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Fermatean Fuzzy Divergences and Their Applications to Decision-making and Pattern Recognition

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ABSTRACT

Fermatean fuzzy sets (FFS) generalized the Intuitionistic fuzzy set and Pythagorean fuzzy set in terms of more space available to choose orthopairs. The manuscript provides Chi-square and Canberra divergence measures for FFSs. Divergence measurements' additional characteristics are looked into to ensure good performance. The entropy and dissimilarity measures from the suggested divergence measures are derived. A technique is developed to transform the real or fuzzy data into Fermatean fuzzy data. An empirically successful VIKOR method is extended for FFSs. The Australasian New Car Assessment Program (ANCAP) provides the star rankings from a safety point of view for each vehicle. The VIKOR method is employed to draw safety rankings of small cars tested from 2019 to 2021 by ANCAP. The numerical examples are given to clarify each method under discussion.

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1. Introduction

The notion of fuzzy sets was developed as a way to mathematically represent the ambiguity we often use to explain events that don't have clearly defined boundaries. The number of

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items encountered in human reasoning that can be the subject of scientific research is enlarged by employing the idea of partial degrees of membership to create a mathematical description of fuzzy sets [30]. However, The negative side of information is required in real-life scenarios and cannot only be retrieved from the positive side. For instance, the use of antibiotic medications, while beneficial in the treatment of various diseases, has some adverse effects on the body. Information's positive and negative aspects can be viewed as membership degree (MD) and non-membership degree (non-MD), respectively. A non-MD is distinct from a MD. In particular, Atanassov pioneered the idea of taking into account both MD and non-MDs and gave it the name intuitionistic fuzzy set (IFS) [2]. Figure 1 provides the geometric explanation of an IFS. In Figure 1, $(1,0)$ and $(0,1)$ symbolize absolute agreement and total disagreement, respectively, whereas $(0,0)$ denotes utter obscurity or ignorance of the situation. The triangular region's ordered pair $(\mu(u),\nu(u))$, also known as intuitionistic fuzzy value (IFV), indicates that the individual is μ percent agrees with the circumstance u and ν percent disagrees with it.

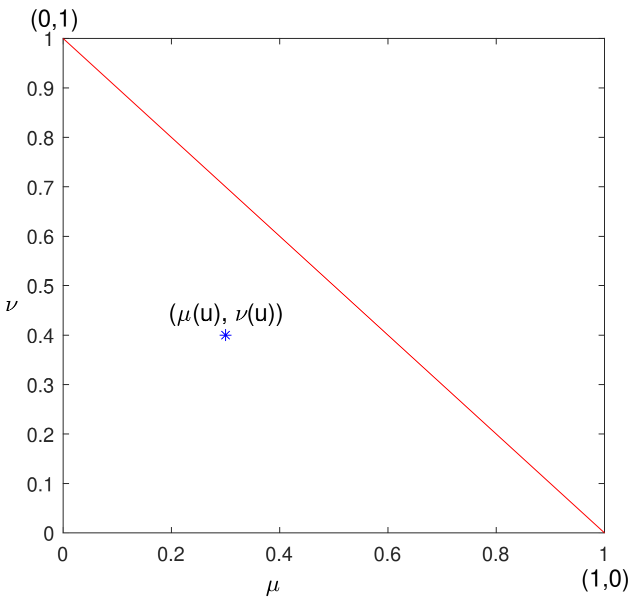

Fig. 1. Geometrical exposition of an IFS

The constraint on membership and non-membership may limit the applicability of IFS, despite the fact that it has been used to solve a variety of issues. This problem is somehow addressed by the introduction of Pythagorean fuzzy sets (PFSs) by Yager [28]. It relaxed the defining conditions, where sum of membership and no-MDs should be less than or equal to one. Recently, Senapati and Yager [27] extended the idea of IFS and PFS to Fermatean fuzzy sets (FFS). FFSs provides the larger space to choose orthopairs. In FFSs, the sum of cube of membership and non-MDs should be less than or equal to one. Figure 2 shows the geometrical interpretations of PFSs and FFSs. A clear picture of expanding the space for orthopairs for FFSs can be seen in Figure 2.

As it provides the larger space to choose orthopairs for membership and non-membership

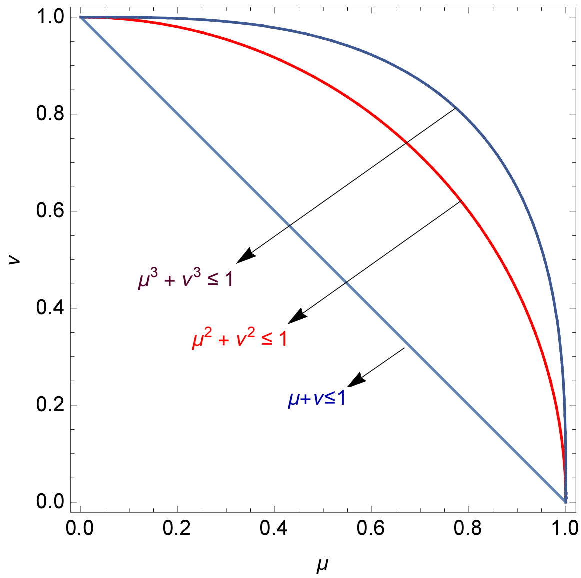

Fig. 2. The geometrical expositions and a comparison between the spaces available for IFS, PFS, and FFS

grades, many scholars have worked on its applications. The Fermatean fuzzy weighted average and geometric aggregation operators were established by Senapati and Yager [25]. They also discussed subtraction, arithmetic mean, and division operations for FFSs [26]. Garg et al. [7] developed different aggregation operators for FFSs and applied them to Covid-19 testing facility. Ghorabaee et al. [13] extended WASPAS for FFSs and discussed their applications to green supplier evaluation. Fermatean fuzzy Einstein aggregation operators and their applications were discuused by Akram et al. [1]. Liu et al. [17] introduced Fermatean fuzzy linguistic term set and their aggregation operators. MULTIMOORA method based on Fermatean fuzzy Einstein aggregation operators was discuused by Rani and Mishra [21]. The interval-valued FFSs and their basic properties were established by Jeevaraj [11]. Sahoo [24] provided new score function for FFSs and discussed fuzzy transportation problems. Fermatean fuzzy SAW, ARAS, and VIKOR method extensions were focused by Gul [8]. The MCDM method was developed based on Dempster–Shafer theory and entropy measure for FFSs [5]. Fermatean fuzzy Hamacher aggregation operators were established by Hadi et al. [9]. Aydemir discussed TOPSIS method using dombi aggregation operators for FFSs [3]. CRITIC-EDAS method for FFS and their applications to sustainable third-party reverse logistics providers were discussed by Mishra et al. [18].

Divergence measures were first proposed in classical probability spaces as a way to compare two probability distributions. The expression to estimate the difference between two fuzzy sets was developed by Bhandari et al. [4]. They suggested the non-negativity, symmetry, and identity of indiscernibles properties for the formula for divergences. Montes et al. [19] later presented an axiomatic definition for fuzzy divergence. Divergence measures play a significant role and are used in a variety of contexts, including image processing, image thresholding, decision-making, edge detection, pattern recognition, clustering, figure skating, et cetera [4, 19, 20, 15]. Zhou et al. [31] established the divergence metrics for the PFS based on the belief function. By using the divergence metric dependent VIKOR approach in a PF domain, renewable energy technologies were evaluated [22]. The axiomatically supported divergence measurements for the q-rung orthopair fuzzy environment were proposed by Khan et al. [14]. Riaz et al. [23] talked about correlation coefficients and how they are used in pattern analysis and clustering. Khan et al. [16] gave the theoretical justifications for the VIKOR method's successful empirical use.

The Australasian New Car Assessment Program (ANCAP) provides the safety ratings for each vehicle in a particular category. ANCAP is an Australasian-based independent vehicle safety authority established in 1993 and issues safety ratings for thousands of new vehicle makes models and variants. It provides relative safety ratings between vehicles of similar size. ANCAP safety ratings are helpful for occupants and pedestrians to avoid or minimize the effects of a crash. Although these ratings are valuable, ANCAP has not provided any specific rankings of vehicles. An interested buyer of a vehicle has many options, if he follow NCAP Safety Ratings. There are many vehicles with maximum (five) stars ratings. Therefore, it is hard for a person to choose a specific vehicle based on ANCAP Safety Ratings. Therefore, a method or technique is required to draw the rankings of vehicles that occurred in the same category. It will surely be helpful for the buyers. For more details, we refer to Section 5.2.

The motivations behind the extensions of fuzzy set theory are to provide the mechanism to deal with the uncertainty that transpires in real-life problems. Due to it, the fuzzification of the real data is of paramount importance. There exist many techniques to fuzzify real-life data. But there does not exist any approach for obtaining the data in Fermatean fuzzy form. Thus we provide a new appproach to transform the data into Fermatean fuzzy form. We are also driven to develop additional functions that extend the VIKOR technique for FFSs while respecting the axiomatic assumptions of divergence measures. The manuscript discusses the fuzziness and dissimilarity measures for FFSs. Lastly, we are motivated to provide the safety rankings to small cars tested by ANCAP during 2019 to 2021.

With that in mind, the purpose of this article is to describe novel divergence functions, uncertainty (entropy), and dissimilarity measurements for FFSs. And to provide a method of transformation of real or fuzzy data into Fermatean fuzzy form. Also, to continue the VIKOR approach for FFSs and to produce the safety rankings for small cars tested by ANCAP from 2019 to 2021.

The manuscript's key contributions are:

- We suggest Canberra divergence and generalized Chi-square measures for FFSs. We establish these measurements' axiomatic validity.
- We examine the extra characteristics of these systems that support mathematical reasoning.
- We talk about the measurements of entropy and dissimilarity for FFSs.
- We present a technique to transform real or fuzzy data into Fermatean fuzzy data.
- The study provides the safety rankings of small cars tested from 2019 to 2021 by ANCAP.
- The manuscript develops the FFSs VIKOR technique.
- This paper gives examples to illustrate our suggested approach.

The remainder of the article is structured as follows: Section 2 covers the fundamentals of FFSs. Section 3 proposes the new divergence measurements and their additional features for FFSs. In Sections 3.1 and 3.2, respectively, it is discussed how divergence metrics transform into entropy and dissimilarity measures. The process of converting real data into Fermatean fuzzy data is explained in Section 4. In Section 5.1, the divergence measure-based VIKOR approach for FFSs is suggested. Section 5.2, it is detailing how the suggested VIKOR approach can be used to rank the safety of small cars that have undergone ANCAP testing. The manuscript's concluding observations are included in Section 7.

2. Preliminaries

The fundamental definitions of FFSs are included in this section. There is mention of the fundamental functions of FFSs. Throughout the entire manuscript, U is a finite non-empty set called the universal set. Besides, the unit interval $[0, 1]$ is represented by $\mathbf{I}$.

Definition 2.1. [2] An IFS A over a universal set U is defined as

$$A = \{(u, \mu_P(u), \nu_P(u)) \mid u \in U\},$$

where $\mu_A : U \to \mathbf{I}$ and $\nu_A : U \to \mathbf{I}$, with the constraint $\mu_A(u) + \nu_A(u) \leq 1$, are the MDs and non-MDs, respectively. The expression $\pi_A(u) = 1 - (\mu_A(u) + \nu_A(u))$ gives the hesitancy degree of an element $u \in U$.

3. Divergence Measures for FFSs

This section provides the axiomatic definition of FFSs' divergence measures. It investigates several functions that are consistent with the axioms underlying the FF divergence measure. We discuss the additional attributes of the suggested divergence measures and place special emphasis on the entropy and dissimilarity measures for FFSs.

Definition 3.1. If a function $Div : C(U) \times C(U) \to \Re$ satisfies the following axioms, it is referred to as a divergence measure for FFSs: for each $A_1, A_2, A_3 \in C(U)$,

(D1) $Div(A_1, A_2) = Div(A_2, A_1)$.

(D2) $Div(A_1, A_2) = 0 \iff A_1 = A_2$.

(D3) $Div(A_1 \cup A_3, A_2 \cup A_3) \leq Div(A_1, A_2)$.

(D4) $Div(A_1 \cap A_3, A_2 \cap A_3) \leq Div(A_1, A_2)$.

Unless otherwise stated, in this section $A_1 = \{(u_i, \mu_{A_1}(u_i), \nu_{A_1}(u_i)) \mid i = 1, \ldots, n\}$ and $A_2 = \{(u_i, \mu_{A_2}(u_i), \nu_{A_2}(u_i)) \mid i = 1, \ldots, n\}$ designate FFSs on the same set $U = \{u_1, \ldots, u_n\}$.

Definition 3.2. The interpretation of the divergence functions based on chi-square distances for FFSs is

$$\bar{D}_1(A_1, A_2) = \frac{1}{n}\sum_{i=1}^{n}\left[\frac{\left(\mu_{A_1}^{\beta}(u_i) - \mu_{A_2}^{\beta}(u_i)\right)^2}{\lambda + \mu_{A_1}^{\beta}(u_i) + \mu_{A_2}^{\beta}(u_i)} + \frac{\left(\nu_{A_1}^{\beta}(u_i) - \nu_{A_2}^{\beta}(u_i)\right)^2}{\lambda + \nu_{A_1}^{\beta}(u_i) + \nu_{A_2}^{\beta}(u_i)}\right], \tag{3.1}$$

where $\lambda > 0$ and $1 \leq \beta \leq 3$.

Remark 3.3. The values of the parameters λ and β chosen empirically. For other values, the function still remain the divergence measure for FFSs. It is better to take the minimum value for λ.

Theorem 3.4. *The mapping $\bar{D}_1 : C(U)\times C(U) \to \Re$ described in (3.1) satisfies the divergence measure axioms.*

Proof. We must confirm the divergence axioms for $\bar{D}_1$ to finish the proof.

- (D1-D2) Straightforward.
- (D3) For any FFSs $A_1 = (\mu_{A_1}, \nu_{A_1})$, $A_2 = (\mu_{A_2}, \nu_{A_2})$ and $A_3 = (\mu_{A_3}, \nu_{A_3})$, the mapping $\bar{D}_1$ requires to fulfill

$$\bar{D}_1(A_1 \cap A_3, A_2 \cap A_3) \leq \bar{D}_1(A_1, A_2). \tag{3.2}$$

Now,

$$\bar{D}_1(A_1 \cap A_3, A_2 \cap A_3) = \frac{1}{n}\sum_{i=1}^{n}\left[\frac{\left((\min\{\mu_{A_1}, \mu_{A_3}\})^{\beta} - (\min\{\mu_{A_2}, \mu_{A_3}\})^{\beta}\right)^2}{\lambda + (\min\{\mu_{A_1}, \mu_{A_3}\})^{\beta} + (\min\{\mu_{A_2}, \mu_{A_3}\})^{\beta}} + \frac{\left((\max\{\nu_{A_1}, \nu_{A_3}\})^{\beta} - (\max\{\nu_{A_2}, \nu_{A_3}\})^{\beta}\right)^2}{\lambda + (\max\{\nu_{A_1}, \nu_{A_3}\})^{\beta} + (\max\{\nu_{A_2}, \nu_{A_3}\})^{\beta}}\right]. \tag{3.3}$$

From $\min\{\mu_{A_1}, \mu_{A_3}\}$, $\min\{\mu_{A_2}, \mu_{A_3}\}$, $\max\{\nu_{A_1}, \nu_{A_3}\}$, and $\max\{\nu_{A_2}, \nu_{A_3}\}$, the following results are deduced,

$$\mu_{A_1} \leq \mu_{A_3} \leq \mu_{A_2} \quad or \quad \mu_{A_2} \leq \mu_{A_3} \leq \mu_{A_1} \quad or \tag{3.4}$$
$$\mu_{A_3} \leq \{\mu_{A_1} \& \mu_{A_2}\} \quad or \quad \mu_{A_3} \geq \{\mu_{A_1} \& \mu_{A_2}\} \quad \& \tag{3.5}$$
$$\nu_{A_1} \leq \nu_{A_3} \leq \nu_{A_2} \quad or \quad \nu_{A_2} \leq \nu_{A_3} \leq \nu_{A_1} \quad or \tag{3.6}$$
$$\nu_{A_3} \leq \{\nu_{A_1} \& \nu_{A_2}\} \quad or \quad \nu_{A_3} \geq \{\nu_{A_1} \& \nu_{A_2}\}. \tag{3.7}$$

The proof becomes straightforward for the inequalities in (3.5) and (3.7). We prove the results for (3.4) and (3.6). If $\mu_{A_1} \leq \mu_{A_3} \leq \mu_{A_2}$ and $\nu_{A_2} \leq \nu_{A_3} \leq \nu_{A_1}$, then (3.3) becomes

$$\bar{D}_1(A_1 \cap A_3, A_2 \cap A_3) = \frac{1}{n}\sum_{i=1}^{n}\left[\frac{\left(\mu_{A_1}^{\beta} - \mu_{A_3}^{\beta}\right)^2}{\lambda + \mu_{A_1}^{\beta} + \mu_{A_3}^{\beta}} + \frac{\left(\nu_{A_1}^{\beta} - \nu_{A_3}^{\beta}\right)^2}{\lambda + \nu_{A_1}^{\beta} + \nu_{A_3}^{\beta}}\right].$$

The divergence between the two FFSs A_1 and A_2 is

$$\bar{D}_1(A_1, A_2) = \frac{1}{n}\sum_{i=1}^{n}\left[\frac{\left(\mu_{A_1}^{\beta} - \mu_{A_2}^{\beta}\right)^2}{\lambda + \mu_{A_1}^{\beta} + \mu_{A_2}^{\beta}} + \frac{\left(\nu_{A_1}^{\beta} - \nu_{A_2}^{\beta}\right)^2}{\lambda + \nu_{A_1}^{\beta} + \nu_{A_2}^{\beta}}\right].$$

Since $\beta \geq 1$, therefore, $\mu^{\beta}_{A_1} \geq \mu^{\beta}_{A_3} \geq \mu^{\beta}_{A_2}$ and $\nu^{\beta}_{A_2} \geq \nu^{\beta}_{A_3} \geq \nu^{\beta}_{A_1}$, we have

$$\frac{\left(\mu^{\beta}_{A_1} - \mu^{\beta}_{A_3}\right)^2}{\lambda + \mu^{\beta}_{A_1} + \mu^{\beta}_{A_3}} \leq \frac{\left(\mu^{\beta}_{A_1} - \mu^{\beta}_{A_2}\right)^2}{\lambda + \mu^{\beta}_{A_1} + \mu^{\beta}_{A_2}} \quad \& \quad \frac{\left(\nu^{\beta}_{A_1} - \nu^{\beta}_{A_3}\right)^2}{\lambda + \nu^{\beta}_{A_1} + \nu^{\beta}_{A_3}} \leq \frac{\left(\nu^{\beta}_{A_1} - \nu^{\beta}_{A_2}\right)^2}{\lambda + \nu^{\beta}_{A_1} + \nu^{\beta}_{A_2}}.$$

This implies

$$\bar{D}_1(A_1 \cap A_3, A_2 \cap A_3) \leq \bar{D}_1(A_1, A_2).$$

We can similarly confirm the remaining parts of axiom (D3).

- (D4) Similar to the evidence of (D3).

The metric $\bar{D}_1$ is a divergence metric for FFSs as a result. It is simple to prove the evidence for $U = \{u_1, u_2, \dots, u_n\}$. ∎

Definition 3.5. The interpretation of divergence functions based on the Canberra distances for FFSs is

$$\bar{D}_2(A_1, A_2) = \frac{1}{n} \sum_{i=1}^{n} \left[\frac{\mid \mu^{\beta}_{A_1}(u_i) - \mu^{\beta}_{A_2}(u_i) \mid}{\lambda + \mu^{\beta}_{A_1}(u_i) + \mu^{\beta}_{A_2}(u_i)} + \frac{\mid \nu^{\beta}_{A_1}(u_i) - \nu^{\beta}_{A_2}(u_i) \mid}{\lambda + \nu^{\beta}_{A_1}(u_i) + \nu^{\beta}_{A_2}(u_i)} \right], \tag{3.8}$$

where $\lambda > 0$ and $1 \leq \beta \leq 3$.

Theorem 3.6. *The expression $\bar{D}_2 : C(U) \times C(U) \to \Re$ described in (3.8) satisfies the divergence axioms.*

Proof. The evidence resembles the evidence of Theorem 3.4. ∎

There are further properties that the chi-square and Canberra divergence measures meet to ensure acceptable theoretical performance. They are shown in the following result.

Theorem 3.7. *The following properties are true for the divergence measures $\bar{D}_k$ $(k = 1, 2)$: If A_1, A_2 and A_3 be three FFSs, then*
T1. *For $\lambda = 1$, $\bar{D}_k(A_1, A_1^c) = 1$ if and only if either $\mu_{A_1} = 1$ or $\nu_{A_1} = 1$.*
T2. $\bar{D}_k(A_1^c, A_2) = \bar{D}_k(A_1, A_2^c)$.
T3. $\bar{D}_k(A_1, A_2) = \bar{D}_k(A_1^c, A_2^c)$.
T4. $\bar{D}_k(A_1 \cup A_2, A_1 \cap A_2) = \bar{D}_k(A_1, A_2)$.
T5. $\bar{D}_k(A_1 \cup A_2, A_3) \leq \bar{D}_k(A_1, A_3) + \bar{D}_k(A_2, A_3)$.
T6. $\bar{D}_k(A_1 \cap A_2, A_3) \leq \bar{D}_k(A_1, A_3) + \bar{D}_k(A_2, A_3)$.
T7. *Whenever $A_1 \subseteq A_2 \subseteq A_3$, we have $\bar{D}_k(A_1, A_3) \geq \max\{\bar{D}_k(A_1, A_2), \bar{D}_k(A_2, A_3)\}$.*

Proof. Suppose $A_1 = (\mu_{A_1}, \nu_{A_1})$, $A_2 = (\mu_{A_2}, \nu_{A_2})$ and $A_3 = (\mu_{A_3}, \nu_{A_3})$ to be three FFSs. Since $\bar{D}_1$ and $\bar{D}_2$ includes the factors $\triangle\mu_{12} = \mu^{\beta}_{A_1} - \mu^{\beta}_{A_2}$ and $\triangle\nu_{12} = \nu^{\beta}_{A_1} - \nu^{\beta}_{A_2}$. These variables form the basis of the divergence measures. With the aid of these elements, we prove the claims (**T1-T7**). If the related factors are identical on both sides, the proof will be considered conclusive.
T1: If $P = (\mu_P, \nu_P)$, then $P^c = (\nu_P, \mu_P)$. The metric $\bar{D}_1$ becomes

$$\bar{D}_1(P, P^c) = \frac{2\left(\mu^{\beta}_P - \nu^{\beta}_P\right)^2}{1 + \mu^{\beta}_P + \nu^{\beta}_P}. \tag{3.9}$$

Equation (3.9) becomes one only if either $\mu_A = 1$ or $\nu_A = 1$.
T2: For $A_1 = (\mu_{A_1}, \nu_{A_1})$ and $A_2 = (\mu_{A_2}, \nu_{A_2})$, their complements become $A_1^c = (\nu_{A_1}, \mu_{A_1})$ and $A_2^c = (\nu_{A_2}, \mu_{A_2})$. The measure $\bar{D}_1(A_1^c, A_2)$ contains $\triangle u_{12} = \nu_{A_1}^{\beta} - \mu_{A_2}^{\beta}$ and $\triangle v_{12} = \mu_{A_1}^{\beta} - \nu_{A_2}^{\beta}$. Also, the measure $\bar{D}_1(A_1, A_2^c)$ includes $\triangle u_{12}^{\star} = \mu_{A_1}^{\beta} - \nu_{A_2}^{\beta}$ and $\triangle v_{12}^{\star} = \nu_{A_1}^{\beta} - \mu_{A_2}^{\beta}$. It indicates that identical elements are present on both sides, guaranteeing that both sides are equal.
T3: Similar to the evidence from the earlier part.
T4: The side $\bar{D}_1(A_1 \cup A_2, A_1 \cap A_2)$ comprises of the following factors $\triangle u_{12} = (\max\{\mu_{A_1}, \mu_{A_2}\})^{\beta} - (\min\{\mu_{A_1}, \mu_{A_2}\})^{\beta}$, and $\triangle v_{12} = (\min\{\nu_{A_1}, \nu_{A_2}\})^{\beta} - (\max\{\nu_{A_1}, \nu_{A_2}\})^{\beta}$. While $\bar{D}_1(A_1, A_2)$ involves $\triangle u_{12}^{\star} = \mu_{A_1}^{\beta} - \mu_{A_2}^{\beta}$ and $\triangle v_{12}^{\star} = \nu_{A_1}^{\beta} - \nu_{A_2}^{\beta}$. It implies that $| \triangle u_{12} | = | \triangle u_{12}^{\star} |$ and $| \triangle v_{12} | = | \triangle v_{12}^{\star} |$, which guarantees that both sides are equal.
T5-T6: Straightforward.
T7: Similar to the Theorem 3.4 proof. ∎

3.1. Entropy Measure for FFSs

We present the divergence-based entropy metrics for FFSs in this section.

Definition 3.8. According to the divergence measures $\bar{D}_1$ and $\bar{D}_2$, the entropy measures of an FFS are as follows: for each $A \in C(U)$,

$$\bar{E}_k(A) = 1 - \frac{1}{F_k}\bar{D}_k(A, A^c), \tag{3.10}$$

where F_k is the constant function used to normalize the values and $\bar{D}_k(P, P^c)$ are

$$\bar{D}_1(A, A^c) = \frac{2}{n}\sum_{i=1}^{n}\left[\frac{\left(\mu_A^{\beta}(u_i) - \nu_A^{\beta}(u_i)\right)^2}{\lambda + \mu_A^{\beta}(u_i) + \nu_A^{\beta}(u_i)}\right],$$

$$\bar{D}_2(A, A^c) = \frac{2}{n}\sum_{i=1}^{n}\left[\frac{| \mu_A^{\beta}(u_i) - \nu_A^{\beta}(u_i) |}{\lambda + \mu_A^{\beta}(u_i) + \nu_A^{\beta}(u_i)}\right].$$

The key assertion of this section is stated in the result that follows.

Theorem 3.9. *The function $\bar{E}_k : C(U) \to [0, 1]$ expounded in (3.10) is an entropy measure.*

Proof. Straightforward. ∎

3.2. Dissimilarity Measure for IFSs

The research that has already been done on intuitionistic fuzzy divergences attests to the fact that they are the proper subset of the IF dissimilarity measures [20]. We do, however, confirm that the outcome holds true for FFSs. In other words, the axioms of dissimilarity measures are satisfied by the divergence measures $\bar{D}_1$ and $\bar{D}_2$.

Definition 3.10. If a mapping $Diss : C(U) \times C(U) \to \Re$ abides by the following axioms, it is referred to as a dissimilarity measure for FFSs: for each $A_1, A_2, A_3 \in C(U)$,

(D1) $Diss(A_1, A_2) = Diss(A_2, A_1)$.

(D2) $Diss(A_1, A_2) = 0 \Longleftrightarrow A_1 = A_2$.

(D3) Whenever $A_1 \subseteq A_2 \subseteq A_3$, we have $\bar{D}_k(A_1, A_3) \geq \max\{\bar{D}_k(A_1, A_2), \bar{D}_k(A_2, A_3)\}$.

Theorem 3.11. *The mappings $\bar{D}_1 \& \bar{D}_2 : C(U) \times C(U) \to \mathbf{I}$ abides the axioms of Definition 3.10.*

Proof. It is directed by reasoning's of Theorems 3.4-3.7. ∎

The next result confirms how well the recommended divergence measures $\bar{D}_1$ and $\bar{D}_2$ perform in comparison to the strict inclusion relation.

Theorem 3.12. *Let A_1, A_2 and A_3 be three FFSs on U. If $A_1 \subseteq A_2 \subseteq A_3$ with $\mu_{A_1} \leq \mu_{A_2} \leq \mu_{A_3}$ and $\nu_{A_1} > \nu_{A_2} > \nu_{A_3}$ or $\mu_{A_1} < \mu_{A_2} < \mu_{A_3}$ and $\nu_{A_1} \geq \nu_{A_2} \geq \nu_{A_3}$, or $\mu_{A_1} < \mu_{A_2} < \mu_{A_3}$ and $\nu_{A_1} > \nu_{A_2} > \nu_{A_3}$. Then $\bar{D}_1$ and $\bar{D}_2$ satisfies*

$$\bar{D}_k(A_1, A_3) > \max\{\bar{D}_k(A_1, A_2), \bar{D}_k(A_2, A_3)\}.$$

Proof. It is directed by reasoning's of Theorems 3.4-3.7. ∎

4. Transformation of Real Data into Fermatean Fuzzy Data

The motivations behind the extensions of fuzzy set theory are to provide the mechanism to deal with the uncertainty occurred in real-life problems. Due to it, the fuzzification of the real data is of paramount importance. There exist many techniques to fuzzify the data. But there does not exist a technique for obtaining the data in Fermatean fuzzy form. Thus we provide a new technique to transform the data into Fermatean fuzzy form. Before going into the main discussion, let L^* denote the set of all ordered pairs such that $L^* = \{(M, N) \mid (M, N) \in [0, 1] \times [0, 1] \ \& \ M + N \leq 1\}$.

Definition 4.1. The mapping $F : [0, 1]^3 \to L^*$ given by

$$F(\mu, c, t) = (M(\mu, c, t), N(\mu, c, t)),$$

where

$$M(\mu, c, t) = ((1 - tc)\mu)^{\frac{1}{3}},$$
$$N(\mu, c, t) = (1 - (1 - tc)\mu - tc)^{\frac{1}{3}},$$

satisfies that

1. If $c_1 > c_2$, then $\pi(F(\mu, c_1, t)) > \pi(F(\mu, c_2, t))$.
2. If $t_1 > t_2$, then $M(\mu, c, t_1) < M(\mu, c, t_2)$ and $N(\mu, c, t_1) < N(\mu, c, t_2)$, for all $\mu, c \in [0, 1]$.
3. $F(0, c, t) = \left(0, (1 - tc)^{\frac{1}{3}}\right)$, for all $c, t \in [0, 1]$.
4. $F(\mu, 0, t) = \left(\mu^{\frac{1}{3}}, (1 - \mu)^{\frac{1}{3}}\right)$, for all $\mu, t \in [0, 1]$.

5. $F(\mu, c, 0) = \left(\mu^{\frac{1}{3}}, (1-\mu)^{\frac{1}{3}}\right)$, for all $\mu, c \in [0, 1]$.

6. $\pi\,(F(\mu, c, t)) = (tc)^{\frac{1}{3}}$.

Theorem 4.2. *Let $B = \{\ (\mu(y))\ \mid\ y \in Y\ \}$ is the real data (or in the form of fuzzy set) and let $\pi, t : Y \to [0, 1]$ be two mappings. Then*

$$\begin{aligned}A &= \{F\,(\mu(y), \pi(y), t(y)) \mid y \in Y\}\\ &= \{(M(\mu(y), \pi(y), t(y)), N(\mu(y), \pi(y), t(y))) \mid y \in Y\}\\ &= \left\{\left(((1-t(y)\pi(y))\mu(y))^{\frac{1}{3}}, (1-(1-t(y)\pi(y))\mu(y)-t(y)\pi(y))^{\frac{1}{3}}\right) \mid y \in Y\right\} \qquad (4.1)\end{aligned}$$

is a Fermatean fuzzy set.

To show A is a FFS, we need to prove that $(M(\mu(y), c(y), t(y)))^3+(N(\mu(y), c(y), t(y)))^3 \leq 1$ or $(M(\mu(y), c(y), t(y)))^3 + (N(\mu(y), c(y), t(y)))^3 + \pi^3(y) = 1$. For simplifications, we write μ, c, and t instead of $\mu(y), c(y)$, and $t(y)$. From (4.1), we have

$$\begin{aligned}(M(\mu, c, t))^3 + (N(\mu, c, t))^3 &= \left(((1-tc)\mu)^{\frac{1}{3}}\right)^3 + \left((1-(1-tc)\mu-tc)^{\frac{1}{3}}\right)^3\\ &= (1-tc)\mu + 1 - (1-tc)\mu - tc\\ &= 1 - tc\\ &\leq 1.\end{aligned}$$

Remark 4.3. It is important to note that the parameter t in Definition 4.1 work as a control parameter which control the values of membership and non-MDs. For lesser values of t, we obtain higher values for the MDs and non-MDs.

Remark 4.4. We are motivated from Jurio et al. [12] work of transferring the fuzzy data into Intuitionistic fuzzy data. He used different parameters to construct Intuitionistic fuzzy sets (IFSs). But his proposed technique does not work for FFSs. Thus we have proposed a refined technique to construct FFSs from real or fuzzy data. Also, we control the values of MDs and non-MDs by controlling the values of the parameter t.

Example 4.5. Let $U = \{u_1, u_2, u_3\}$ be the set of universe and let $B = \{(u_1, 0.5), (u_2, 0.2), (u_3, 0.8)\}$. We fix parameters c and t as 0.2 and 0.3, respectively. The resultant FFS is obtained which is

$$A = \{(u_1, 0.7775, 0.7775), (u_2, 0.5729, 0.9094), (u_3, 0.9094, 0.5729)\}.$$

We change the values of the parameter t over its range and further observe the behavior of membership and non-membership values. We represent the resultant FFS correspond to t as A_t, that is, A_0 for $t = 0$ and so on.

$$\begin{aligned}A_0 &= \{(u_1, 0.7937, 0.7937), (u_2, 0.5848, 0.9283), (u_3, 0.9283, 0.5848)\}\\ A_{0.2} &= \{(u_1, 0.7830, 0.7830), (u_2, 0.5769, 0.9158), (u_3, 0.9158, 0.5769)\}\\ A_{0.4} &= \{(u_1, 0.7719, 0.7719), (u_2, 0.5688, 0.9029), (u_3, 0.9029, 0.5688)\}\\ A_{0.6} &= \{(u_1, 0.7606, 0.7606), (u_2, 0.5604, 0.8896), (u_3, 0.8896, 0.5604)\}\\ A_{0.8} &= \{(u_1, 0.7489, 0.7489), (u_2, 0.5518, 0.8759), (u_3, 0.8759, 0.5518)\}\\ A_1 &= \{(u_1, 0.7368, 0.7368), (u_2, 0.5429, 0.8618), (u_3, 0.8618, 0.5429)\}\end{aligned}$$

Using this data, we draw MDs and non-MDs in Figures 3 and 4, respectively. Figure 3 consist of six lines and each line connecting three points. The six lines draw against the membership values of $A_0, A_{0.2}, A_{0.4}, A_{0.6}, A_{0.8}$, and A_1. The lowest line in the graph generated against A_1 and upper one against A_0. Figure 3 provides the graphical verification that the variations in the values of parameter t result different MDs. Similarly, we observe the behavior of non-MDs in Figures 4, where the lowest and highest lines generated against A_1 and A_0, respectively.

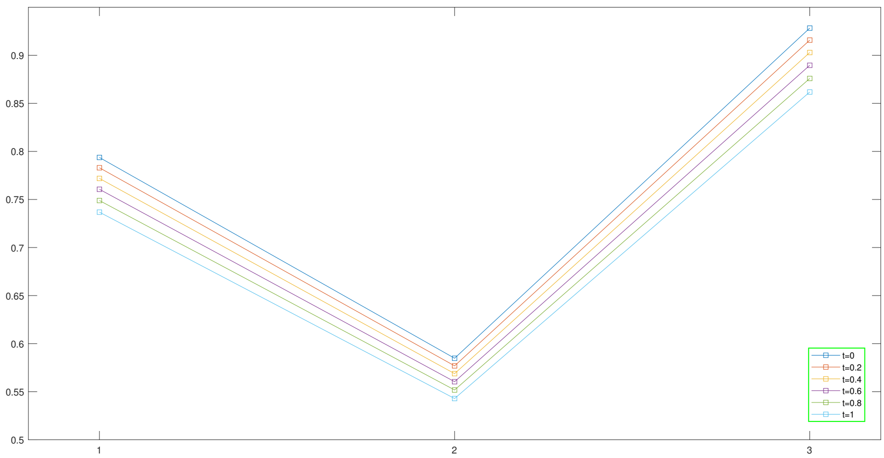

Fig. 3. Change in membership values for different values of t

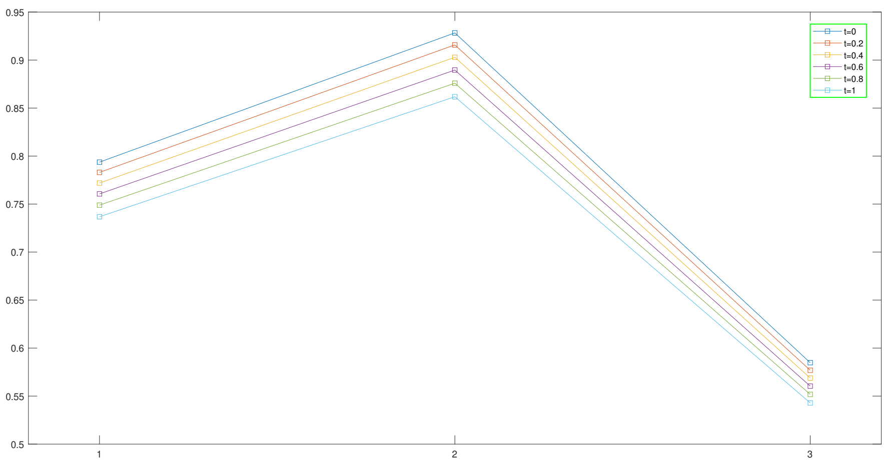

Fig. 4. Change in non-membership values for different values of t

5. Applications

The applications of the suggested divergence measures are included in this section. There are two subsections in it: In Section 5.1, the VIKOR strategy for FFSs is expanded. The ranking of small cars subjected to ANCAP safety testing from 2019 to 2021 is presented in Section 5.2.

5.1. Fermatean Fuzzy VIKOR Method

In this section, the VIKOR approach is expanded to include FFSs. This approach is created using the suggested divergence measurements. It accepts input in the form of FFS and linguistic variables.

1. The list of options is evaluated by the experts E_k, $k = \{1, 2, \dots, \ell\}$ using the suggested criteria. To depict the assessments, they can either directly input FFSs or use language variables. Fermatean fuzzy (FF) data are created from the linguistic matrix data. Thus, we get ℓ FF matrices $\left(M^k = \left[\chi_{ij}^k\right] = \left[\left(\mu_{ij}^k, \nu_{ij}^k\right)\right], k \in \{1, 2, \dots, \ell\}\right)$ from ℓ decision makers.

2. Any aggregation operator specified in [25] can be used to aggregate the data in the FF matrices.

3. The weight of the criterion varies in MCDM problems that are encountered in real life. For this reason, the weight vector $\omega = \{\omega_1, \omega_2, \dots, \omega_n\}$ is affixed to the list of requirements, where $0 \leq \omega_j \leq 1$ and $\sum_{j=1}^{n} \omega_j = 1$. Using divergence metrics on a combined FF matrix, we can construct the weight vector as follows:

$$\omega_j = \frac{\frac{1}{m-1}\left(\sum_{i=1}^{m}\sum_{k=1}^{m} \bar{D}_k\left(\chi_{ij}, \chi_{kj}\right)\right)}{\sum_{j=1}^{n}\left[\frac{1}{m-1}\left(\sum_{i=1}^{m}\sum_{k=1}^{m} \bar{D}_k\left(\chi_{ij}, \chi_{kj}\right)\right)\right]}, \quad j = 1, 2, \dots, n. \tag{5.1}$$

4. Calculate the positive ideal (PI) and negative ideal (NI) FFVs for each attribute. These characteristics direct us toward the best option and away from the least desirable one. The PI and NI FFVs are discovered using Equations (5.2) and (5.3), respectively.

$$PI_j = \left\{\left(\max_{i=1}^{m}(\mu_{ij}), \min_{i=1}^{m}(\nu_{ij})\right)\right\}, \tag{5.2}$$

$$NI_j = \left\{\left(\min_{i=1}^{m}(\mu_{ij}), \max_{i=1}^{m}(\nu_{ij})\right)\right\}, \quad j = 1, 2, \dots, n. \tag{5.3}$$

5. Using $\bar{D}_1$ and $\bar{D}_2$, the divergence between each FFV and PI FFV, as well as PI and NI FFVs, is estimated.

6. Equations (5.4), (5.5), and (5.6), respectively, construct the group utility index $\bar{S}$,

individual regret index $\bar{R}$, and compromise index $\bar{Q}$, where

$$\bar{S}(u_i) = \sum_{j=1}^{n} \omega_j \cdot \frac{\bar{D}_k(PI_j, \chi_{ij})}{\bar{D}_k(PI_j, NI_j)}, \tag{5.4}$$

$$\bar{R}(u_i) = \max_{j=1}^{n} \omega_j \cdot \frac{\bar{D}_k(PI_j, \chi_{ij})}{\bar{D}_k(PI_j, NI_j)}, \tag{5.5}$$

$$\bar{Q}(u_i) = \lambda \cdot \frac{\bar{S}(u_i) - \min_{i'=1}^{m} \bar{S}(u_{i'})}{\max_{i'=1}^{m} \bar{S}(u_{i'}) - \min_{i'=1}^{m} \bar{S}(u_{i'})} + (1-\lambda) \cdot \frac{\bar{R}(u_i) - \min_{i'=1}^{m} \bar{R}(u_{i'})}{\max_{i'=1}^{m} \bar{R}(u_{i'}) - \min_{i'=1}^{m} \bar{R}(u_{i'})}, \tag{5.6}$$

where χ_{ij} is the FFV for i^{th} alternative against j^{th} criterion.

7. By placing $\bar{S}$, $\bar{R}$, and $\bar{Q}$ in increasing order, three ranking lists are created. If the following two conditions hold true, the option u_i associated with the smallest value ($u^1 = \min_{i=1}^{m} \bar{Q}(u_i)$) of $\bar{Q}$ is ranked best (compromise solution):

C1. The ideal option found with "acceptable advantage" if

$$u^2 - u^1 \geq \frac{1}{m-1},$$

where u^2 is the next-to-last minimum in the $\bar{Q}$ list.

C2. If the best option appears on both the $\bar{S}$ and $\bar{R}$ lists, it is come with "acceptable stability" (that is, minimum in both lists).

If one of the aforementioned requirements is violated, a collection of compromise solutions is procured.

– If C2 violates, the compromise solution is composed of the equivalent u^1 and u^2 options.

– A compromise solution set has L members as a result of the C1 violation, where L is the maximum for which

$$u^L - u^1 < \frac{1}{m-1}.$$

5.2. Selection of Small Car Based on ANCAP Safety Ratings

The application of the suggested method is provided in this section. The information was obtained from the ANCAP's official website. ANCAP is an independent automotive safety organization with a base in Australasia that was founded in 1993 and has rated the safety of tens of thousands of new vehicle makes models and variants. It offers comparisons of the relative levels of safety of various-sized automobiles. In the case of a collision, ANCAP safety ratings offer protection for passengers and pedestrians as well as the vehicle's technological capacity to prevent or lessen the impacts of a collision.

On there official website, we have searched the data with the following parameters: Category: Small cars, Safety Ratings (All), Ratings years 2019-21, and Fuel type: Conventional. The thirteen vehicles lie in the above mentioned search, that is, we obtained the data of 13 small cars tested from 2019 to 2021 and their fuel type is conventional. It includes the Citroen C4, Audi A3, SEAT Leon, Cupra Leon, Kia Cerato (S & sport variant), Kia Cerato, BMW 2 Series Gran Coupé, Volkswagen Golf, Skoda Scala, BMW 1 Series, Ford Focus, Mercedes-Benz B-Class, and Mazda 3. All of the small cars included in the search has five stars safety rating except Kia Cerato (S & sport variant). Kia Cerato (S & sport variant) has four star safety ratings. Their rating is based on the set of following criteria: Adult Occupant Protection, Child Occupant Protection, Vulnerable Road User Protection and Safety Assist. The details about the each criterion is provided in Figure 5. We have collected the data of thirteen small cars from their official website.

The available data is in the form of percentage, we have converted it into decimal form and present in Table 1. We have transformed the real data of Table 1 into FF data using the method discussed in Section 4. We have taken the values of parameters c and t equal to 0.2 and 1, respectively. The obtained FFS is displayed in Table 2.

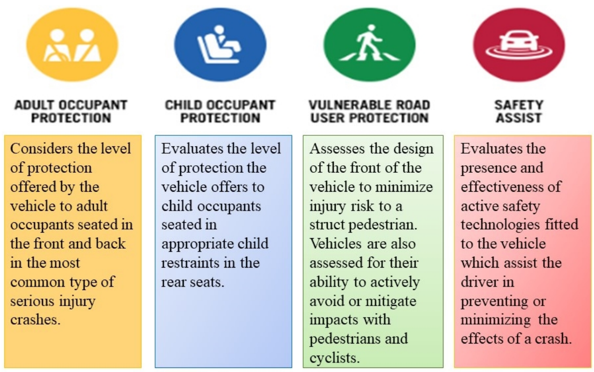

Fig. 5. Criteria explanations

Table 1. ANCAP Safety Ratings

Model	Adult OP	Child OP	VRUP	Safety Assist
Citroen C4 (u_1)	0.76	0.81	0.57	0.62
Audi A3 (u_2)	0.89	0.81	0.68	0.73
SEAT Leon (u_3)	0.92	0.88	0.71	0.80
Cupra Leon (u_4)	0.91	0.88	0.71	0.80
Kia Cerato (S & sport variant) (u_5)	0.90	0.83	0.55	0.71
Kia Cerato (u_6)	0.90	0.83	0.72	0.73
BMW 2 Series Gran Coupé (u_7)	0.94	0.89	0.76	0.73
Volkswagen Golf (u_8)	0.95	0.89	0.76	0.80
Skoda Scala (u_9)	0.97	0.87	0.81	0.76
BMW 1 Series (u_{10})	0.83	0.89	0.76	0.73
Ford Focus (u_{11})	0.96	0.87	0.72	0.72
Mercedes-Benz B-Class (u_{12})	0.96	0.92	0.78	0.77
Mazda 3 (u_{13})	0.98	0.89	0.81	0.76

OP = Occupant Protection, VRUP = Vulnerable road user protection

Table 2. ANCAP Safety Ratings in FFSs for parameters $c = 0.2$

Model	Adult OP	Child OP	VRUP	Safety Assist
u_1	$(0.8472, 0.5769)$	$(0.8653, 0.5337)$	$(0.7697, 0.7007)$	$(0.7916, 0.6724)$
u_2	$(0.8929, 0.4448)$	$(0.8653, 0.5337)$	$(0.8163, 0.6350)$	$(0.8359, 0.6000)$
u_3	$(0.9029, 0.4000)$	$(0.8896, 0.4579)$	$(0.8282, 0.6145)$	$(0.8618, 0.5429)$
u_4	$(0.8996, 0.4160)$	$(0.8896, 0.4579)$	$(0.8282, 0.6145)$	$(0.8618, 0.5429)$
u_5	$(0.8963, 0.4309)$	$(0.8724, 0.5143)$	$(0.7606, 0.7114)$	$(0.8282, 0.6145)$
u_6	$(0.8963, 0.4309)$	$(0.8724, 0.5143)$	$(0.8320, 0.6073)$	$(0.8359, 0.6000)$
u_7	$(0.9094, 0.3634)$	$(0.8929, 0.4448)$	$(0.8472, 0.5769)$	$(0.8359, 0.6000)$
u_8	$(0.9126, 0.3420)$	$(0.8929, 0.4448)$	$(0.8472, 0.5769)$	$(0.8618, 0.5429)$
u_9	$(0.9189, 0.2884)$	$(0.8862, 0.4703)$	$(0.8653, 0.5337)$	$(0.8472, 0.5769)$
u_{10}	$(0.8724, 0.5143)$	$(0.8929, 0.4448)$	$(0.8472, 0.5769)$	$(0.8359, 0.6000)$
u_{11}	$(0.9158, 0.3175)$	$(0.8862, 0.4703)$	$(0.8320, 0.6073)$	$(0.8320, 0.6073)$
u_{12}	$(0.9158, 0.3175)$	$(0.9029, 0.4000)$	$(0.8545, 0.5604)$	$(0.8509, 0.5688)$
u_{13}	$(0.9221, 0.2520)$	$(0.8929, 0.4448)$	$(0.8653, 0.5337)$	$(0.8472, 0.5769)$

OP = Occupant Protection, VRUP = Vulnerable road user protection

Next, we apply the proposed VIKOR method to the FF data. The details steps of VIKOR method for FF data are explained in Section 5.1. We have acquire the FF data or FF matrix, therefore, we start the process from Step 3. $\omega = \{\omega_1 = 0.2, \omega_2 = 0.3, \omega_3 = 0.25, \omega_4 = 0.25\}$ is taken as weight vector to distinguish the importance of criteria. The interested candidate alter the weight vector according to his need and experience. Also, there are different subjective and objective methods to get the criteria weights. It's depend on the person who want the ranking of small cars. He can choose any method to assign the criteria weights.

In the next step (Step 4), the PI and NI FFVs are determined by Equations (5.2) and (5.3). We employ these equations on Table 2 to get PI and NI FFVs.

$$\begin{aligned} PI = \{&PI_1 = (0.9221, 0.2520), PI_2 = (0.9029, 0.4000), PI_3 = (0.8653, 0.5337),\\ &PI_4 = (0.8618, 0.5429)\}\\ NI = \{&NI_1 = (0.8472, 0.5769), NI_2 = (0.8653, 0.5337), NI_3 = (0.7606, 0.7114),\\ &NI_4 = (0.7916, 0.6724)\}. \end{aligned}$$

In Step 5, the divergence between each FFV (Table 2) and PI FFV is calculated by the measure $\bar{D}_2$ presented in (3.8). Also, the divergence between PI and NI FFVs are calculated. We have taken $\lambda = 0.1$ and $\beta = 2$ for $\bar{D}_2$ in (3.8). The results are presented in matrix

$D = [d_{ij}]_{13\times 4}$, (5.7), where $d_{ij} = \bar{D}_2(PI_j, \chi_{ij})$.

$$D = \begin{bmatrix} 0.6221 & 0.2690 & 0.3439 & 0.2649 \\ 0.4021 & 0.2690 & 0.2046 & 0.1150 \\ 0.3182 & 0.1197 & 0.1627 & 0 \\ 0.3489 & 0.1197 & 0.1627 & 0 \\ 0.3767 & 0.2314 & 0.3677 & 0.1444 \\ 0.3767 & 0.2314 & 0.1482 & 0.1150 \\ 0.2451 & 0.0931 & 0.0868 & 0.1150 \\ 0.2004 & 0.0931 & 0.0868 & 0 \\ 0.0831 & 0.1446 & 0 & 0.0684 \\ 0.5217 & 0.0931 & 0.0868 & 0.1150 \\ 0.1476 & 0.1446 & 0.1482 & 0.1299 \\ 0.1476 & 0 & 0.0536 & 0.0520 \\ 0 & 0.0931 & 0 & 0.0684 \end{bmatrix} \tag{5.7}$$

The divergence measures between PI and NI FFVs are presented in (5.8).

$$E = [0.6221 \quad 0.2690 \quad 0.3677 \quad 0.2649] \tag{5.8}$$

Matrix F in (5.9) is obtained by dividing the Matrix D by E and then multiplying with weight vector, that is, $F = \omega \times \frac{D}{E} = \omega \times \frac{\bar{D}_2(PI_j, \chi_{ij})}{\bar{D}_2(PI_j, NI_j)}$.

$$F = \begin{bmatrix} 0.2000 & 0.3000 & 0.2338 & 0.2500 \\ 0.1293 & 0.3000 & 0.1391 & 0.1086 \\ 0.1023 & 0.1335 & 0.1106 & 0 \\ 0.1122 & 0.1335 & 0.1106 & 0 \\ 0.1211 & 0.2581 & 0.2500 & 0.1363 \\ 0.1211 & 0.2581 & 0.1008 & 0.1086 \\ 0.0788 & 0.1038 & 0.0590 & 0.1086 \\ 0.0644 & 0.1038 & 0.0590 & 0 \\ 0.0267 & 0.1613 & 0 & 0.0645 \\ 0.1677 & 0.1038 & 0.0590 & 0.1086 \\ 0.0475 & 0.1613 & 0.1008 & 0.1226 \\ 0.0475 & 0 & 0.0365 & 0.0491 \\ 0 & 0.1038 & 0 & 0.0645 \end{bmatrix} \tag{5.9}$$

In Step 6, the $\bar{S}$, $\bar{R}$, and $\bar{Q}$ are calculated by (5.4), (5.5), and (5.6), respectively.

$$\begin{array}{c} \\ \bar{S} \\ \bar{R} \\ \bar{Q} \\ \\ \bar{S} \\ \bar{R} \\ \bar{Q} \end{array} \begin{bmatrix} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & u_7 \\ 0.984 & 0.677 & 0.346 & 0.356 & 0.766 & 0.589 & 0.350 \\ 0.300 & 0.300 & 0.134 & 0.134 & 0.258 & 0.258 & 0.109 \\ 1.000 & 0.820 & 0.294 & 0.299 & 0.788 & 0.684 & 0.246 \\ u_8 & u_9 & u_{10} & u_{11} & u_{12} & u_{13} & \\ 0.227 & 0.253 & 0.439 & 0.432 & 0.133 & 0.168 & \\ 0.104 & 0.161 & 0.168 & 0.161 & 0.049 & 0.104 & \\ 0.164 & 0.294 & 0.416 & 0.399 & 0 & 0.130 & \end{bmatrix} \tag{5.10}$$

In Step 7, three ranking arrangements are acquired by classifying $\bar{S}$, $\bar{R}$, and $\bar{Q}$ from (5.10). Preference given to the minimum values of $\bar{S}$, $\bar{R}$, and $\bar{Q}$. $\bar{S}$, $\bar{R}$, and $\bar{Q}$ has minimum values for the alternative u_{12} (Mercedes-Benz B-Class). It ensures that the small car Mercedes-Benz B-Class acquired with acceptable stability. Also, by sorting $\bar{Q}$, the last two minimum values are $\bar{Q}(u_{12}) = 0$ and $\bar{Q}(u_{13}) = 0.130$. Since $\bar{Q}(u_{13}) - \bar{Q}(u_{12}) = 0.130 - 0 = 0.130 > \frac{1}{m-1} = \frac{1}{13-1} = 0.0833$, that is, acceptable advantage achieved. Thus, the optimum alternative u_{12} acquire with acceptable stability and acceptable advantage.

$$\begin{array}{c} \bar{S} \\ \bar{R} \\ \bar{Q} \end{array} \begin{bmatrix} u_{12} \succ u_{13} \succ u_8 \succ u_9 \succ u_3 \succ u_7 \succ u_4 \succ u_{11} \succ u_{10} \succ u_6 \succ u_2 \succ u_5 \succ u_1 \\ u_{12} \succ u_{13} \succ u_8 \succ u_7 \succ u_3 \succ u_4 \succ u_9 \succ u_{11} \succ u_{10} \succ u_6 \succ u_5 \succ u_2 \succ u_1 \\ u_{12} \succ u_{13} \succ u_8 \succ u_7 \succ u_3 \succ u_9 \succ u_4 \succ u_{11} \succ u_{10} \succ u_6 \succ u_5 \succ u_2 \succ u_1 \end{bmatrix} \tag{5.11}$$

The final ranking of the small cars based on ANCAP Safety Ratings is obtained from (5.11) and given as

$$u_{12} \succ u_{13} \succ u_8 \succ u_7 \succ u_3 \succ u_9 \succ u_4 \succ u_{11} \succ u_{10} \succ u_6 \succ u_5 \succ u_2 \succ u_1.$$

Remark 5.1. It is important to note that by changing the value of the parameter t to transform the data into FF data does not effect the ranking of small cars. Please kept in mind, we are using $\bar{D}_2$ to calculate divergence measures. Also, the different values of β and δ in (3.8) do not alter the ranking of small cars.

Remark 5.2. We have solved the problem discussed in Section 5.2 by the divergence measure $\bar{D}_1$ defined in (3.1). After calculations, we obtain the small car u_{12} with acceptable stability, but lack the acceptable advantage. Thus, the set of compromise solution is obtained and written as

$$\{u_{12} \succ u_{13} \succ u_8\} \succ u_7 \succ u_3 \succ u_4 \succ u_9 \succ u_{11} \succ u_{10} \succ u_6 \succ u_5 \succ u_2 \succ u_1.$$

6. Comparison

In this section, we compare the suggested MCDM method with the MCDM techniques that have already been published. We choose the FF environment for comparison and the comparison details are shown in Table 3. It shows the author's details, their proposed methods, and rankings. We have seen small changes in the rankings of small cars using different methods in Table 3. The abbreviations used in Table 3 come from the cited papers.

6.1. Comparison in pattern recognition problems

The classification of an unfamiliar pattern into some recognized patterns is known as pattern recognition. When working in a fuzzy environment, compatibility measurements like divergence, distance, correlation, similarity, accuracy, etc. are used to accomplish pattern recognition. Here, for pattern identification, we use some of the existing distance measures with the suggested divergence measures. Senapati and Yager [27] extended Euclidean distance ($\bar{D}_3$), and Deng and Wang [6] proposed Hellinger ($\bar{D}_4$) and triangular ($\bar{D}_5$) distances for FFSs.

Table 3. Comparison with existing MCDMs Methods

Authors & Methods	Rankings
Senapati & Yager [25] FFWA operator	$u_{13} \succ u_{12} \succ u_9 \succ u_8 \succ u_7 \succ u_3 \succ u_4 \succ u_{11} \succ u_{10} \succ u_6 \succ u_2 \succ u_5 \succ u_1$
Senapati & Yager [25] FFWG operator	$u_{13} \succ u_{12} \succ u_9 \succ u_8 \succ u_7 \succ u_{11} \succ u_3 \succ u_4 \succ u_{10} \succ u_6 \succ u_2 \succ u_5 \succ u_1$
Senapati & Yager [25] FFWPA operator	$u_{13} \succ u_{12} \succ u_9 \succ u_8 \succ u_7 \succ u_3 \succ u_4 \succ u_{11} \succ u_{10} \succ u_6 \succ u_2 \succ u_5 \succ u_1$
Senapati & Yager [25] FFWPG operator	$u_{13} \succ u_{12} \succ u_9 \succ u_8 \succ u_7 \succ u_3 \succ u_4 \succ u_{11} \succ u_{10} \succ u_6 \succ u_2 \succ u_5 \succ u_1$
Akram et al. [7] FFYWA operator	$u_{13} \succ u_{12} \succ u_9 \succ u_8 \succ u_7 \succ u_3 \succ u_4 \succ u_{11} \succ u_{10} \succ u_6 \succ u_2 \succ u_5 \succ u_1$
Akram et al. [7] FFYOWA operator	$u_{13} \succ u_{12} \succ u_9 \succ u_8 \succ u_7 \succ u_3 \succ u_4 \succ u_{11} \succ u_{10} \succ u_6 \succ u_2 \succ u_5 \succ u_1$
Akram et al. [7] FFYWG operator	$u_{12} \succ u_{13} \succ u_9 \succ u_8 \succ u_7 \succ u_3 \succ u_4 \succ u_{11} \succ u_{10} \succ u_6 \succ u_2 \succ u_5 \succ u_1$
Akram et al. [7] FFYOWG operator	$u_{12} \succ u_{13} \succ u_9 \succ u_8 \succ u_7 \succ u_3 \succ u_4 \succ u_{11} \succ u_{10} \succ u_6 \succ u_2 \succ u_5 \succ u_1$
Ghorabaee et al. [13] WASPAS method	$u_{13} \succ u_{12} \succ u_9 \succ u_8 \succ u_7 \succ u_3 \succ u_4 \succ u_{11} \succ u_{10} \succ u_6 \succ u_2 \succ u_5 \succ u_1$
Akram et al. [1] FFEWA operator	$u_{13} \succ u_{12} \succ u_9 \succ u_8 \succ u_7 \succ u_{11} \succ u_3 \succ u_4 \succ u_{10} \succ u_6 \succ u_2 \succ u_5 \succ u_1$
Akram et al. [1] FFEOWA operator	$u_{12} \succ u_{13} \succ u_9 \succ u_8 \succ u_7 \succ u_3 \succ u_4 \succ u_{11} \succ u_{10} \succ u_6 \succ u_2 \succ u_5 \succ u_1$
Akram et al. [1] GFFEWA operator	$u_{13} \succ u_{12} \succ u_9 \succ u_8 \succ u_7 \succ u_{11} \succ u_3 \succ u_4 \succ u_{10} \succ u_6 \succ u_2 \succ u_5 \succ u_1$
Akram et al. [1] GFFEOWA operator	$u_{12} \succ u_{13} \succ u_9 \succ u_8 \succ u_7 \succ u_3 \succ u_4 \succ u_{11} \succ u_{10} \succ u_6 \succ u_2 \succ u_5 \succ u_1$
Gul [8] SAW method	$u_{12} \succ u_{13} \succ u_8 \succ u_7 \succ u_3 \succ u_9 \succ u_4 \succ u_{11} \succ u_{10} \succ u_6 \succ u_5 \succ u_2 \succ u_1$
Gul [8] ARAS method	$u_{13} \succ u_{12} \succ u_8 \succ u_7 \succ u_3 \succ u_4 \succ u_9 \succ u_{11} \succ u_{10} \succ u_6 \succ u_5 \succ u_2 \succ u_1$
Gul [8] VIKOR method	$u_{12} \succ u_{13} \succ u_8 \succ u_7 \succ u_3 \succ u_4 \succ u_9 \succ u_{11} \succ u_{10} \succ u_6 \succ u_5 \succ u_2 \succ u_1$
Deng & Wang [5] EFF method	$u_{12} \succ u_{13} \succ u_8 \succ u_7 \succ u_3 \succ u_9 \succ u_4 \succ u_{11} \succ u_{10} \succ u_6 \succ u_5 \succ u_2 \succ u_1$
Hadi et al. [9] FFHWA operator	$u_{13} \succ u_{12} \succ u_9 \succ u_8 \succ u_7 \succ u_{11} \succ u_3 \succ u_4 \succ u_{10} \succ u_6 \succ u_2 \succ u_5 \succ u_1$
Hadi et al. [9] FFHOWA operator	$u_{13} \succ u_{12} \succ u_9 \succ u_8 \succ u_7 \succ u_3 \succ u_4 \succ u_{11} \succ u_{10} \succ u_6 \succ u_2 \succ u_5 \succ u_1$
Hadi et al. [9] FFHHWA operator	$u_{13} \succ u_{12} \succ u_9 \succ u_8 \succ u_7 \succ u_{11} \succ u_3 \succ u_4 \succ u_{10} \succ u_6 \succ u_2 \succ u_5 \succ u_1$
Mishra et al. [18] CRITIC-EDAS method	$u_{13} \succ u_{12} \succ u_9 \succ u_8 \succ u_7 \succ u_{11} \succ u_3 \succ u_4 \succ u_{10} \succ u_6 \succ u_2 \succ u_5 \succ u_1$
Proposed VIKOR method by $\bar{D}_1$	$\{u_{12} \succ u_{13} \succ u_8\} \succ u_7 \succ u_3 \succ u_4 \succ u_9 \succ u_{11} \succ u_{10} \succ u_6 \succ u_5 \succ u_2 \succ u_1$
Proposed VIKOR method by $\bar{D}_2$	$u_{12} \succ u_{13} \succ u_8 \succ u_7 \succ u_3 \succ u_9 \succ u_4 \succ u_{11} \succ u_{10} \succ u_6 \succ u_5 \succ u_2 \succ u_1$

$$\bar{D}_3(A_1, A_2) = \sqrt{\frac{1}{2n}\sum_{i=1}^{n}\left[\left(\mu_{A_1}^3(u_i) - \mu_{A_2}^3(u_i)\right)^2 + \left(\nu_{A_1}^3(u_i) - \nu_{A_2}^3(u_i)\right)^2 + \left(\pi_{A_1}^3(u_i) - \pi_{A_2}^3(u_i)\right)^2\right]}$$

$$\bar{D}_4(A_1, A_2) = \sqrt{\frac{1}{2n}\sum_{i=1}^{n}\left[\left(\sqrt{\mu_{A_1}^3(u_i)} - \sqrt{\mu_{A_2}^3(u_i)}\right)^2 + \left(\sqrt{\nu_{A_1}^3(u_i)} - \sqrt{\nu_{A_2}^3(u_i)}\right)^2\right]}$$

$$\bar{D}_5(A_1, A_2) = \sqrt{\frac{1}{2n}\sum_{i=1}^{n}\left[\frac{\left(\mu_{A_1}^3(u_i) - \mu_{A_2}^3(u_i)\right)^2}{\mu_{A_1}^3(u_i) + \mu_{A_2}^3(u_i)} + \frac{\left(\nu_{A_1}^3(u_i) - \nu_{A_2}^3(u_i)\right)^2}{\nu_{A_1}^3(u_i) + \nu_{A_2}^3(u_i)}\right]}$$

The question arises, what will be the final decision when various measures categorize unidentified patterns into various classes? How do we choose the metrics when they yield disparate results? We will see that the divergence measure $\bar{D}_2$ is a more certain and reliable measure in this regard. The example that follows demonstrates our findings.

Example 6.1. Let Q_1, Q_2, Q_3, Q_4, and P be the five patterns given in the form of FFSs:

$$\begin{aligned}
P &= \{(u_1, 0.8, 0.3), (u_2, 0.6, 0.2), (u_3, 0.7, 0.5), (u_4, 0.6, 0.1)\},\\
Q_1 &= \{(u_1, 0.7, 0.4), (u_2, 0.5, 0.2), (u_3, 0.7, 0.3), (u_4, 0.4, 0.2)\},\\
Q_2 &= \{(u_1, 0.8, 0.2), (u_2, 0.6, 0.1), (u_3, 0.6, 0.1), (u_4, 0.6, 0.1)\},\\
Q_3 &= \{(u_1, 0.5, 0.1), (u_2, 0.4, 0.3), (u_3, 0.5, 0.2), (u_4, 0.8, 0.3)\},\\
Q_4 &= \{(u_1, 0.6, 0.4), (u_2, 0.7, 0.2), (u_3, 0.3, 0.4), (u_4, 0.6, 0.1)\}.
\end{aligned}$$

We can determine whether P and $Q_i, i = 1, \ldots, 4$ are similar. If there is a minimum divergence between P and Q_i, the pattern P belongs to Q_j. Table 4 and Figure 6 both display the calculated values of divergence between P and Q_i using the various divergence metrics. Labels **1**, ..., **4** on X-axis in Figure 6 represents the FFSs Q_1 to Q_4, respectively.

Now, according to Table 4, P is nearest to Q_1 from the views of $\bar{D}_1$ and $\bar{D}_4$, but P is closest to Q_2 from the perspectives of $\bar{D}_2$, $\bar{D}_3$, and $\bar{D}_5$. So what will be our final decision? Also from Figure 6, the graph of divergence between P and Q_1 remains below for measures $\bar{D}_1$ and $\bar{D}_4$, and the minimum value of divergence is attained between P and Q_2 by $\bar{D}_2$, $\bar{D}_3$, and $\bar{D}_5$. In order to evaluate the effectiveness of the divergence metric, Hatzimichailidis et al. [10] defined the Degree of Confidence (DoC). For a divergence metric, the higher the value of DoC, the more reliable the measure is. A divergence measure's DoC is determined as follows:

$$DoC^{i^*} = \sum_{i=1, i\neq i^*}^{n} \left|\bar{D}_k(P, Q_{i^*}) - \bar{D}_k(P, Q_i)\right|,$$

where i^* corresponds to the FFS Q_i having minimum divergence from unknown pattern P.

Thus, we have calculated the DoC for each divergence measure and written it in Table 4. It can be seen that the DoC of $\bar{D}_2$ is much higher than other measures. This guarantee that the pattern recognition by $\bar{D}_2$ is much more confident than others.

Table 4. Divergence measures calculation for Example 6.1

	(P, Q_1)	(P, Q_2)	(P, Q_3)	(P, Q_4)	Result	DoC
$\bar{D}_1$	0.0196	0.0471	0.0687	0.0275	Q_1	0.0844
$\bar{D}_2$	0.3286	0.2471	0.4749	0.3515	Q_2	**0.4139**
$\bar{D}_3$	0.1261	0.1092	0.3016	0.2329	Q_2	0.3332
$\bar{D}_4$	0.1230	0.1261	0.2245	0.1847	Q_1	0.1662
$\bar{D}_5$	0.1673	0.1443	0.2960	0.2382	Q_2	0.2685

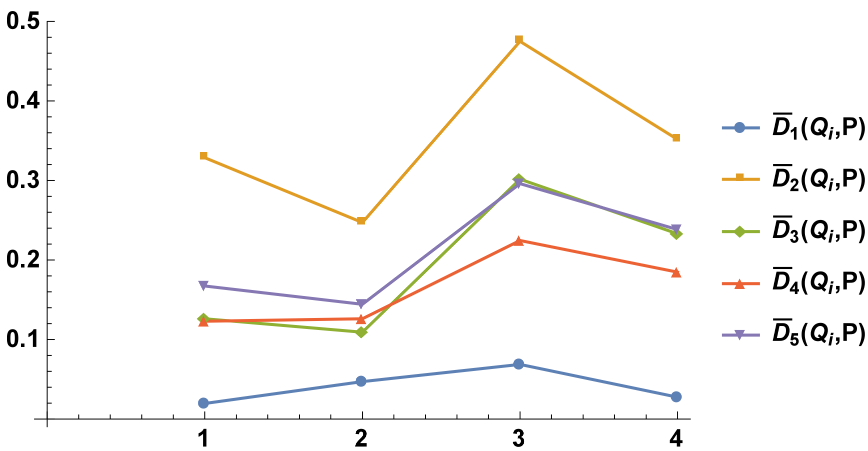

Fig. 6. Divergence between the known and unknown patterns

7. Conclusion

The study established the generalized chi-square and generalized Canberra distances-based divergence measures for FFSs. The hypotheses have been constructed to determine whether a function is a divergence measure. The proposed divergence functions' axiomatic qualities have been confirmed. Divergence measures other characteristics have been looked at to ensure their effectiveness. Additionally, research has been done on the C-IF entropy and dissimilarity metrics. Multi-criteria decision-making problems have been solved by extending and utilizing the divergence-measure-based VIKOR approach for FFSs. A method to transform the real data into FF data has been presented and justified by numerical examples. In the end, the proposed method has been employed to rank the small cars tested from 2019 to 2021 by ANCAP. ANCAP provides the safety ratings for the vehicles tested in a particular year. But it is hard for someone to choose the vehicle based on their provided rankings. Thus, the proposed MCDM method has been used to rank the thirteen small cars with conventional fuel types tested from 2019 to 2021. The suggested divergence measurements will be used in various fields, including pattern recognition, classification, and image processing. We shall define the knowledge, entropy, similarity, and dissimilarity measures for FFSs.

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