &= \|\Theta_{k+1} - \Theta_k\|^2 + 2\alpha_k\langle\Theta_{k+1} - \Theta_k, \acute{\Phi}(\Theta_k) - \acute{\Phi}(\Theta_{k+1})\rangle \\ &\quad +2\alpha_k\langle\Theta_{k+1} - \Theta_k, \acute{\Phi}(\Theta_{k+1})\rangle + \alpha_k^2\|\acute{\Phi}(\Theta_k)\|^2. \end{aligned} \tag{2.13}$$

Using the L-Lipschitz continuity of $\acute{\Phi}$ and the convexity of Φ, we obtain

$$\langle \Theta_{k+1} - \Theta_k, \acute{\Phi}(\Theta_k) - \acute{\Phi}(\Theta_{k+1}) \rangle \geq -L\|\Theta_{k+1} - \Theta_k\|^2$$

and

$$\langle \Theta_{k+1} - \Theta_k, \acute{\Phi}(\Theta_{k+1}) \rangle \geq \Phi(\Theta_{k+1}) - \Phi(\Theta_k).$$

Utilizing the above estimates in (2.13), we get

$$\begin{aligned}\|\Theta_{k+1} - b_k\|^2 &\geq \|\Theta_{k+1} - \Theta_k\|^2 - 2\alpha_k L\|\Theta_k - \Theta_{k+1}\|^2 + 2\alpha_k(\Phi(\Theta_{k+1}) - \Phi(\Theta_k)) \\ &\quad +\alpha_k^2\|\acute{\Phi}(\Theta_k)\|^2 \\ &= (1 - 2L\alpha_k)\|\Theta_{k+1} - \Theta_k\|^2 + 2\alpha_k(\Phi(\Theta_{k+1}) - \Phi(\Theta_k)) \\ &\quad +\alpha_k^2\|\acute{\Phi}(\Theta_k)\|^2. \end{aligned} \tag{2.14}$$

So therefore, from (2.8), (2.12) in (2.14), we get

$$\begin{aligned}\|\Theta_{k+1} - \bar{u}\|^2 &\leq \|\Theta_k - \bar{u}\|^2 - 2\alpha_k\langle \Theta_k - \bar{u}, \acute{\Phi}(\Theta_k)\rangle \\ &\quad -(1 - 2L\alpha_k)\|\Theta_{k+1} - \Theta_k\|^2 - 2\alpha_k(\Phi(\Theta_{k+1}) - \Phi(\Theta_k)).\end{aligned}$$

Rearranging the above statement, we have

$$\begin{aligned}&\|\Theta_{k+1} - \bar{u}\|^2 + 2\alpha_{k+1}(\Phi(\Theta_{k+1}) - \inf \Phi) \\ &\qquad \leq \|\Theta_k - \bar{u}\|^2 + 2\alpha_k(\Phi(\Theta_k) - \inf \Phi) - 2\alpha_k\langle \Theta_k - \bar{u}, \acute{\Phi}(\Theta_k)\rangle \\ &\qquad\quad - (1 - 2L\alpha_k)\|\Theta_{k+1} - \Theta_k\|^2 - 2(\alpha_k - \alpha_{k+1})(\Phi(\Theta_{k+1}) - \inf \Phi).\end{aligned}$$

Note that, if α_k is non-increasing, we have $(\alpha_k - \alpha_{k+1})(\Phi(\Theta_{k+1}) - \inf \Phi) \geq 0$, that is

$$\begin{aligned}&\frac{1}{2}\|\Theta_{k+1} - \bar{u}\|^2 + \alpha_{k+1}(\Phi(\Theta_{k+1}) - \inf \Phi) \\ &\qquad \leq \frac{1}{2}\|\Theta_k - \bar{u}\|^2 + \alpha_k(\Phi(\Theta_k) - \inf \Phi) - \alpha_k\langle \Theta_k - \bar{u}, \acute{\Phi}(\Theta_k)\rangle \\ &\qquad\quad - \frac{1}{2}(1 - 2L\alpha_k)\|\Theta_{k+1} - \Theta_k\|^2.\end{aligned}$$

This is the required result. ∎

The following results can easily be adopted from [18, Lemma 2.2 & 2.3].

Lemma 2.10. *Let $\tilde{g}_i : \mathcal{D} \times \mathcal{D} \to \mathbb{R} \cup \{+\infty\}$, $i \in \{1, 2, \cdots, M\}$ be a finite family of bifunctions satisfying Assumption 2.1. Let $S_j, j \in \{1, 2, \cdots, N\}$ be a finite family of η-demimetric mappings on $\mathcal{H}$ and Φ be a convex, bounded below and Gâteaux differentiable function on $\mathcal{H}$ with derivative $\acute{\Phi}$. If the condition (A6) holds, then for any $\bar{u} \in Fix(S_j)$ and any $\varepsilon \in (0, 2)$, then sequence (Θ_k) given by (2.1) satisfies for all $k \geq 0$,*

$$\langle \Theta_k - \bar{u}, \acute{\Phi}(\Theta_k)\rangle \geq \frac{1}{1 + \Psi\varepsilon\alpha_k}(\Psi\varepsilon U_k - (D_\varepsilon + d\Psi\varepsilon\alpha_k)), \tag{2.15}$$

where

$$\begin{aligned}U_k &:= \frac{1}{2}\|\Theta_k - \bar{u}\|^2 + \alpha_k(\Phi(\Theta_k) - \inf \Phi), \\ d &:= \Phi(\bar{u}) - \inf \Phi, \\ D_\varepsilon &:= \frac{\|\acute{\Phi}(\bar{u})\|^2}{2(2 - \varepsilon)\Psi}.\end{aligned}$$

Assume that the conditions (A1)-(A2) and (A5) hold and suppose $(\alpha_k) \subset (0, \frac{1}{2L}]$ *(when* $L \neq 0$*). Then we have for all* $k \geq 0$,

$$U_k \leq U_0 e^{-\frac{\Psi\varepsilon}{1+\Psi\varepsilon\alpha_0}(\sum_{r=0}^{k}\alpha_r - \alpha_0)} + (D_\varepsilon + d\Psi\varepsilon\alpha_0)\frac{1+2\Psi\varepsilon\alpha_0}{\Psi\varepsilon} e^{\frac{2\Psi\varepsilon}{1+\Psi\varepsilon\alpha_0}}. \tag{2.16}$$

Proof. See proof in [18]. ∎

Lemma 2.11. *Let* $\tilde{g}_i : \mathcal{D} \times \mathcal{D} \to \mathbb{R} \cup \{+\infty\}$, $i \in \{1, 2, \cdots, M\}$ *be a finite family of bifunctions satisfying Assumption 2.1. Let* $S_j, j \in \{1, 2, \cdots, N\}$ *be a finite family of* η*-demimetric mapping on* $\mathcal{H}$ *and* Φ *be a convex, bounded below and Gâteaux differentiable function on* $\mathcal{H}$ *with derivative* $\acute{\Phi}$*. If the conditions (A1)-(A3), (A5) and (A6) hold, then the sequence* (Θ_k) *generated by (2.1) is bounded.*

Proof. This result is easily can see in consequence of Lemma 2.10. ∎

3. Strong Convergence Analysis

In this section, we first establish the following results for the strong convergence analysis of the approximants (2.1).

Lemma 3.1. *Let* $\tilde{g}_i : \mathcal{D} \times \mathcal{D} \to \mathbb{R} \cup \{+\infty\}$, $i \in \{1, 2, \cdots, M\}$ *be a finite family of bifunctions satisfying Assumption 2.1. Let* $S_j : \mathcal{D} \to \mathcal{H}$ *be a finite family of* η_j*-demimetric mappings and* Φ *be a convex, bounded below and Gâteaux differentiable function on* $\mathcal{H}$ *with derivative* $\acute{\Phi}$*. Assume that* $\Pi := \bigcap_{i=1}^{M}(EP(\tilde{g}_i)) \cap \bigcap_{j=1}^{N} Fix(S_j) \neq \emptyset$*. Suppose that the demiclosed principle, (A3) and (A5) hold and assume the sequence* (Θ_k) *generated by (2.1) is bounded and satisfies* $\|\Theta_{k+1} - \Theta_k\| \to 0$*. Then* $\Theta_k \rightharpoonup \bar{u}$, $\bar{u} \in \Pi$ *and we have*

$$\lim_{k\to\infty}\inf \langle \Theta_k - \bar{u}, \acute{\Phi}(\bar{u})\rangle \geq 0,$$

where $\bar{u}$ *is the solution of (1.7) or (1.8).*

Proof. Let (x_{k_m}) be a subsequence of (Θ_k) which converges weakly to an element x^* in $\mathcal{H}$. Assume that $\|\Theta_{k+1} - \Theta_k\| \to 0$, $\alpha_k \to 0$ and (Θ_k) is bounded, consequently, Θ_{k_m} is weakly converges to x^* and $y_{k_m} := \Theta_{k_m} - \alpha_{k_m}\acute{\Phi}(\Theta_{k_m})$ converges weakly to x^*. Utilizing (A4) and boundedness of $\acute{\Phi}(\Theta_{k_m})$, we have $\alpha_{k_m}\|\acute{\Phi}(\Theta_{k_m})\| \to 0$. From (2.1), we get

$$\sum_{j=1}^{N} \gamma_j \|S_j q_{k_m} - q_{k_m}\| = \|\Theta_{k_m+1} - q_{k_m}\| \to 0, \; j = 1, 2, \cdots, N.$$

From the demiclosed principle of S_j, we obtain $x^* = S_j x^*$, $j \in \{1, 2, \cdots, N\}$. Next, we show that $x^* \in \bigcap_{i=1}^{M} EP(\tilde{g}_i)$.
Note that

$$p_k = \arg\min\{\tau\tilde{g}_i(\Theta_k, p) + \frac{1}{2}\|\Theta_k - p\|^2 : p \in \mathcal{D}\}.$$

Using Lemma 2.7, we get

$$0 \in \partial_2 \left\{\tau\tilde{g}_i(\Theta_k, p) + \frac{1}{2}\|\Theta_k - p\|^2\right\}(p_k) + N_{\mathcal{D}}(p_k).$$

Then, there exist $s \in \partial_2 \tilde{g}_i(\Theta_k, p_k)$ and $\bar{s} \in N_{\mathcal{D}}(p_k)$ such that

$$\tau s + \Theta_k - p_k + \bar{s}. \tag{3.1}$$

Since $\bar{s} \in N_{\mathcal{D}}(p_k)$ and $\langle \bar{s}, p - p_k \rangle \leq 0$ for all $p \in \mathcal{D}$. So, by using (3.1), we have

$$\tau \langle s, p - p_k \rangle \geq \langle p_k - \Theta_k, p - p_k \rangle, \ \forall p \in \mathcal{D}. \tag{3.2}$$

Since $s \in \partial_2 \tilde{g}_i(\Theta_k, p_k)$, therefore we have

$$\tilde{g}_i(\Theta_k, p) - \tilde{g}_i(\Theta_k, p_k) \geq \langle s, p - p_k \rangle, \ \forall p \in \mathcal{D}. \tag{3.3}$$

Utilizing (3.2) and (3.3), we obtain

$$\tau(\tilde{g}_i(\Theta_k, p) - \tilde{g}_i(\Theta_k, p_k)) \geq \langle p_k - \Theta_k, p - p_k \rangle, \ \forall p \in \mathcal{D}. \tag{3.4}$$

Since $\Theta_{k_m} \rightharpoonup x^* \in \mathcal{H}$ as $m \to \infty$, therefore we have $\Theta_{k_m+1} \rightharpoonup x^*$ and $\Theta_{k_m} \rightharpoonup x^*$ as $m \to \infty$. Moreover, from $b_k \rightharpoonup x^*$ and $\|b_k - p_k\| \to 0$ as $k \to \infty$ imply that $p_k \rightharpoonup x^*$. By using (L3) and from (3.4), letting $k \to \infty$, we deduce that $\tilde{g}_i(x^*, p) \geq 0$ for all $p \in \mathcal{D}$, $i \in \{1, 2, \cdots, M\}$. Therefore, $x^* \in \bigcap_{i=1}^{M} EP(\tilde{g}_i)$. Hence $x^* \in \Pi$.

The term $\langle \Theta_k - \bar{u}, \acute{\Phi}(\bar{u}) \rangle$ is bounded, as (Θ_k) is bounded. So there exists a subsequence (Θ_{k_m}) weakly converges to a point $x^* \in \mathcal{H}$, so therefore $x^* \in \Pi$ and such that

$$\liminf_{k \to \infty} \langle \Theta_k - \bar{u}, \acute{\Phi}(\bar{u}) \rangle = \lim_{m \to \infty} \langle \Theta_{k_m} - \bar{u}, \acute{\Phi}(\bar{u}) \rangle,$$

hence $\liminf_{k \to \infty} \langle \Theta_k - \bar{u}, \acute{\Phi}(\bar{u}) \rangle = \langle x^* - \bar{u}, \acute{\Phi}(\bar{u}) \rangle$. As $\bar{u}$ is a solution of (1.8), we have $\langle x^* - \bar{u}, \acute{\Phi}(\bar{u}) \rangle \geq 0$. This is the required result. ∎

Lemma 3.2. *Let $\tilde{g}_i : \mathcal{D} \times \mathcal{D} \to \mathbb{R} \cup \{+\infty\}$, $i \in \{1, 2, \cdots, M\}$ be a finite family of bifunctions satisfying Assumption 2.1. Let $S_j : \mathcal{D} \to \mathcal{H}$ is a finite family of η_j-demimetric mappings and Φ be a convex, bounded below and Gâteaux differentiable function on $\mathcal{H}$ with derivative $\acute{\Phi}$. Assume that $\Pi := \bigcap_{i=1}^{M}(EP(\tilde{g}_i)) \cap \bigcap_{j=1}^{N} Fix(S_j) \neq \emptyset$. Suppose that the demiclosed principle, (A3), (A5) and (A6) hold and let the sequence (Θ_k) generated by (2.1) has a subsequence (Θ_{k_m}) such that:*

(I) $(\Theta_{k_m}) \subset \Gamma := \{x \in \mathcal{H} : \langle x - \bar{u}, \acute{\Phi}(x) \rangle \leq 0\}$, *where $\bar{u}$ is the solution of (1.7) or (1.8).*

(II) $\|\Theta_{k_m+1} - \Theta_{k_m}\| \to 0$ *as* $k \to \infty$.

Then (Θ_{k_m}) converges strongly to $\bar{u}$.

Proof. It is observed that using (A6), we have $\Psi \|\Theta_{k_m} - \bar{u}\|^2 \leq \langle \Theta_{k_m} - \bar{u}, \acute{\Phi}(\Theta_{k_m}) - \acute{\Phi}(\bar{u}) \rangle$. So (I) implies that

$$\Psi \|\Theta_{k_m} - \bar{u}\|^2 \leq -\langle \Theta_{k_m} - \bar{u}, \acute{\Phi}(\bar{u}) \rangle. \tag{3.5}$$

From (3.5), we obtain $\|\Theta_{k_m} - \bar{u}\| \leq \frac{\acute{\Phi}(\bar{u})}{\Psi}$. So therefore, (Θ_{k_m}) and as well Γ are bounded. Consequently, a subsequence $(\Theta_{k_m}) \in \mathcal{H}$ converges weakly to a point $x^* \in \mathcal{H}$ and utilizing (II), we obtain $\|\Theta_{k_m} - \Theta_{k_m+1}\| \to 0$ as $k \to \infty$. Moreover, from (2.1), we have

$$\begin{aligned} \beta_k \|q_{k_m} - \sum_{j=1}^{N} \gamma_j w_{k_m}^{(j)}\| &\leq \beta_k \sum_{j=1}^{N} \gamma_j \|q_{k_m} - w_{k_m}^{(j)}\| \qquad (3.6) \\ &= \frac{1}{\beta_k} \|\Theta_{k_m+1} - q_{k_m}\| \to 0, \ \text{as } k \to \infty. \end{aligned}$$

By using (A5) and since $(\alpha_k) \to 0$, q_{k_m} converges weakly to $\bar{u}$. Note that, $x^* \in \Pi$, (as proved in Lemma 3.1) and utilizing (3.5) and (1.7) entails

$$\limsup_{k\to+\infty} \|\Theta_{k_m} - \bar{u}\|^2 \leq -(\frac{1}{\Psi})\langle x^* - \bar{u}, \acute{\Phi}(\bar{u})\rangle \leq 0,$$

hence $\lim_{m\to+\infty} \|\Theta_{k_m} - \bar{u}\| = 0$. This is the required result. ∎

Lemma 3.3. *Let $\tilde{g}_i : \mathcal{D}\times\mathcal{D} \to \mathbb{R}\cup\{+\infty\}$, $i \in \{1, 2, \cdots, M\}$ be a finite family of bifunctions satisfying Assumption 2.1. Let $S_j : \mathcal{D} \to \mathcal{H}$ is a finite family of η-demimetric mappings and Φ be a convex, bounded below and Gâteaux differentiable function on $\mathcal{H}$ with derivative $\acute{\Phi}$. Assume that $\Pi := \bigcap_{i=1}^{M}(EP(\tilde{g}_i)) \cap \bigcap_{j=1}^{N} Fix(S_j) \neq \emptyset$. Suppose that the demiclosed principle and (A1)-(A6) hold and let the sequence (Θ_k) given by (2.1) satisfies:*

(I) $\|\Theta_{k+1} - \Theta_k\| \to 0$.

(II) $\lim_{k\to\infty} \|\Theta_k - \bar{u}\|$ *exists,*

where $\bar{u}$ is the solution of (1.7) or (1.8). Then (Θ_k) converges strongly to $\bar{u}$.

Proof. It is observed that from Lemma 2.11, (Θ_k) is a bounded sequence. Suppose that $\lim_{k\to\infty} \|\Theta_k - \bar{u}\| = \mu > 0$ and utilizing Lemma 3.1, we have $\liminf_{k\to\infty}\langle\Theta_k - \bar{u}, \acute{\Phi}(\bar{u})\rangle \geq 0$ and also from (A6), we get

$$\langle\Theta_k - \bar{u}, \acute{\Phi}(\Theta_k)\rangle \geq \Psi\|\Theta_k - \bar{u}\|^2 + \langle\Theta_k - \bar{u}, \acute{\Phi}(\bar{u})\rangle.$$

After simplification, we obtain

$$\liminf_{k\to}\langle\Theta_k - \bar{u}, \acute{\Phi}(\Theta_k)\rangle \geq \Psi\mu^2.$$

It deduced from Lemma 2.9 that there exists $k_0 \geq 0$ such that for $k \geq k_0$,

$$V_{k+1} - V_k \leq -\alpha_k(\frac{1}{2}\Psi\mu^2),$$

where $V_k := \frac{1}{2}\|\Theta_k - \bar{u}\|^2 + \alpha_k(\Phi(\Theta_k) - \inf\Phi)$. It yields

$$(\frac{1}{2}\Psi\mu^2)\sum_{m=k_0}^{k}\alpha_k \leq V_{k_0} - V_{k+1}.$$

It is observe from the above estimate, if $\sum\alpha_k = \infty$, then the last inequality is inappropriate as $k \to \infty$, because (Θ_k) is bounded, so its right hand side is supposed to be bounded, while the left hand side approaches to $+\infty$. Hence, as consequence $\mu = 0$. This is the required result. ∎

Theorem 3.4. *Let $\tilde{g}_i : \mathcal{D}\times\mathcal{D} \to \mathbb{R}\cup\{+\infty\}$, $i \in \{1, 2, \cdots, M\}$ be a finite family of bifunctions satisfying Assumption 2.1. Let $S_j : \mathcal{D} \to \mathcal{H}$ is a finite family of η_j-demimetric mappings and Φ be a convex, bounded below and Gâteaux differentiable function on $\mathcal{H}$ with derivative $\acute{\Phi}$. Assume that $\Pi := \bigcap_{i=1}^{M}(EP(\tilde{g}_i)) \cap \bigcap_{j=1}^{N} Fix(S_j) \neq \emptyset$. Suppose that (A1)-(A6) hold then the sequence (Θ_k) given by (2.1) converges strongly to $\bar{u}$, where $\bar{u}$ is the unique solution of (1.7) or (1.8).*

Proof. It follows from Lemma 2.10 that if $V_k = \frac{1}{2}\|\Theta_k - \bar{u}\|^2 + \alpha_k^*(\Phi(\Theta_k) - \inf \Phi)$, then both (V_k) and (Θ_k) are bounded. Hence, there exists a constant $M \geq 0$ such that $\|\langle \Theta_k - \bar{u}, \acute{\Phi}(\Theta_k)\rangle\| \leq M$ for all $k \geq 0$. Utilizing Lemma 2.9, it yields

$$V_{k+1} - V_k + \frac{1}{2}(1 - 2L\alpha_k^*)\|\Theta_{k+1} - \Theta_k\|^2 \leq M\alpha_k^*. \tag{3.7}$$

For simplification, we consider the following two cases:

Case A. In the first instance, we assume that (V_k) is monotone, i.e., for large enough k_0, $(V_k)_{k \geq k_0}$ is either non-increasing or non-decreasing. In addition, (V_k) is bounded and hence it is convergent. Using (C2), that $\lim_{k \to +\infty} \|\Theta_k - \bar{u}\|$ exists. Utilizing (3.7) and $\lim_{k \to \infty} \|V_{k+1} - V_k\| = 0$, we have

$$\lim_{k \to \infty} \|\Theta_{k+1} - \Theta_k\| = 0. \tag{3.8}$$

Now, consider the re-arranged version of the estimate (2.4) and using (A1), we have

$$\begin{aligned} \beta_k(1 - \eta_j - \beta_k)\sum_{j=1}^{M} \gamma_j \|q_k - S_j q_k\|^2 &\leq \|\Theta_k - \bar{u}\|^2 - \|\Theta_{k+1} - \bar{u}\|^2 \\ &\leq (\|\Theta_k - \bar{u}\| + \|\Theta_{k+1} - \bar{u}\|)\|\Theta_k - \Theta_{k+1}\|. \end{aligned}$$

Letting $k \to \infty$ and utilizing (3.8), we have

$$\beta_k(1 - \eta_j - \beta_k)\sum_{j=1}^{M} \gamma_j \|b_k - S_j q_k\|^2 = 0. \tag{3.9}$$

It is observed that

$$\sum_{j=1}^{N} \gamma_j \|S_j q_k - q_k\| = \frac{1}{\beta_k}\|\Theta_{k+1} - q_k\| \to 0, \; j = 1, 2, \cdots, N.$$

The above estimate implies that

$$\lim_{k \to \infty} \|q_k - \Theta_{k+1}\| = 0. \tag{3.10}$$

From (3.8), (3.10) and the following triangular inequality:

$$\|q_k - \Theta_k\| \leq \|q_k - \Theta_{k+1}\| + \|\Theta_{k+1} - \Theta_k\|,$$

we get

$$\lim_{k \to \infty} \|q_k - \Theta_k\| = 0. \tag{3.11}$$

Hence from Lemma 3.3, we deduce that $\bar{u} \in \Pi$.

Case B. Conversely, suppose (V_k) is not monotone sequence and for all $k \geq k_0$ (for some k_0 large enough). Let a mapping $\omega : \mathbb{N} \to \mathbb{N}$ defined by

$$\omega(k) := \max\{m \in \mathbb{N}; m \leq k, V_k \leq V_{k+1}\}. \tag{3.12}$$

Note that, ω is a non-decreasing sequence imply that $\omega(k) \to +\infty$ as $k \to +\infty$ and $V_{\omega_k} \leq V_{\omega(k)+1}$ for $k \geq k_0$, so therefor by using (3.7), it yields

$$\frac{1}{2}(1-2L\alpha_{\omega(k)})\|\Theta_{\omega(k)+1}-\Theta_{\omega(k)}\|^2 \leq M\alpha_{\omega_k} \to 0, \tag{3.13}$$

hence, $\|\Theta_{\omega(k)+1}-\Theta_{\omega(k)}\| \to 0$. Utilizing Lemma 2.9, for any $n \geq 0$, the inequality $V_{n+1} < V_n$ holds provided that $\Theta_n \notin \Gamma := \{\Theta \in \mathcal{H}; \langle \Theta - \bar{u}, \acute{\Phi}(\Theta)\rangle \leq 0\}$. Consequently, we have $\Theta_{\omega(k)} \in \Gamma$ for all $k \geq k_0$ (since $V_{\omega(k)} \leq V_{\omega(k)+1}$). By using Lemma 3.2, we conclude that $\|\Theta_{\omega(k)} - \bar{u}\| \to 0$ and it follows that

$$\lim_{k\to\infty} V_{\omega(k)} = \lim_{k\to\infty} V_{\omega(k)+1} = 0.$$

Moreover, for $k \geq k_0$, it is mention that $V_k \leq V_{\omega(k)+1}$ if $k \neq \omega(k)$, that is, $\omega(k) < k$, because we have $V_n > V_{n+1}$ for $\omega(k)+1 \leq n \leq k-1$. It follows that for all $k \geq k_0$, $0 \leq V_k \leq \max\{V_{\omega(k)}, V_{\omega(k)+1}\} \to 0$, hence $\lim_{k\to\infty} V_k = 0$. This completes the proof. ∎

4. Numerical Experiment and Results

This section provides effective viability of our approximants supported by a suitable example.

Example 4.1. Let $\mathcal{H} = \mathbb{R}$, $\mathcal{D} \subset \mathcal{H}$ the set of all real numbers, with the inner product defined by $\langle x, y\rangle = xy$, for all $x, y \in \mathbb{R}$ and induced usual norm $|\cdot|$. For each $i \in \{1, 2, 3, ..., M\}$, let $\tilde{g}_i : \mathcal{H}_1 \to \mathcal{H}_1$ be a finite family of bifunctions satisfying Assumption 2.1, and let the bifunctions $\tilde{g}_i(x, y) : \mathbb{R} \to \mathbb{R}$ be defined by $\tilde{g}_i(x, y) = E_i(x)(y - x)$, where

$$E_i(x) = 0, \text{ if } 0 \leq x \leq \tau_i,$$
and
$$E_i(x) = \sin(x-\tau_i) + \exp(x-\tau_i) - 1, \quad \text{if } \tau_i \leq x \leq 1,$$

where $0 < \tau_1 < \tau_2 < \cdots < \tau_M < 1$. Suppose $\Phi : \mathbb{R} \to (-\infty, \infty]$ is defined as $\Phi(x) = \frac{1}{2}|\tilde{A}x - \epsilon|^2$, with $\tilde{A}x = 0 = \epsilon$. Then Φ is a proper, convex and lower semicontinuous mapping, since $\tilde{A}$ is a continuous linear mapping (see[20]). For each $j \in \{1, 2, \cdots, N\}$, let the family of operators $S_j : \mathbb{R} \to \mathbb{R}$ be defined by

$$S_j(s) = \begin{cases} -\frac{3s}{j}, & s \in [0, \infty); \\ s, & s \in (-\infty, 0). \end{cases}$$

Clearly, S_j defines a finite family of η-demimetric operators with $\bigcap_{j=1}^{N} Fix(S_j) = \{0\}$. Hence $\Gamma = (\bigcap_{i=1}^{M} EP(g_i)) \cap (\bigcap_{j=1}^{N} Fix(S_j)) = 0$.

It is easy to prove that the conditions $(L3)$ and $(L4)$ for the bifunctions $\tilde{g}_i$ are satisfied. Since $E_i(x)$ is nondecreasing on $[0, 1]$, we have

$$\tilde{g}_i(x, y) + \tilde{g}_i(y, x) = (x - y)(E_i(y) - E_i(x)) \leq 0.$$

It is noted that every monotone function is also pseudomonotone, so $\tilde{g}_i$ is monotone and it also pseudomonotone. Furthermore, $E_i(x)$ is 4-Lipschitz continuous. After simple calculation, it yields,

$$\begin{aligned}\tilde{g}_i(x,y)+\tilde{g}_i(y,z)-\tilde{g}_i(x,z) &= (y-z)(E_i(x)-E_i(y)) \\ &\geq -4|x-y||y-z| \\ &\geq -2(x-y)^2-2(y-z)^2,\end{aligned}$$

which shows that the Lipschitz-type continuity of $\tilde{g}_i$ with $d_1 = d_2 = 2$. Thus, we have

$$\tilde{g}_i(x,y) = E_i(x)(y-x) \geq 0,$$

for all $y \in [0,1]$, if and only if $0 \leq x \leq \tau_i$, i.e., $EP(\tilde{g}_i) = [0,\tau_i]$. It is noted that $\bigcap_{i=1}^{M} EP(\tilde{g}_i) = [0,\tau_1]$. Hence $\Gamma = \Omega \cap \bigcap_{i=1}^{M} EP(g_i) = 0$. In order to compute Θ_{k+1}, for each $j \in \{1,2,\cdots,N\}$, take $S_j = S$. We know that S is η-demimetric mapping with a constant $\eta = \frac{96}{121}$. Choose $S_j q_k = -5q_k$, $\beta_k = \frac{1}{1+100k}$, $\gamma = \frac{1}{7}$, $M = 2\times10^5$, and $N = 3\times10^4$. For the numerical experiment of the HSDA 2.1, the stopping criteria is defined as Error $= E_k = \|\Theta_k - \Theta_{k-1}\| < 10^{-5}$. The different cases of x_0 are giving as following:
Case I: $x_0 = 2$,
Case II: $x_0 = -3.7$.

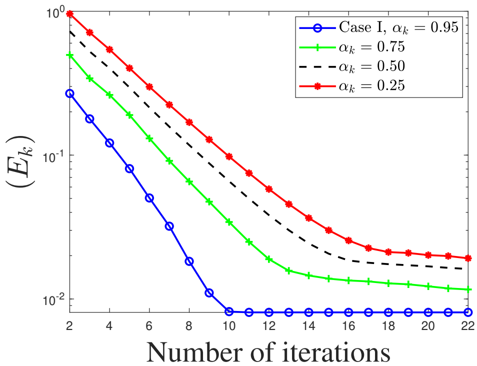
Table 1. Computations of the approximants 2.1 with different values of α_k.

	No. of Iterations		CPU Time	
	Case I	Case II	Case I	Case II
Thm. 3.4 ($\alpha = 0.95$)	10	30	0.043279	0.048213
Thm. 3.4 ($\alpha = 0.75$)	29	41	0.046856	0.054741
Thm. 3.4 ($\alpha = 0.50$)	37	51	0.051886	0.061019
Thm. 3.4 ($\alpha = 0.25$)	45	69	0.064893	0.089532

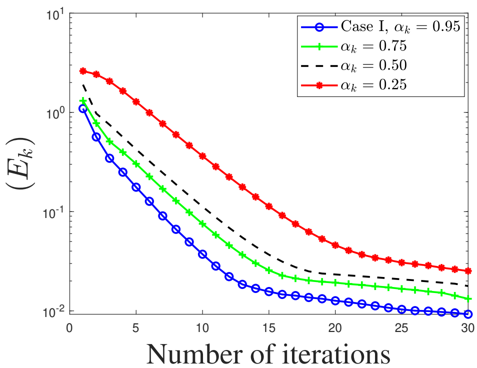
The error plotting $\|\Theta_k - \Theta_{k-1}\|$ against the approximants 2.1 for each case in Table 1 has shown in Figure 1.

Remark 4.2.

1. *The example presented above elaborate the impact of different values of α_k on our proposed approximants.*
2. *The numerical results presented in Table 1 and Figures 1 indicate that our proposed approximants is efficient, easy to implement and does well for any values of $\alpha \neq 0$ in both number of iterations and CPU time required.*
3. *We observe that the CPU time of the approximants 2.1 increases, but the number of iterations decreases when the parameter α approaches 1.*



(a) Case 1



(b) Case 1

Fig. 1. Computations of No. of Iterations and Error for iterative schemce 2.1

5. Conclusions

In this paper, we have devised a modified HSDA for computing the convex minimization problems over the solution set of pseudomontone equilibrium problem and the set of fixed point set of a finite family of η_j-demimetric mappings in Hilbert space. The theoretical framework of the algorithm has been strengthened with an appropriate numerical example. As far as we know, such results have not so far appeared in the literature and as a consequence, our theoretical framework constitutes an important topic of future research.

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Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

Availability of Data and Material

Data sharing not applicable to this article as no data-sets were generated or analysed during the current study.

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