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An Approximation Technique for General Split Feasibility Problems Based on Projection onto the Intersection of Half-spaces

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ABSTRACT

This paper presents a novel relaxed *CQ* algorithm for solving the multiple-sets split feasibility problem with multiple output sets (MSSFPMOS) in infinite-dimensional real Hilbert spaces. The proposed method replaces the projection to half-space with the projection to the intersection of two half-spaces, resulting in accelerated convergence by utilizing previous half-spaces. The present study introduces a novel algorithm that dynamically determines the stepsize, without any a priori knowledge of the operator norm required. Furthermore, the algorithm is proven to exhibit strong convergence to the minimum-norm solution of the MSSFPMOS. Finally, a number of numerical experiments have been conducted to showcase the impressive performance of the proposed algorithm.

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1. Introduction

Let H_1 and H_2 be two real Hilbert spaces. Let $F : H_1 \to H_2$ be a bounded linear operator and $F^* : H_2 \to H_1$ be its adjoint. The split feasibility problem (SFP, for short) is to find a point

$$u^* \in C \text{ such that } Fu^* \in Q, \tag{1.1}$$

where C and Q are non-empty, closed, and convex subsets of H_1 and H_2, respectively.

Due to it's practical application, the SFP has received a great attention by many researchers, and several generalizations of it have been studied by many authors, see, for in-

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stance, the multiple-sets split feasibility problem (MSSFP) [7], the split feasibility problem with multiple output sets (SFPMOS)[17], the split variational inequality problem (SVIP) [5].

Let H_1 and H_2 be two real Hilbert spaces. Let $F : H_1 \to H_2$ be a bounded linear operator and $F^* : H_2 \to H_1$ its adjoint. The multiple-sets split feasibility problem (MSSFP, for short) consists of finding a point $u^* \in H_1$ such that

$$u^* \in \bigcap_{i=1}^{n} C_i \text{ such that } Fu^* \in \bigcap_{j=1}^{m} Q_j,$$

where $C_1, \dots, C_n$ and $Q_1, \dots, Q_m$ are non-empty, closed, and convex subsets of H_1 and H_2, respectively and $n \geq 1$ and $m \geq 1$ are given integers.

The MSSFP, a general way to characterize various inverse problems arising in many real-world application problems, such as medical image reconstruction and intensity-modulated radiation therapy is to find a point in the intersection of a finite family of closed convex sets in one space such that its image under a linear transformation belongs to the intersection of another finite family of closed convex sets in the image space.

In the present study, we address the following problem in infinite-dimensional real Hilbert spaces.

Let H, H_j, $j = 1, 2, \dots, m$, be real Hilbert spaces and let $F_j : H \to H_j$, $j = 1, 2, \dots, m$, be bounded linear operators. The multiple-sets split feasibility problem with multiple output sets (MSSFPMOS, for short) is to find an element u^* such that

$$u^* \in \Omega := \Big(\cap_{i=1}^{n} C_i \Big) \cap \Big(\cap_{j=1}^{m} F_j^{-1}\Big(Q_j\Big)\Big) \neq \emptyset \tag{1.2}$$

where C_i, $i = 1, 2, \dots, n$, and Q_j, $j = 1, 2, \dots, m$, are non-empty, closed, and convex subsets of H and H_j, $j = 1, 2, \dots, m$, respectively, $n, m \geq 1$ are given integers. That is, $u^* \in C_i$ for each $i = 1, 2, \dots, n$, and $F_j u^* \in Q_j$ for each $j = 1, 2, \dots, m$.

When considering the specific scenario where $n = m = 1$, the MSSFPMOS (1.2) simplifies to the more focused SFP (1.1) [2, 3, 6].

The SFP was initially introduced by Censor and Elfving [6] as a mathematical framework for addressing inverse problems in finite-dimensional Hilbert spaces. This problem formulation has found applications in various areas, including phase retrievals and medical image reconstruction, enabling the development of effective algorithms and techniques in these fields. The SFP has also attracted significant attention, leading to the development of various iterative methods for its solution. Several references [15, 2, 8, 16, 19, 21, 22, 23, 24, 25, 26, 30] and others provide insights into these iterative approaches. However, the initial algorithm proposed by Censor and Elfving [6], which relied on computing the inverse of F at each iteration, did not gain much popularity.

A more widely adopted algorithm for solving SFP is the CQ algorithm introduced by Byrne [2]:

$$u_{t+1} = P_C(u_t - \lambda F^*(I - P_Q)Fu_t), \tag{1.3}$$

where P_C and P_Q are the metric projections onto C and Q, respectively, F^* is the adjoint of F, and the stepsize $\lambda \in \big(0,\ 2\|F\|^{-2}\big)$. Xu [23] proved the weak convergence of (1.3) in the framework of infinite-dimensional Hilbert space. To acquire strong convergence, Wang and Xu [20] presented an alternative method:

$$u_{t+1} = P_C\big((1 - \sigma_t)(u_t - \lambda F^*(I - P_Q)Fu_t)\big), \tag{1.4}$$

where $\lambda \in \left(0,\ 2\|F\|^{-2}\right)$ and $\{\sigma_t\} \subset (0,1)$ such that $\lim_{t\to\infty} \sigma_t = 0$; $\sum_{t=0}^{\infty} |\sigma_{t+1} - \sigma_t| < \infty$. They proved that the sequence $\{u_t\}$ generated by (1.4) converges strongly to the minimum-norm solution of the SFP (1.1). Later, Yu et al. [27] proved that $\{u_t\}$ generated by (1.4) is strongly convergent without the assumption $\sum_{t=0}^{\infty} |\sigma_{t+1} - \sigma_t| < \infty$.

The *CQ* algorithm, classified as a gradient-projection method (GPM) in convex minimization, gained recognition due to its ability to avoid the computation of the inverse of F. However, implementing this algorithm requires prior knowledge of the operator norm $\|F\|$, which can be challenging to estimate accurately due to its global invariance. Additionally, computing a projection onto a closed convex subset is generally a nontrivial task.

To overcome these challenges, Fukushima [10] proposed a method to compute the projection onto a level set of a convex function by iteratively projecting onto half-spaces that contain the original level set. This idea was extended by Yang [24] and Lopez et al. [14] to solve SFPs in finite- and infinite-dimensional Hilbert spaces, respectively. Their research focused on SFPs where the sets C and Q are represented as sublevel sets of convex functions with bounded subdifferential operators. To be specific, the sets C and Q are defined by

$$C = \{u \in H_1 : c(u) \le 0\} \quad \text{and} \quad Q = \{v \in H_2 : q(v) \le 0\}, \tag{1.5}$$

where $c : H_1 \to \mathbb{R}$ and $q : H_2 \to \mathbb{R}$ are convex and differentiable functions. Given the iterative point u_t, Yang [24] constructed the super sets (half-spaces) C_t and Q_t of the original set C and Q, respectively. The half-spaces C_t and Q_t are defined by

$$C_t = \{u \in H_1 : c(u_t) + \langle \xi_t,\ u - u_t\rangle \le 0\}, \quad \text{where } \xi_t \in \partial c(u_t), \tag{1.6}$$

$$Q_t = \{v \in H_2 : q(Fu_t) + \langle \eta_t,\ v - Fu_t\rangle \le 0\}, \quad \text{where } \eta_t \in \partial q(Fu_t). \tag{1.7}$$

Yang [24] introduced a relaxed *CQ* algorithm of the form:

$$u_{t+1} = P_{C_t}\Big(u_t - \lambda F^*(I - P_{Q_t})Fu_t\Big), \tag{1.8}$$

where $\lambda \in \left(0,\ 2\|F\|^{-2}\right)$ and a weak convergence of it is proved.

However, it should be noted that the stepsize λ in equation (1.8) is dependent on the operator norm $\|F\|$ or its estimation, which is typically a complex calculation. To circumvent this issue, numerous self-adaptive stepsizes have been developed. Specifically, López et al. [14] introduced a relaxed version of the *CQ* algorithm based on Yang's relaxed *CQ* algorithm for solving the SFP, where closed convex subsets C and Q are considered as level sets of convex functions. They proposed an adaptive approach to determine the stepsize sequence, addressing a limitation in the original relaxed *CQ* algorithm proposed by Yang [24]. In the adjusted algorithm proposed by López et al. [14], the parameter λ was substituted with a dynamic stepsize sequence $\{\tau_t\}$ that was ingeniously defined as follows:

$$\tau_t = \frac{\rho_t \|(I - P_{Q_t})Fu_t\|^2}{\|F^*(I - P_{Q_t})Fu_t\|^2}, \tag{1.9}$$

where $\rho_t \subset (0,2)$, $\forall t \ge 1$ such that $\liminf_{t\to\infty} \rho_t(2-\rho_t) > 0$. It is imperative to acknowledge that although their algorithm achieves weak convergence, it is limited to the framework of infinite-dimensional Hilbert spaces. From its inception, equation (1.9) has garnered significant attention owing to its favorable numerical efficacy and uncomplicated structure.

In a recent publication, Yu and Wang [28] introduced a series of novel relaxed CQ algorithms. The fundamental concept underlying these algorithms involves substituting the projections onto the half-spaces C_t and Q_t with the projections onto the intersection of C_t and C_{t-1}, and the intersection of Q_t and Q_{t-1}, respectively. By leveraging the previous half-spaces, the algorithms' convergence rate is enhanced. One of the algorithms proposed by the authors takes the following form:

$$u_{t+1} = P_{C_t^2}\Big(u_t - \tau_t F^*(I - P_{Q_t^2})Fu_t\Big), \tag{1.10}$$

where $C_t^2 = C_t \cap C_{t-1}$, $Q_t^2 = Q_t \cap Q_{t-1}$, and $\tau_t = \frac{\rho_t \|(I-P_{Q_t^2})Fu_t\|^2}{\|F^*(I-P_{Q_t^2})Fu_t\|^2}$ with $\rho_t \subset (0,2)$. They proved that the algorithm (1.10) is a weakly convergent to the solution of the SFP (1.1). In the setting of infinite-dimensional spaces, however, strong convergence is frequently preferable than weak convergence for efficiently solving our problems. This naturally leads to the following question.

Question 1.1. Can we design a strongly convergent iterative scheme for the algorithm (1.10) and extend it for solving the MSSFPMOS (1.2) within the framework of infinite-dimensional real Hilbert spaces?

This paper presents a comprehensive response to Question 1.1, wherein we draw inspiration from the aforementioned works and suggest a strongly convergent relaxed CQ method for effectively solving the MSSFPMOS (1.2) in infinite-dimensional real Hilbert spaces. Our proposed algorithm offers several notable benefits, which are enumerated below.

(1) The algorithm we propose addresses a broader problem, namely the MSSFPMOS (1.2).

(2) The selection of the stepsize is dynamically determined and not contingent upon the operator norm.

(3) We substitute the projection onto the half-space with the projection onto the intersection of two half-spaces. This results in expedited convergence of the algorithm.

(4) The algorithm we propose guarantees a strong convergence to the minimum-norm solution of the MSSFPMOS (1.2).

The subsequent sections of this document are structured as follows. Sect. 2 provides an introduction to fundamental definitions and lemmas that will be utilized throughout the paper. In Sect. 3, a novel iterative method is proposed and its strong convergence is proven. Sect. 4 presents numerical experiments aimed at illustrating the effectiveness of the proposed method.

2. Preliminaries

In this section, we recall some definitions and basic results which are needed in the sequel. Throughout this paper, let

- H, H_1 or H_2 be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$, and induced norm $\|\cdot\|$,

- I denote the identity operator on H, H_1 or H_2,
- the symbols " $\rightharpoonup$ " and " $\to$ ", denote the weak and strong convergence, respectively,
- for any sequence $\{u_t\} \subseteq H$, $\omega_w(u_t) = \Big\{u \in H : \exists \{u_{t_l}\} \subseteq \{u_t\}$ such that $u_{t_l} \rightharpoonup u\Big\}$ denotes the weak ω-limit set of $\{u_t\}$.

Definition 2.1. (see [1]) Let $S \subseteq H$ be a non-empty, closed, and convex set. An operator $F : S \to H$ is called

(1) Lipschitz continuous with constant $\theta > 0$ on S if

$$\|Fu - Fv\| \leq \theta \|u - v\|, \ \forall u, v \in S;$$

(2) nonexpansive on S if

$$\|Fu - Fv\| \leq \|u - v\|, \ \forall u, v \in S;$$

(3) firmly nonexpansive on S if

$$\|Fu - Fv\|^2 \leq \|u - v\|^2 - \|(I - F)u - (I - F)v\|^2, \ \forall u, v \in S,$$

which is equivalent to

$$\|Fu - Fv\|^2 \leq \langle Fu - Fv, \ u - v\rangle, \ \forall u, v \in S;$$

(4) θ-inverse strongly monotone ($\theta - ism$) on S if there is $\theta > 0$ such that

$$\langle Fu - Fv, \ u - v\rangle \geq \theta \|Fu - Fv\|^2, \ \forall u, v \in S.$$

Definition 2.2. (see [1]) Let $S \subseteq H$ be a non-empty, closed, and convex set. For each $u \in H$, there is a unique nearest point in S, denoted by $P_S(u)$ such that

$$\|u - P_S(u)\| = \min\{\|u - v\| : v \in S\}.$$

The operator $P_S : H \to S$ is called a metric projection of H onto S.

Lemma 2.3. *(see [1]) Let $S \subseteq H$ be a non-empty, closed, and convex set. The following assertions hold for all $u, v \in H$ and $w \in S$:*

(1) $\langle u - P_S(u), \ w - P_S(u)\rangle \leq 0;$

(2) $\|P_S(u) - P_S(v)\| \leq \|u - v\|;$

(3) $\|P_S(u) - P_S(v)\|^2 \leq \langle P_S(u) - P_S(v), \ u - v\rangle;$

(4) $\|P_S(u) - w\|^2 \leq \|u - w\|^2 - \|u - P_S(u)\|^2.$

From Lemma 2.3, we conclude that the mappings P_S and $I - P_S$ are both $1 - ism$, firmly nonexpansive, and nonexpansive.

Lemma 2.4. *For all $u, v \in H$ and for all $\theta \in \mathbb{R}$, we have*

(1) $\|u \pm v\|^2 \leq \|u\|^2 \pm 2\langle v,\ u+v\rangle$;

(2) $\|u \pm v\|^2 = \|u\|^2 + \|v\|^2 \pm 2\langle u,\ v\rangle$;

(3) $\|\theta u + (1-\theta)v\|^2 = \theta\|u\|^2 + (1-\theta)\|v\|^2 - \theta(1-\theta)\|u-v\|^2$.

Definition 2.5. (see [1]) Let $\phi : H \to (-\infty, +\infty]$ be a given function. Then,

(1) The function ϕ is proper if

$$\{u \in H : \phi(u) < +\infty\} \neq \emptyset.$$

(2) A proper function ϕ is convex if for each $\theta \in (0,1)$,

$$\phi(\theta u + (1-\theta)y) \leq \theta\phi(u) + (1-\theta)\phi(v), \forall u, v \in H.$$

Definition 2.6. Let $\phi : H \to (-\infty, +\infty]$ be a proper function.

(1) A vector $\xi \in H$ is a subgradient of ϕ at a point u if

$$\phi(v) \geq \phi(u) + \langle \xi,\ v-u\rangle,\ \forall v \in H.$$

(2) The set of all subgradients of ϕ at $u \in H$, denoted by $\partial\phi(u)$, is called the subdifferential of ϕ, and

$$\partial\phi(u) = \{\xi \in H : \phi(v) \geq \phi(u) + \langle \xi,\ v-u\rangle,\ \text{for each } v \in H\}. \tag{2.1}$$

(3) If $\partial\phi(u) \neq \emptyset$, ϕ is said to be subdifferentiable at u. If ϕ is continuously differentiable, then

$$\partial\phi(u) = \{\nabla\phi(u)\}.$$

Definition 2.7. Let $\phi : H \to (-\infty, +\infty]$ be a proper function. Then,

(1) ϕ is lower semi-continuous (lsc) at u if $u_t \to u$ implies

$$\phi(u) \leq \liminf_{t\to\infty} \phi(u_t).$$

(2) ϕ is weakly lower semi-continuous (w-lsc) at u if $u_t \rightharpoonup t$ implies

$$\phi(u) \leq \liminf_{t\to\infty} \phi(u_t).$$

(3) ϕ is weakly/lower semi-continuous on H if it is weakly/lower semi-continuous at every point $u \in H$.

Lemma 2.8. *(see [1]) Let $\phi : H \to (-\infty, +\infty]$ be a proper convex function. Then ϕ is lower semi-continuous if and only if it is weakly lower semi-continuous.*

Lemma 2.9. *(see [23]) Let $C \subseteq H_1$ and $Q \subseteq H_2$ be non-empty, closed, and convex sets, and $\phi : H_1 \to (-\infty, +\infty]$ is given by*

$$\phi(u) = \frac{1}{2}\|(I - P_Q)Fu\|^2,$$

where $F : H_1 \to H_2$ is a bounded linear operator. Then, for $\theta > 0$ and $u^* \in H_1$, the following statements are equivalent.

(1) The point u^* solves the SFP (1.1).

(2) $u^* = P_C(u^* - \theta\nabla\phi(u^*))$.

(3) The point u^* solves the variational inequality problem: find a point $w \in C$ such that

$$\langle \nabla\phi(w),\ u - w\rangle \geq 0, \forall u \in C.$$

Lemma 2.10. *(see [3]) Let the function ϕ be given as in Lemma 2.9. Then,*

(1) ϕ is convex and weakly lower semi-continuous on H_1;

(2) $\nabla\phi(u) = F^*(I - P_Q)Fu$, for $u \in H_1$;

(3) $\nabla\phi$ is $\|F\|^2$-Lipschitz.

Lemma 2.11. *(see [12]) Let $\{\chi_t\}$ be a non-negative real sequence, such that for all $t \in \mathbb{N}$*

$$\begin{aligned} \chi_{t+1} &\leq (1 - \varrho_t)\chi_t + \varrho_t\mu_t, \\ \chi_{t+1} &\leq \chi_t - \varsigma_t + \varphi_t, \end{aligned} \tag{2.2}$$

where $\{\varrho_t\} \subset (0, 1)$, $\{\varsigma_t\}$ is a non-negative, real sequence, and $\{\mu_t\}$ and $\{\varphi_t\}$ are real sequences such that

(1) $\sum_{t=1}^{\infty} \varrho_t = \infty$;

(2) $\lim_{t\to\infty} \varphi_t = 0$;

(3) $\lim_{l\to\infty} \varsigma_{t_l} = 0$ implies $\limsup_{l\to\infty} \mu_{t_l} \leq 0$ for any subsequence $\{t_l\}$ of $\{t\}$.

Then, $\lim_{t\to\infty} \chi_t = 0$.

3. The Algorithm and its convergence analysis

For simplicity, hereafter, denote $I^* := \{1, 2, \dots, n\}$ and $J^* := \{1, 2, \dots, m\}$. We consider the MSSFPMOS (1.2) in which the set C_i $(i \in I^*)$ and the set Q_j $(j \in J^*)$ are defined by

$$C_i = \{u \in H : c_i(u) \leq 0\} \quad \text{and} \quad Q_j = \{v \in H_j : q_j(v) \leq 0\}, \tag{3.1}$$

where $c_i : H \to (-\infty, +\infty]$ for all $i \in I^*$ and $q_j : H_j \to (-\infty, +\infty]$ for all $j \in J^*$ are convex functions. Moreover, we assume (standard assumptions) that

(1) both $c_i (i \in I^*)$ and $q_j (j \in J^*)$ are subdifferentiable on H and H_j, respectively;

(2) for any $u \in H$ and for each $i \in I^*$, a subgradient $\xi_i \in \partial c_i(u)$ can be calculated;

(3) for any $v \in H_j$ and for each $j \in J^*$, a subgradient $\eta_j \in \partial q_j(v)$ can be calculated;

(4) both $\partial c_i (i \in I^*)$ and $\partial q_j (j \in J^*)$ are bounded operators (bounded on bounded sets).

Based on the standard assumptions, the functions $c_i(i \in I^*)$ and $q_j(j \in J^*)$ are clearly lower semi-continuous. Moreover, since $c_i(i \in I^*)$ and $q_j(j \in J^*)$ are also convex, it then follows from Lemma 2.8 that $c_i(i \in I^*)$ and $q_j(j \in J^*)$ are weakly lower semi-continuous. In our algorithm, given the t^{th} iterative point u_t, we construct "n" sets C_i^t $(i \in I^*)$ which contains the original sets C_i $(i \in I^*)$ and "m" sets Q_j^t $(j \in J^*)$ which contains the original sets Q_j $(j \in J^*)$, as follows. The set C_i^t $(i \in I^*)$ is constructed as

$$C_i^t = \Big\{u \in H : c_i(u_t) + \langle \xi_i^t,\ u - u_t\rangle \leq 0\Big\}, \tag{3.2}$$

where $\xi_i^t \in \partial c_i(u_t)$ and it follows from the fact that $C_i^t \supseteq C_i \neq \emptyset$ $(i \in I^*)$ the set C_i^t is non-empty (see in [29]). The set Q_j^t $(j \in J^*)$ is defined as

$$Q_j^t = \Big\{v \in H_j : q_j(F_j u_t) + \langle \eta_j^t,\ v - F_j u_t\rangle \leq 0\Big\}, \tag{3.3}$$

where $\eta_j^t \in \partial q_j(F_j u_t)$. Indeed, Q_j^t is non-empty because $Q_j^t \supseteq Q_j \neq \emptyset$ $(j \in J^*)$. Therefore, both C_i^t and Q_j^t are nothing but non-empty half-spaces and it is easy to verify that (see [29]) $C_i^t \supseteq C_i$ $(i \in I^*)$ and $Q_j^t \supseteq Q_j$ $(j \in J^*)$ hold for every $t \geq 0$.

Note that in contrast to an algorithm that involves metric projections onto the given sets C_i and Q_j, which is more complex, an algorithm that utilizes metric projections onto the half-spaces C_i^t and Q_j^t defined in (3.2) and (3.3) is easier to implement due to the explicit formula for projecting onto a half-space. In this case, at each step t, the algorithm only needs to compute a projection onto the current sets C_i^t and Q_j^t, rather than utilizing the previous half-spaces in their entirety. This article draws inspiration from the works of Yu and Wang [28]. Our proposed algorithm replaces the projections to the half-spaces C_i^t and Q_j^t with the projections to the intersection of C_i^t and C_i^{t-1}, and the intersection of Q_j^t and Q_j^{t-1}, respectively. This modification enables us to make full use of the previous half-spaces, resulting in a faster convergence rate of our algorithm. Furthermore, we introduce a dynamically chosen stepsize that is not reliant on the operator norm.

We hereby introduce a highly effective self-adaptive relaxed CQ method that exhibits strong convergence properties for solving the MSSFPMOS (1.2) within the context of infinite-dimensional real Hilbert spaces.

Algorithm 1: A self-adaptive approximation technique for MSSFPMOS (1.2)

Step 0. Choose two sequences $\{\sigma_t\} \subset (0,1)$ and $\{\rho_t\} \subset (0,2)$ and select $\beta > 0$. Let $u_0 \in H$ be arbitrary initial guess and set $t := 0$. Take the weights α_i^t $(i \in I^*) > 0$ and the constant parameters β_j $(j \in J^*) > 0$ such that

$$\sum_{i=1}^{n} \alpha_i^t = 1 \quad \text{and} \quad \inf_{i \in I_t} \alpha_i^t > \alpha > 0, \text{ where } I_t = \{i \in I^* : \alpha_i^t > 0\}, \quad \text{and} \quad \sum_{j=1}^{m} \beta_j = 1.$$

Step 1. Given the current iterate u_t, compute the next iterate u_{t+1} via the formula

$$u_{t+1} = \sum_{i=1}^{n} \alpha_i^t P_{C_{i,t}^{int}} \left((1-\sigma_t) \left(u_t - \tau_t \sum_{j=1}^{m} \beta_j F_j^* \left(I - P_{Q_{j,t}^{int}} \right) F_j u_t \right) \right)$$

where $C_{i,t}^{int}$, $Q_{j,t}^{int}$, and τ_t are respectively defined as follows:

$$C_{i,t}^{int} = C_i^t \cap C_i^{t-1}, \text{ for each } i = 1,2,\dots,n,$$
$$Q_{j,t}^{int} = Q_j^t \cap Q_j^{t-1}, \text{ for each } j = 1,2,\dots,m,$$

and

$$\tau_t := \frac{\rho_t \sum_{j=1}^{m} \beta_j \left\| \left(I - P_{Q_{j,t}^{int}} \right) F_j u_t \right\|^2}{\left(\max\left\{ \beta, \left\| \sum_{j=1}^{m} \beta_j F_j^* \left(I - P_{Q_{j,t}^{int}} \right) F_j u_t \right\| \right\} \right)^2}. \tag{3.4}$$

Step 2. If $u_{t+1} = u_t$, then stop; otherwise, set $t := t+1$ and return to **Step 1.**

Remark 3.1. Since C_i^t, C_i^{t-1} for each $i \in I^*$ and Q_j^t, Q_j^{t-1} for each $j \in J^*$ are both half-spaces, $C_{i,t}^{int}$ and $Q_{j,t}^{int}$ are both intersection of two half-spaces. From the subdifferentiable inequality (2.1), it is clear that $C_i \subseteq C_i^{t-1}$, $C_i \subseteq C_i^t$ for each $i \in I^*$ and $Q_j \subseteq Q_j^{t-1}$, $Q_j \subseteq Q_j^t$ for each $j \in J^*$. Hence, we have $C_i \subseteq C_{i,t}^{int}$ for each $i \in I^*$ and $Q_j \subseteq Q_{j,t}^{int}$ for each $j \in J^*$. Moreover, the explicit formula for projecting onto the intersection of two half-spaces can be found in [1], making the implementation of Algorithm 1 a straightforward task.

Theorem 3.2. *Assume that the set of solutions Ω of the MSSFPMOS (1.2) is non-empty and suppose that the sequences $\{\sigma_t\}$ and $\{\rho_t\}$ in Algorithm 1 satisfy the conditions:*

(a1) $\{\sigma_t\} \subset (0,1)$ *such that* $\lim_{t\to\infty} \sigma_t = 0$ *and* $\sum_{t=0}^{\infty} \sigma_t = \infty$,

(a2) $\{\rho_t\} \subset (0,2)$ *such that* $\liminf_{t\to\infty} \rho_t(2-\rho_t) > 0$.

Then, the sequence $\{u_t\}$ generated by Algorithm 1 converges strongly to an element $u^* \in \Omega$, where $u^* = P_\Omega 0$.

Proof. Let $u^* \in \Omega$. Since $C_i \subseteq C_{i,t}^{int}$ for each $i \in I^*$, then $u^* = P_{C_i} u^* = P_{C_{i,t}^{int}} u^*$. Let

$$v_t = (1-\sigma_t)\Big(u_t - \tau_t \sum_{j=1}^{m} \beta_j F_j^* \Big(I - P_{Q_{j,t}^{int}}\Big) F_j u_t\Big)$$

for all $i \in I^*$. Since $P_{C_{i,t}^{int}}$ for each $i \in I^*$ is firmly nonexpansive, we get

$$\begin{aligned}
\|u_{t+1} - u^*\|^2 &= \Big\| \sum_{i=1}^{n} \alpha_i^t \Big(P_{C_{i,t}^{int}} v_t - u^* \Big) \Big\|^2 \\
&\leq \sum_{i=1}^{n} \alpha_i^t \Big\| P_{C_{i,t}^{int}} v_t - u^* \Big\|^2 \\
&\leq \sum_{i=1}^{n} \alpha_i^t \Big(\|v_t - u^*\|^2 - \Big\| (I - P_{C_{i,t}^{int}}) v_t \Big\|^2 \Big) \\
&= \|v_t - u^*\|^2 - \sum_{i=1}^{n} \alpha_i^t \Big\| (I - P_{C_{i,t}^{int}}) v_t \Big\|^2 \\
&= \Big\| (1-\sigma_t)\Big(u_t - \tau_t \sum_{j=1}^{m} \beta_j F_j^* \Big(I - P_{Q_{j,t}^{int}}\Big) F_j u_t\Big) - u^* \Big\|^2 \\
&\quad - \sum_{i=1}^{n} \alpha_i^t \Big\| (I - P_{C_{i,t}^{int}}) v_t \Big\|^2 \\
&= \Big\| \sigma_t(-u^*) + (1-\sigma_t)\Big(u_t - \tau_t \sum_{j=1}^{m} \beta_j F_j^* \Big(I - P_{Q_{j,t}^{int}}\Big) F_j u_t - u^*\Big) \Big\|^2 \\
&\quad - \sum_{i=1}^{n} \alpha_i^t \Big\| (I - P_{C_{i,t}^{int}}) v_t \Big\|^2 \\
&\leq \sigma_t \|u^*\|^2 + (1-\sigma_t) \Big\| u_t - \tau_t \sum_{j=1}^{m} \beta_j F_j^* \Big(I - P_{Q_{j,t}^{int}}\Big) F_j u_t - u^* \Big\|^2 \\
&\quad - \sum_{i=1}^{n} \alpha_i^t \Big\| (I - P_{C_{i,t}^{int}}) v_t \Big\|^2. \qquad (3.5)
\end{aligned}$$

Since $Q_j \subseteq Q_{j,t}^{int}$ for each $j \in J^*$, then $F_j u^* = P_{Q_j} F_j u^* = P_{Q_{j,t}^{int}} F_j u^*$. Note that for each $j \in J^*$, $I - P_{Q_{j,t}^{int}}$ is $1-ism$ and $\sum_{j=1}^{m} \beta_j F_j^* \Big(I - P_{Q_{j,t}^{int}}\Big) F_j u^* = 0$, it means for each $j \in J^*$ that

$$\begin{aligned}
&\Big\langle \sum_{j=1}^{m} \beta_j F_j^* \Big(I - P_{Q_{j,t}^{int}}\Big) F_j u_t,\ u_t - u^* \Big\rangle \\
= &\sum_{j=1}^{m} \beta_j \Big\langle F_j^* \Big(I - P_{Q_{j,t}^{int}}\Big) F_j u_t,\ u_t - u^* \Big\rangle \\
= &\sum_{j=1}^{m} \beta_j \Big\langle \Big(I - P_{Q_{j,t}^{int}}\Big) F_j u_t - \Big(I - P_{Q_{j,t}^{int}}\Big) F_j u^*,\ F_j u_t - F_j u^* \Big\rangle
\end{aligned}$$

$$\geq \sum_{j=1}^{m} \beta_j \left\| \left(I - P_{Q_{j,t}^{int}}\right) F_j u_t \right\|^2. \tag{3.6}$$

Set $\Xi_t := \max\left\{\beta, \left\| \sum_{j=1}^{m} \beta_j F_j^* \left(I - P_{Q_{j,t}^{int}}\right) F_j u_t \right\|\right\}$. This implies $\left\| \sum_{j=1}^{m} \beta_j F_j^* \left(I - P_{Q_{j,t}^{int}}\right) F_j u_t \right\| \leq \Xi_t$. This together with (3.6) gives that

$$\begin{aligned}
&\left\| u_t - \tau_t \sum_{j=1}^{m} \beta_j F_j^* \left(I - P_{Q_{j,t}^{int}}\right) F_j u_t - u^* \right\|^2 \\
= \; & \left\| (u_t - u^*) - \tau_t \sum_{j=1}^{m} \beta_j F_j^* \left(I - P_{Q_{j,t}^{int}}\right) F_j u_t \right\|^2 \\
\leq \; & \|u_t - u^*\|^2 + \tau_t^2 \left\| \sum_{j=1}^{m} \beta_j F_j^* \left(I - P_{Q_{j,t}^{int}}\right) F_j u_t \right\|^2 \\
& -2\tau_t \left\langle \sum_{j=1}^{m} \beta_j F_j^* \left(I - P_{Q_{j,t}^{int}}\right) F_j u_t, \; u_t - u^* \right\rangle \\
\leq \; & \|u_t - u^*\|^2 + \tau_t^2 \Xi_t^2 - 2\tau_t \sum_{j=1}^{m} \beta_j \left\| \left(I - P_{Q_{j,t}^{int}}\right) F_j u_t \right\|^2 \\
= \; & \|u_t - u^*\|^2 - \rho_t(2 - \rho_t) \frac{\left(\sum_{j=1}^{m} \beta_j \left\| \left(I - P_{Q_{j,t}^{int}}\right) F_j u_t \right\|^2 \right)^2}{\Xi_t^2}.
\end{aligned} \tag{3.7}$$

Substituting, (3.7) into (3.5), we get

$$\begin{aligned}
\|u_{t+1} - u^*\|^2 \leq \; & \sigma_t \|u^*\|^2 + (1 - \sigma_t)\|u_t - u^*\|^2 \\
& -\rho_t(2 - \rho_t)(1 - \sigma_t) \frac{\left(\sum_{j=1}^{m} \beta_j \left\| \left(I - P_{Q_{j,t}^{int}}\right) F_j u_t \right\|^2 \right)^2}{\Xi_t^2} \\
& -\sum_{i=1}^{n} \alpha_i^t \left\| (I - P_{C_{i,t}^{int}}) v_t \right\|^2.
\end{aligned} \tag{3.8}$$

By (a1) and (a2), we obtain from (3.8) that

$$\begin{aligned}
\|u_{t+1} - u^*\|^2 \leq \; & \sigma_t \|u^*\|^2 + (1 - \sigma_t)\|u_t - u^*\|^2 \\
\leq \; & \max\left\{ \|u^*\|^2, \; \|u_t - u^*\|^2 \right\} \\
& \vdots \\
\leq \; & \max\left\{ \|u^*\|^2, \; \|u_0 - u^*\|^2 \right\}.
\end{aligned} \tag{3.9}$$

For that reason, the sequence $\{\|u_t - u^*\|\}_{t=0}^{t=\infty}$ is bounded. As a consequence, $\{u_t\}_{t=0}^{t=\infty}$ and $\{F_j u_t\}_{t=0}^{t=\infty}$ for each $j \in J^*$ are bounded.

By (a2), we obtain from (3.7) that

$$\left\| u_t - \tau_t \sum_{j=1}^{m} \beta_j F_j^* \left(I - P_{Q_{j,t}^{int}} \right) F_j u_t - u^* \right\|^2 \leq \| u_t - u^* \|^2. \tag{3.10}$$

By Lemma 2.3, we also obtain the following estimation.

$$\begin{aligned}
& \| u_{t+1} - u^* \|^2 \\
= & \left\| \sum_{i=1}^{n} \alpha_i^t P_{C_{i,t}^{int}}(v_t) - u^* \right\|^2 \\
= & \left\| \sum_{i=1}^{n} \alpha_i^t P_{C_{i,t}^{int}}(v_t) - \sum_{i=1}^{n} \alpha_i^t P_{C_{i,t}^{int}}(u^*) \right\|^2 \\
\leq & \left\| v_t - u^* \right\|^2 \\
= & \left\| (1-\sigma_t) \left(u_t - \tau_t \sum_{j=1}^{m} \beta_j F_j^* \left(I - P_{Q_{j,t}^{int}} \right) F_j u_t \right) - u^* \right\|^2 \\
= & \left\| (1-\sigma_t) \left(u_t - \tau_t \sum_{j=1}^{m} \beta_j F_j^* \left(I - P_{Q_{j,t}^{int}} \right) F_j u_t - u^* \right) - \sigma_t u^* \right\|^2 \\
\leq & (1-\sigma_t) \left\| u_t - \tau_t \sum_{j=1}^{m} \beta_j F_j^* \left(I - P_{Q_{j,t}^{int}} \right) F_j u_t - u^* \right\|^2 + \sigma_t \| u^* \|^2.
\end{aligned} \tag{3.11}$$

From (3.11), we also get

$$\begin{aligned}
\| u_{t+1} - u^* \|^2 \leq & \left\| (1-\sigma_t) \left(u_t - \tau_t \sum_{j=1}^{m} \beta_j F_j^* \left(I - P_{Q_{j,t}^{int}} \right) F_j u_t \right) - u^* \right\|^2 \\
= & \left\| \sigma_t(-u^*) + (1-\sigma_t) \left(u_t - \tau_t \sum_{j=1}^{m} \beta_j F_j^* \left(I - P_{Q_{j,t}^{int}} \right) F_j u_t - u^* \right) \right\|^2 \\
= & \sigma_t^2 \| u^* \|^2 + (1-\sigma_t)^2 \left\| u_t - \tau_t \sum_{j=1}^{m} \beta_j F_j^* \left(I - P_{Q_{j,t}^{int}} \right) F_j u_t - u^* \right\|^2 \\
+ & 2\sigma_t (1-\sigma_t) \left\langle u_t - \tau_t \sum_{j=1}^{m} \beta_j F_j^* \left(I - P_{Q_{j,t}^{int}} \right) F_j u_t - u^*, \; -u^* \right\rangle.
\end{aligned} \tag{3.12}$$

By combining (3.10) and (3.12), we obtain that

$$\begin{aligned}
\| u_{t+1} - u^* \|^2 \leq & \sigma_t^2 \| u^* \|^2 + (1-\sigma_t)^2 \| u_t - u^* \|^2 + 2\sigma_t (1-\sigma_t) \langle u_t - u^*, \; -u^* \rangle \\
+ & 2\sigma_t \tau_t (1-\sigma_t) \left\langle \sum_{j=1}^{m} \beta_j F_j^* \left(I - P_{Q_{j,t}^{int}} \right) F_j u_t, \; u^* \right\rangle \\
\leq & (1-\sigma_t) \| u_t - u^* \|^2 + \sigma_t \Big[\sigma_t \| u^* \|^2 + 2(1-\sigma_t) \langle u_t - u^*, \; -u^* \rangle \\
+ & 2\tau_t (1-\sigma_t) \left\langle \sum_{j=1}^{m} \beta_j F_j^* \left(I - P_{Q_{j,t}^{int}} \right) F_j u_t, \; u^* \right\rangle \Big]
\end{aligned}$$

$$
\begin{aligned}
&\leq (1-\sigma_t)\|u_t-u^*\|^2+\sigma_t\Big[\sigma_t\|u^*\|^2+2(1-\sigma_t)\langle u_t-u^*,\ -u^*\rangle \\
&+ 2\tau_t(1-\sigma_t)\Big\|\sum_{j=1}^{m}\beta_j F_j^*\Big(I-P_{Q_{j,t}^{int}}\Big)F_j u_t\Big\|\|u^*\|\Big]. \qquad (3.13)
\end{aligned}
$$

Subsequently, we demonstrate that the sequence $\{u_t\}$ produced by Algorithm 1 converges strongly to the minimum norm element $u^*=P_\Omega 0$.

Let $u^*=P_\Omega 0$. Based on (al) and (a2), without loss of generality, we can assume that $\exists\kappa>0$ such that $\rho_t(2-\rho_t)(1-\sigma_t)\geq\kappa$ for all $t\in\mathbb{N}$. Hence, we obtain from (3.8) that

$$
\begin{aligned}
\|u_{t+1}-u^*\|^2 &\leq \sigma_t\|u^*\|^2+\|u_t-u^*\|^2-\kappa\frac{\left(\sum_{j=1}^m\beta_j\Big\|\Big(I-P_{Q_{j,t}^{int}}\Big)F_j u_t\Big\|^2\right)^2}{\Xi_t^2} \\
&-\sum_{i=1}^{n}\alpha_i^t\Big\|(I-P_{C_{i,t}^{int}})v_t\Big\|^2. \qquad (3.14)
\end{aligned}
$$

Using (3.13) and (3.14), for all $t\in\mathbb{N}$, we derive the two inequalities in (3.15):

$$
\begin{aligned}
\|u_{t+1}-u^*\|^2 &\leq (1-\sigma_t)\|u_t-u^*\|^2+\sigma_t\mu_t, \\
\|u_{t+1}-u^*\|^2 &\leq \|u_t-u^*\|^2-\varsigma_t+\sigma_t\|u^*\|^2.
\end{aligned} \qquad (3.15)
$$

Now, relating (3.15) to (2.2), we obtain the following settings for all positive integer t:

$$
\begin{aligned}
&\chi_t=\|u_t-u^*\|^2; \\
&\mu_t=\sigma_t\|u^*\|^2+2(1-\sigma_t)\langle u_t-u^*,\ -u^*\rangle+2\tau_t(1-\sigma_t)\Big\|\sum_{j=1}^m\beta_jF_j^*\Big(I-P_{Q_{j,t}^{int}}\Big)F_ju_t\Big\|\|u^*\|; \\
&\varsigma_t:=\kappa\frac{\left(\sum_{j=1}^m\beta_j\Big\|\Big(I-P_{Q_{j,t}^{int}}\Big)F_ju_t\Big\|^2\right)^2}{\Xi_t^2}+\sum_{i=1}^n\alpha_i^t\Big\|(I-P_{C_{i,t}^{int}})v_t\Big\|^2
\end{aligned} \qquad (3.16)
$$

Furthermore, set $\varrho_t:=\sigma_t$, $\varphi_t:=\sigma_t\|u^*\|^2$ and thus $\{\varrho_t\}\subset(0,1)$, $\lim_{t\to\infty}\varrho_t=0$, $\sum_{t=0}^{\infty}\varrho_t=\infty$, $\lim_{t\to\infty}\varphi_t=0$.

To utilize Lemma 2.11 for the convergence analysis of the sequence $\{\chi_t\}$, it suffices to illustrate that for any subsequence $\{t_l\}$ of $\{t\}$

$$
\lim_{l\to\infty}\varsigma_{t_l}=0 \text{ implies } \limsup_{l\to\infty}\mu_{t_l}\leq 0.
$$

Let $\{t_l\}$ be a subsequence of $\{t\}$ and suppose $\lim_{l\to\infty}\varsigma_{t_l}=0$. Then, we have

$$
\lim_{l\to\infty}\left[\kappa\frac{\left(\sum_{j=1}^m\beta_j\Big\|\Big(I-P_{Q_{j,t_l}^{int}}\Big)F_ju_{t_l}\Big\|^2\right)^2}{\Xi_{t_l}^2}+\sum_{i=1}^n\alpha_i^t\Big\|(I-P_{C_{i,t_t}^{int}})v_{t_l}\Big\|^2\right]=0. \qquad (3.17)
$$

Since $\kappa > 0$, (3.17) implies that

$$\lim_{l\to\infty} \frac{\left(\sum_{j=1}^m \beta_j \left\| \left(I - P_{Q^{int}_{j,t_l}}\right) F_j u_{t_l} \right\|^2\right)^2}{\Xi^2_{t_l}} = 0, \tag{3.18}$$

and

$$\lim_{l\to\infty} \left\| (I - P_{C^{int}_{i,t_l}}) v_{t_l} \right\|^2 = 0. \tag{3.19}$$

Since $I - P_{Q^{int}_{j,t_l}}$ for each $j \in J^*$ is nonexpansive, $\{u_{t_l}\}$ is bounded, and F_j for each $j \in J^*$ is a bounded linear operator, the sequence $\{\left\| (I - P_{Q^{int}_{j,t_l}}) F_j u_{t_l} \right\|\}$ for each $j \in J^*$ is bounded, and thus the sequence $\{\Xi_{t_l}\}$ is bounded. Hence, we obtain from (3.18) that

$$\lim_{l\to\infty} \sum_{j=1}^m \beta_j \left\| \left(I - P_{Q^{int}_{j,t_l}}\right) F_j u_{t_l} \right\| = 0. \tag{3.20}$$

Next, we prove that each weak cluster point of $\{u_{t_l}\}$ belongs to Ω, that is $\omega_w(u_{t_l}) \subseteq \Omega$. Let $\bar{u} \in H$ be a weak cluster point of $\{u_{t_l}\}$. Since $\{u_{t_l}\}$ is bounded, we may assume that there exists a subsequence $\{u_{t_{l_r}}\}$ of $\{u_{t_l}\}$ that weakly convergent to $\bar{u}$. Furthermore, since each F_j for each $j \in J^*$ is linear and bounded, this yields that $\{F_j u_{t_{l_r}}\}$ weakly converges to $F_j \bar{u}$. We claim here that $\bar{u} \in \Omega$. To show this, it suffices to show that $\bar{u} \in C_i$ for all $i \in I^*$ and $F_j \bar{u} \in Q_j$ for all $j \in J^*$.

Firstly, we show that $F_j \bar{u} \in Q_j$ for all $j \in J^*$. Since ∂q_j for each $j \in J^*$ is bounded on bounded set, we may assume that there is a constant $\mathring{\eta} > 0$ such that $\|\eta_j^{t_{l_r}}\| \leq \mathring{\eta}$, where $\eta_j^{t_{l_r}} \in \partial q_j(F_j u_{t_{l_r}})$ for each $j \in J^*$. That is the sequence $\{\eta_j^{t_{l_r}}\}$ is bounded. Since $P_{Q^{int}_{j,t_{l_r}}}(F_j u_{t_{l_r}}) \in Q^{int}_{j,t_{l_r}} \subseteq Q_j^{t_{l_r}}$ for each $j \in J^*$, it follows from (3.3) and (3.20) for all $j \in J^*$ and as $r \to \infty$ that

$$\begin{aligned} q_j\left(F_j u_{t_{l_r}}\right) \leq \left\langle \eta_j^{t_{l_r}},\ F_j u_{t_{l_r}} - P_{Q^{int}_{j,t_{l_r}}}(F_j u_{t_{l_r}}) \right\rangle &\leq \left\|\eta_j^{t_{l_r}}\right\| \left\| \left(I - P_{Q^{int}_{j,t_{l_r}}}\right) F_j u_{t_{l_r}} \right\| \\ &\leq \mathring{\eta} \left\| \left(I - P_{Q^{int}_{j,t_{l_r}}}\right) F_j u_{t_{l_r}} \right\| \to 0. \end{aligned} \tag{3.21}$$

The weakly lower semi-continuity of q_j together with (3.21) implies for all $j \in J^*$ that

$$q_j(F_j \bar{u}) \leq \liminf_{r\to\infty} q_j\left(F_j u_{t_{l_r}}\right) \leq \lim_{r\to\infty} \mathring{\eta} \left\| \left(I - P_{Q^{int}_{j,t_{l_r}}}\right) F_j u_{t_{l_r}} \right\| = 0. \tag{3.22}$$

It turns out that, $F_j \bar{u} \in Q_j$, for all $j \in J^*$.

Next, we prove that $\bar{u} \in C_i$ for all $i \in I^*$. By (a1) and (3.18), we obtain that

$$\begin{aligned} \|v_{t_{l_r}} - u_{t_{l_r}}\|^2 &\leq \left\| (1 - \sigma_{t_{l_r}}) \left(u_{t_{l_r}} - \tau_{t_{l_r}} \sum_{j=1}^m \beta_j F_j^* \left(I - P_{Q^{int}_{j,t_{l_r}}}\right) F_j u_{t_{l_r}} \right) - u_{t_{l_r}} \right\|^2 \\ &= \left\| \sigma_{t_{l_r}}(-u_{t_{l_r}}) + (1 - \sigma_t) \left(u_{t_{l_r}} - \tau_{t_{l_r}} \sum_{j=1}^m \beta_j F_j^* \left(I - P_{Q^{int}_{j,t}}\right) F_j u_{t_{l_r}} - u_{t_{l_r}} \right) \right\|^2 \end{aligned}$$

$$\leq \sigma_{t_{l_r}}\left\|u_{t_{l_r}}\right\|^2 + (1-\sigma_{t_{l_r}})\tau_{t_{l_r}}^2\left\|\sum_{j=1}^m \beta_j F_j^*\left(I - P_{Q_{j,t}^{int}}\right)F_j u_{t_{l_r}}\right\|^2$$

$$\leq \sigma_{t_{l_r}}\left\|u_{t_{l_r}}\right\|^2 + (1-\sigma_{t_{l_r}})\rho_{t_{l_r}}^2\frac{\left(\sum_{j=1}^m \beta_j\left\|\left(I - P_{Q_{j,t_{l_r}}^{int}}\right)F_j u_{t_{l_r}}\right\|^2\right)^2}{\Xi_{t_{l_r}}^2} \to 0 \tag{3.23}$$

as $r \to \infty$, that is

$$\lim_{r\to\infty}\left\|v_{t_{l_r}} - u_{t_{l_r}}\right\| = 0. \tag{3.24}$$

In fact, since $P_{C_{i,t_{l_r}}^{int}}(u_{t_{l_r}}) \in C_{i,t_{l_r}}^{int} \subseteq C_i^{t_{l_r}}$ for each $i \in I^*$, it follows from (3.3) for all $i \in I^*$ that

$$\begin{aligned}
c_i(u_{t_{l_r}}) &\leq \left\langle \xi_i^{t_{l_r}},\ u_{t_{l_r}} - P_{C_{i,t_{l_r}}^{int}}(v_{t_{l_r}})\right\rangle \\
&= \left\langle \xi_i^{t_{l_r}},\ (u_{t_{l_r}} - v_{t_{l_r}}) + \left(I - P_{C_{i,t_{l_r}}^{int}}\right)v_{t_{l_r}}\right\rangle \\
&= \left\langle \xi_i^{t_{l_r}},\ (u_{t_{l_r}} - v_{t_{l_r}})\right\rangle + \left\langle \xi_i^{t_{l_r}},\ \left(I - P_{C_{i,t_{l_r}}^{int}}\right)v_{t_{l_r}}\right\rangle \\
&\leq \left\|\xi_i^{t_{l_r}}\right\|\left[\left\|u_{t_{l_r}} - v_{t_{l_r}}\right\| + \left\|\left(I - P_{C_{i,t_{l_r}}^{int}}\right)v_{t_{l_r}}\right\|\right].
\end{aligned} \tag{3.25}$$

Since ∂c_i for each $i \in I^*$ is bounded on bounded set, we may again assume that for all $t_{l_r} \geq 0$, there is a constant $\mathring{\xi} > 0$ such that $\|\xi_i^{t_{l_r}}\| \leq \mathring{\xi}$, where $\xi_i^{t_{l_r}} \in \partial c_i(u_{t_{l_r}})$ for each $i \in I^*$. Hence, it follows from (3.19), (3.24), and (3.25) for all $i \in I^*$ as $r \to \infty$ that

$$\begin{aligned}
c_i(u_{t_{l_r}}) &\leq \left\|\xi_i^{t_{l_r}}\right\|\left[\left\|u_{t_{l_r}} - v_{t_{l_r}}\right\| + \left\|\left(I - P_{C_{i,t_{l_r}}^{int}}\right)v_{t_{l_r}}\right\|\right] \\
&\leq \mathring{\xi}\left[\left\|u_{t_{l_r}} - v_{t_{l_r}}\right\| + \left\|\left(I - P_{C_{i,t_{l_r}}^{int}}\right)v_{t_{l_r}}\right\|\right] \to 0.
\end{aligned} \tag{3.26}$$

The weakly lower semi-continuity of c_i together with (3.26) implies for all $i \in I^*$ that

$$c_i(\bar{u}) \leq \liminf_{r\to\infty} c_i(u_{t_{l_r}}) \leq 0, \tag{3.27}$$

consequently, $\bar{u} \in C_i$, $\forall i \in I^*$. Altogether, we conclude that $\bar{u} \in \Omega$. Since $\bar{u}$ is arbitrary, we conclude that each weak cluster point of $\{u_{t_l}\}$ belongs to Ω. That is $w_\omega(u_{t_l}) \subseteq \Omega$. This implies there exists a subsequence $\{u_{t_{l_r}}\}$ of $\{u_{t_l}\}$ such that $u_{t_{l_r}} \rightharpoonup \bar{u}$.

In addition, from Lemma 2.3 (1) and (a1), we obtain that

$$\begin{aligned}
\limsup_{r\to\infty}\mu_{t_{l_r}} &= \limsup_{r\to\infty}\Big[\sigma_{t_{l_r}}\|u^*\|^2 + 2(1-\sigma_{t_{l_r}})\langle u_{t_{l_r}} - u^*,\ -u^*\rangle \\
&+ 2\tau_{t_{l_r}}(1-\sigma_{t_{l_r}})\left\|\sum_{j=1}^m \beta_j F_j^*\left(I - P_{Q_{j,t_{l_r}}^{int}}\right)F_j u_{t_{l_r}}\right\|\|u^*\|\Big] \\
&= 2\limsup_{r\to\infty}\langle u_{t_{l_r}} - u^*,\ -u^*\rangle
\end{aligned}$$

$$\begin{aligned} &= 2 \max_{\bar{u}\in\omega_w(u_{t_{l_r}})} \langle \bar{u} - u^*, \ -u^* \rangle \\ &\leq 0. \end{aligned} \tag{3.28}$$

Therefore, from Lemma 2.11, we conclude that any sequence $\{u_t\}$ generated by Algorithm 1 converges strongly to the minimum-norm element $u^* = P_\Omega 0$. The proof is complete. ∎

3.1. Corollaries

It is readily seen that, for the case where $n = 1$, the MSSFPMOS (1.2) reduced to the following problem: introduced and studied by Reich et al. [17] in infinite-dimensional Hilbert spaces.

Let H, H_j, $j = 1, 2, \dots, m$, be real Hilbert spaces and let $F_j : H \to H_j$, $j = 1, 2, \dots, m$, be bounded linear operators. The split feasibility problem with multiple output sets (SFPMOS, for short) is to find a point u^* such that

$$u^* \in \Gamma := C \cap \left(\cap_{j=1}^m F_j^{-1}(Q_j) \right) \neq \emptyset, \tag{3.29}$$

where C and Q_j, $j = 1, 2, \dots, m$, are non-empty, closed and convex subsets of H and H_j, $j = 1, 2, \dots, m$, respectively.

Reich et al. [17] introduced the following two approximation iterative methods for solving the SFPMOS (3.29). For any given point $u_0 \in H$, $\{u_t\}$ is a sequence generated by

$$u_{t+1} := P_C\left(u_t - \tau_t \sum_{j=1}^m F_j^*(I - P_{Q_j})F_j u_t\right) \tag{3.30}$$

and for any initial point $v_0 \in H$, $\{v_t\}$ is a sequence generated by

$$v_{t+1} := \sigma_t f(v_t) + (1 - \sigma_t)P_C\left(v_t - \tau_t \sum_{j=1}^m F_j^*(I - P_{Q_j})F_j v_t\right), \tag{3.31}$$

where $f : C \to C$ is a $\theta \in [0, 1)$-strict contraction mapping of H into itself, $\tau_t \in (0, \infty)$ and $\{\sigma_t\} \subset (0, 1)$. It was proved that, if the sequence $\{\tau_t\}$ satisfies the condition:

$$0 < a \leq \tau_t \leq b < \frac{2}{m \max_{j=1,2,\dots,m}\{\|F_j\|^2\}}$$

for all $t \geq 1$, then the sequence $\{u_t\}$ generated by (3.30) converges weakly to a solution point $u^* \in \Gamma$ of the SFPMOS (3.29). Furthermore, if the sequence $\{\sigma_t\}$ satisfies the conditions:

$$\lim_{t\to\infty} \sigma_t = 0 \quad \text{and} \quad \sum_{t=1}^\infty \sigma_t = \infty,$$

then the sequence $\{v_t\}$ generated by (3.31) converges strongly to a solution point $u^* \in \Gamma$ of the SFPMOS (3.29), which is a unique solution of the variational inequality

$$\langle (I - f)u^*, u - u^* \rangle \geq 0 \ \forall u \in \Gamma.$$

Note that the iterative methods given by (3.30) and (3.31) require to compute the metric projections on to the sets C and Q_j and need to compute the operator norm, in which is difficult to do so.

If $n = 1$ in the MSSFPMOS (1.2), accordingly in Algorithm 1, as an immediate consequence of Theorem 3.2, we obtain the following result which solves the SFPMOS (3.29).

Corollary 3.3. *Assume that the set of solutions Γ of the SFPMOS (3.29) is non-empty and suppose that the sequences $\{\sigma_t\}$ and $\{\rho_t\}$ in Algorithm 2 satisfy the assumptions (a1) and (a2) in Theorem 3.2. Then, the sequence $\{u_t\}$ generated by Algorithm 2 converges strongly to an element $u^* \in \Gamma$, where $u^* = P_\Gamma 0$.*

Algorithm 2: A self-adaptive approximation technique for SFPMOS (3.29)

Step 0. *Choose two sequences $\{\sigma_t\} \subset (0,1)$ and $\{\rho_t\} \subset (0,2)$ and select $\beta > 0$. Let $u_0 \in H$ be arbitrary initial guess and set $t := 0$. Take the constant parameters β_j $(j = 1, 2, \dots, m) > 0$ such that $\sum_{j=1}^{m} \beta_j = 1$.*

Step 1. *Given the current iterate u_t, compute the next iterate u_{t+1} via the formula*

$$u_{t+1} = P_{C_t^2}\Big((1-\sigma_t)\big(u_t - \tau_t \sum_{j=1}^{m} \beta_j F_j^* (I - P_{Q_{j,t}^{int}}) F_j u_t\big)\Big),$$

where $C_t^2 = C_t \cap C_{t-1}$, $Q_{j,t}^{int} = Q_j^t \cap Q_j^{t-1}$ for each $j = 1, 2, \dots, m$, and

$$\tau_t := \frac{\rho_t \sum_{j=1}^{m} \beta_j \big\| (I - P_{Q_{j,t}^{int}}) F_j u_t \big\|^2}{\Big(max\Big\{\beta,\ \big\| \sum_{j=1}^{m} \beta_j F_j^* (I - P_{Q_{j,t}^{int}}) F_j u_t \big\|\Big\}\Big)^2}.$$

Step 2. *If $u_{t+1} = u_t$, then stop; otherwise, set $t := t+1$ and return to* **Step 1.**

It is readily seen that, for the case where $n = 1 = m$, in the MSSFPMOS (1.2), accordingly in Algorithm 1, as an immediate consequence of Theorem 3.2, we obtain the following result which solves the SFP (1.1).

Corollary 3.4. *Assume that the set of solutions $\Pi = C \cap F^{-1}(Q)$ of the SFP (1.1) is non-empty and suppose that the sequences $\{\sigma_t\}$ and $\{\rho_t\}$ in Algorithm 3 satisfy the assumptions (a1) and (a2) in Theorem 3.2. Then, the sequence $\{u_t\}$ generated by Algorithm 3 converges* *strongly to an element $u^* \in \Pi$, where $u^* = P_{\Pi}0$.*

Algorithm 3: A self-adaptive approximation technique for SFP (1.1)

Step 0. *Choose two sequences $\{\sigma_t\} \subset (0,1)$ and $\{\rho_t\} \subset (0,2)$ and select $\beta > 0$. Let $u_0 \in H_1$ be arbitrary initial guess and set $t := 0$.*

Step 1. *Given the current iterate u_t, compute the next iterate u_{t+1} via the formula*

$$u_{t+1} = P_{C_t^2}\Big((1-\sigma_t)\big(u_t - \tau_t F^*(I - P_{Q_t^2})Fu_t\big)\Big),$$

where $C_t^2 = C_t \cap C_{t-1}$, $Q_t^2 = Q_t \cap Q_{t-1}$, and the stepsize τ_t is self-adaptively defined by

$$\tau_t := \frac{\rho_t \big\|(I - P_{Q_t^2})Fu_t\big\|^2}{\Big(max\{\beta,\ \|F^*(I - P_{Q_t^2})Fu_t\|\}\Big)^2}.$$

Step 2. *If $u_{t+1} = u_t$, then stop; otherwise, set $t := t+1$ and return to* **Step 1.**

4. Numerical Experiments

In this section, we perform some computational tests to illustrate the implementation and efficiency of our proposed algorithm and we compare it with several existing methods in the literature.

The numerical results are completed on a standard TOSHIBA laptop with Intel(R) Core(TM) i5-2450M CPU@2.5GHz 2.5GHz with memory 4GB. The code is implemented in MATLAB R2020a.

Example 4.1. Let $H = \mathbb{R}^3$, $H_1 = \mathbb{R}^6$, $H_2 = \mathbb{R}^9$, $H_3 = \mathbb{R}^{12}$ and $H_4 = \mathbb{R}^{15}$. Find a point $u^* \in \mathbb{R}^3$ such that

$$u^* \in \Omega := C_1 \cap \Big(\cap_{j=1}^4 F_j^{-1}(Q_j)\Big) \neq \emptyset, \tag{4.1}$$

where

$$\begin{aligned} C_1 &= \{u \in \mathbb{R}^3 : \|u - o_1\|^2 \le r_1^2\},\\ Q_1 &= \{F_1 u \in \mathbb{R}^6 : \|F_1 u - O_1\|^2 \le R_1^2\},\\ Q_2 &= \{F_2 u \in \mathbb{R}^9 : \|F_2 u - O_2\|^2 \le R_2^2\},\\ Q_3 &= \{F_3 u \in \mathbb{R}^{12} : \|F_3 u - O_3\|^2 \le R_3^2\},\\ Q_4 &= \{F_4 u \in \mathbb{R}^{15} : \|F_4 u - O_4\|^2 \le R_4^2\}, \end{aligned}$$

where o_1, $O_1 \in \mathbb{R}^6$, $O_2 \in \mathbb{R}^9$, $O_3 \in \mathbb{R}^{12}$, $O_4 \in \mathbb{R}^{15}$, $r_1, R_1, R_2, R_3, R_4 \in \mathbb{R}$, and $F_1 : \mathbb{R}^3 \to \mathbb{R}^6$, $F_2 : \mathbb{R}^3 \to \mathbb{R}^9$, $F_3 : \mathbb{R}^3 \to \mathbb{R}^{12}$, and $F_4 : \mathbb{R}^3 \to \mathbb{R}^{15}$.

For any $u \in \mathbb{R}^3$, we have $c_1(u) = \|u - o_1\|^2 - r_1^2$ and $q_j(F_j u) = \|F_j u - O_j\|^2 - R_j^2$ for $j = 1,2,3,4$. In what follows the subgradients ξ_1^t and η_j^t of respectively $c_1(u_t)$ and $q_j(F_j u_t)$ can be calculated respectively at the points u_t and $T_j u_t$ by $\xi_1^t(u_t) = 2(u_t - o_1)$ and $\eta_j^t(F_j u_t) = 2(F_j u_t - O_j)$. Thus, according to (3.2) and (3.3), the half-spaces C_1^t and Q_j^t ($j = 1,2,3,4$), respectively of the sets C_1 and Q_j can be easily determined at a point u_t and $F_j u_t$, respectively, and the metric projections onto the half-paces C_1^2 and $Q_{j,t}^{int}$ ($j = 1,2,3,4$), can be easily computed.

Now, we take, the radii $r_1 = 4$, $R_1 = 8$, $R_2 = 15$, $R_3 = 22$, $R_4 = 18$, the elements of the representing matrices F_j are randomly generated in the closed interval $[-5, 5]$, and the centers

$$o_1 = (0.4, 0.6, 0.6)^T, O_1 = (0.1, -0.5, 0.4, -0.5, -0.1, -0.2)^T,$$
$$O_2 = (0.1, 1.0, 0.5, 1.0, -0.5, 0.1, -0.9, 0.5, 0.2)^T,$$
$$O_3 = (0.7, 1.0, 0.9, -0.2, -1.0, 0.1, -0.6, -0.6, -0.3, -0.9, 0.5, 0.5)^T,$$
$$O_4 = (0.1, -0.3, 0.7, 0.1, 0.9, 0.8, -0.3, 0.1, -0.3, 0.26, 0.6, 0.5, -0.7, 0.6, -0.9)^T.$$

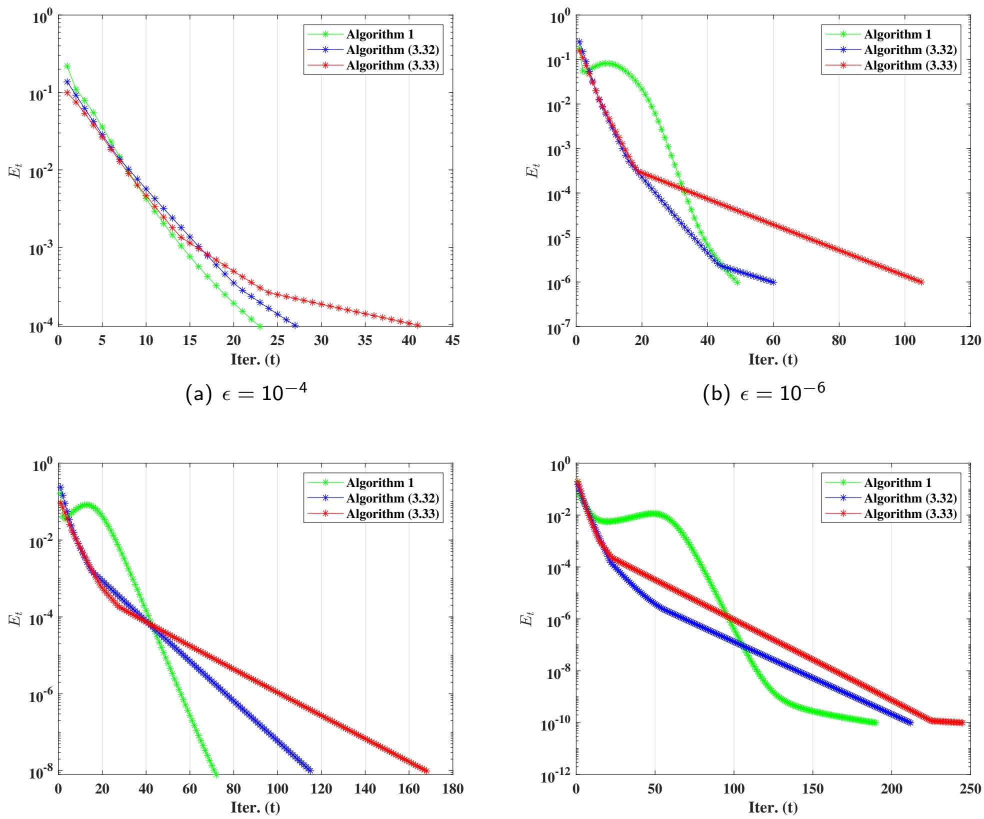
In example 4.1, we examine the convergence of the sequence $\{u_t\}$ generated by Algorithm 1 compared to the iterative methods given by Algorithm (3.30) and Algorithm (3.31). For this purpose, we consider the values of the parameters appeared in the methods as follows. We take $\beta = 0.3$, $\rho_t = \frac{t}{2t+1}$, $\sigma_t = \frac{1}{10t}$, $\alpha_1^t = 1$, $\beta_j = \frac{j}{10}$ $(j = 1, 2, 3, 4)$, $x_0 = (-1, 3, -2)^T$. Moreover, in Algorithms (3.30) and (3.31), we take $\tau_t = 0.0005$ and $f(u) = 0.975u$ in Algorithm (3.31).

In this experiment, we use $E_t = \|u_{t+1} - u_t\|^2 < \epsilon$ for small enough $\epsilon > 0$ as a stopping criteria. In Table 1 and Figure 1, we report the numerical results of the compared methods for different values of ϵ.

Table 1. Numerical results of compared methods for different values of ϵ.

		Algorithm 1	Algorithm (3.30)	Algorithm (3.31)
$\epsilon = 10^{-4}$	Iter. (t)	23	27	41
	cpu(s)	0.001590	0.016056	0.007037
$\epsilon = 10^{-6}$	Iter. (t)	49	60	105
	cpu(s)	0.001393	0.001567	0.001662
$\epsilon = 10^{-8}$	Iter. (t)	72	115	168
	cpu(s)	0.015945	0.011189	0.017034
$\epsilon = 10^{-10}$	Iter. (t)	190	212	245
	cpu(s)	0.032656	0.041615	0.062844

It is readily apparent from Table 1 and Figure 1 that Algorithm 1 exhibits superior performance compared to the other algorithms, as evidenced by its lower number of iterations and shorter runtime in seconds.



(a) $\epsilon = 10^{-4}$ (b) $\epsilon = 10^{-6}$

(c) $\epsilon = 10^{-8}$ (d) $\epsilon = 10^{-10}$

Fig. 1. Iter. (t) against Error, experimental results of compared methods for different values of ϵ.

Example 4.2. Let $H_1 = H_2 = L_2([0, 1])$ with the inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$ defined by

$$\langle u, v \rangle = \int_0^1 u(s)v(s)ds, \ \forall u, v \in L_2([0, 1]),$$

$$\|u\|_2 := \sqrt{\int_0^1 |u(s)|^2 ds}, \ \forall u \in L^2([0, 1]).$$

Furthermore, we consider the following half-spaces

$$C := \left\{ u \in L^2([0, 1]) : \langle u(s), 3s^2 \rangle = 0 \right\} \text{ and}$$

$$Q := \left\{ v \in L^2([0, 1]) : \langle v, \frac{s}{3} \rangle \geq -1 \right\}.$$

In addition, we consider a linear continuous operator $F : L_2([0, 1]) \to L_2([0, 1])$, where

$(Fu)(s) = u(s)$. Then, $(F^*u)(s) = u(s)$ and $\|F\| = 1$. That is, F is an identity operator. The metric projection onto an half-space has an explicit formula [4]. Now, we solve the following problem

$$\text{find } u^* \in C \text{ such that } Fu^* \in Q. \tag{4.2}$$

In Example 4.2, we examine the numerical behaviour of our proposed method: Algorithm 3 and compare it with the following strongly convergent iterative algorithms, repesctively introduced by López et al. [14] and He et al. [13] by solving problem (4.2). For $u, u_0 \in H_1$;

$$u_{t+1} := \sigma_t u + (1-\sigma_t)P_{C_t}\Big(u_t - \tau_t F^*(I - P_{Q_t})Fu_t\Big), \forall t \geq 1, \tag{4.3}$$

$$u_{t+1} := P_{C_t}\Big(\sigma_t u + (1-\sigma_t)\big(u_t - \tau_t F^*(I - P_{Q_t})Fu_t\big)\Big), \tag{4.4}$$

where C_t and Q_t are given as in (1.6) and (1.7), respectively, $\{\sigma_t\} \subset (0,1)$, and $\tau_t = \frac{\rho_t\|(I-P_{Q_t})Fu_t\|^2}{\|F^*(I-P_{Q_t})Fu_t\|^2}$ with $\rho_t \subset (0,2)$.

For comparison purpose, we take the following data: In all methods, $\rho_t = \frac{t}{2t+1}$ and $\sigma_t = 0.5$. Moreover, we take $\beta = 0.3$ in Algorithm 3 and fix $u = cos(s)$ in Algorithms (4.3) and (4.4).

Now, using $E_t = \|u_{t+1} - u_t\| < 10^{-4}$ as stopping criteria for all methods, for different choices of the initial point u_0, the outcomes of the numerical experiments of the compared methods are reported in Table 2 and Figure 2.

Table 2. Comparison of Algorithm 3 with Algorithms (4.3) and (4.4) for different choices of u_0

		Algorithm 3	Algorithm (4.3)	Algorithm (4.4)
	Iter. (t)	18	21	20
$u_0 = s^3$	cpu(s)	0.178345	0.337353	0.468288
	Iter. (t)	18	20	20
$u_0 = s$	cpu(s)	0.176935	0.322715	0.458368
	Iter. (t)	17	21	20
$u_0 = se^s$	cpu(s)	0.164977	0.319452	0.446301
	Iter. (n)	18	20	20
$u_0 = sin(s)$	cpu(s)	0.165513	0.306389	0.447171

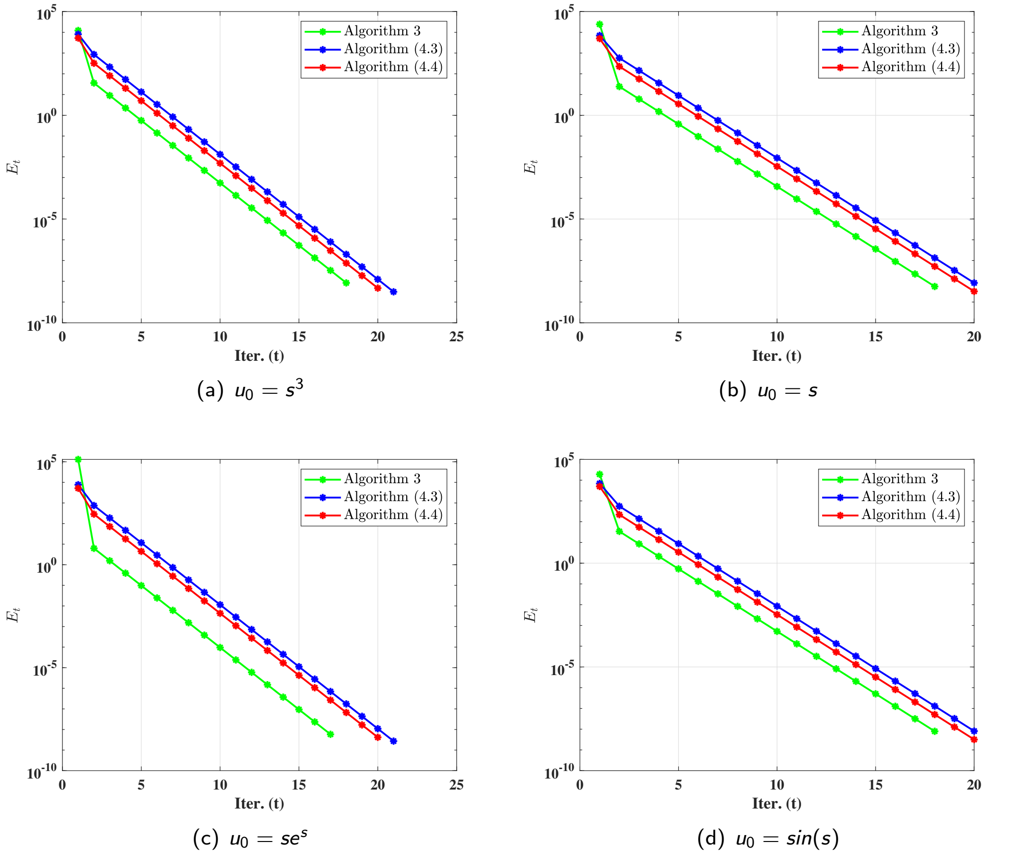


Fig. 2. Comparison of Algorithm 3 with Algorithms (4.3) and (4.4) for different choices of u_0

It can be observed from Table 2 and Figure 2 that for each choices of u_0, Algorithm 3 is faster in terms of less number of iterations (Iter. (t)) and cpu-run time in seconds (cpu(s)) than the compared algorithms.

Example 4.3. In this Example, we consider numerical experiments to illustrate the application of the proposed algorithm to inverse problems arising from signal processing. Compressed sensing is a very active domain of research and applications, based on the fact that an N-sample signal u with exactly K nonzero components can be recovered from $K \ll M < N$ measurements as long as the number of measurements is smaller than the number of signal samples and at the same time much larger than the sparsity level of u. Likewise, the measurements are required to be incoherent, which means that the information contained in the signal is spread out in the domain. Since $M < N$, the problem of recovering u from M measurements is ill conditioned because we encounter an underdeterminated system of linear equations. But, using a sparsity prior, it turns out that reconstructing u from b is possible as long as the number of nonzero elements is small enough (see [18]). More specifically, compressed sensing can be formulated as inverting the equation system

$$b = Fu + \Sigma, \tag{4.5}$$

where $u \in \mathbb{R}^N$ is a vector with K nonzero components to be recovered, $b \in \mathbb{R}^M$ is the vector of noisy observations or measurements (the measured data) with noisy Σ (when $\Sigma = 0$, it means that there is no noise to the observed data), and $F : \mathbb{R}^N \to \mathbb{R}^M$ is a bounded linear observation operator, often ill-conditioned because it models a process with loss of information. A powerful approach for problem (4.5) consists in considering a solution u represented by a sparse expansion, that is, represented by a series expansion with respect to an orthonormal basis that has only a small number of large coefficients. When attempting to find sparse solutions to linear inverse problems of type (4.5), successful model is the convex unconstrained minimization problem

$$\min_{u\in\mathbb{R}^N} \frac{1}{2}\|Fu - b\|_2^2 + \varrho\|u\|_1, \tag{4.6}$$

where ϱ is positive parameter and $\|.\|_1$ is the ℓ_1 norm. Problem (4.6) consists in minimizing an objective function, which includes a quadratic error term combined with a sparseness-including ℓ_1 regularization term, which is to make small component of u to become zero. Problem (4.5) can be seen as the following least absolute shrinkage and selection operator (LASSO), which is commonly used in the theory of signal processing (see [11])

$$\min_{u\in\mathbb{R}^N} \frac{1}{2}\|Fu - b\|_2^2 \text{ subject to } \|u\|_1 \le \varpi, \tag{4.7}$$

where $\varpi > 0$ is a given constant. By the theory of convex analysis, one is able to show that a solution to the LASSO problem (4.7), for appropriate choices $\varpi > 0$, is a minimizer of (4.6) (see [9]). It can be observed that (4.7) indicates the potential of finding a sparse solution of the SFP (1.1) due to the ℓ_1 constraint. More precisely, it is readily seen that problem (4.7) is a particular case of the SFP (1.1) with $C := \{u : \|u\|_1 \le \varpi\}$ and $Q = \{b\}$, and thus can be solved by Algorithm 3 and the iterative methods given by Algorithms (4.3) and (4.4). We define the convex function $c(u) = \|u\|_1 - \varpi$, and according (1.6), the level set C_t is defined by

$$C_t = \{u \in \mathbb{R}^N : c(u_t) + \langle \xi_t, u - u_t\rangle \le 0\},$$

where $\xi_t \in \partial c(u_t)$. Observe that the metric projection onto C_t can be computed by the following manner,

$$P_{C_t}(v)) = \begin{cases} v, & \text{if} \quad c(u_t) + \langle \xi_t, v - u_t\rangle \le 0, \\ v - \frac{\langle c(u_t)+\langle \xi_t, v-u_t\rangle\rangle}{\|\xi_t\|_{L^2}^2}\xi_t, & \text{otherwise.} \end{cases}$$

We choose a subgradient $\xi_t \in \partial c(u_t)$ as

$$(\xi_t)_i = \begin{cases} 1 & \text{if} \quad (\xi_t)_i > 0, \\ 0 & \text{if} \quad (\xi_t)_i = 0, \\ -1 & \text{if} \quad (\xi_t)_i < 0. \end{cases}$$

In a special case where $Q = Q_t = \{b\}$, Algorithm 3 converges to the solution of (4.7). Moreover, Algorithm 3 can be implemented easily, because the projection onto the level set has an explicit formula. In order to show the efficiency of Algorithm 3, a comparative sparse signal recovery experiments were carried-out with Algorithms (4.3) and (4.4).

The vector u is a K sparse signal with non-zero K elements that are generated from uniform distribution within an interval of $[-2, 2]$, F is a matrix generated from normal distribution with mean zero and variance of one and b is an observation generated by white Gaussian noise with signal-to-noise ratio $SNR = 40$. The process of sparse signal recovery start by randomly generating $\varpi = K$ and the anchor (in Algorithms (4.3) and (4.4)) and initial point u_0 are $N \times 1$ vectors. The main target is then to recover the K sparse signal by solving (4.7) for u. The restoration accuracy is then measured by mean squared error (MSE) as follows:

$$MSE = \frac{\|u_{t+1} - u\|}{N} \leq \epsilon, \tag{4.8}$$

where u_t is an estimated signal of u, and $\epsilon > 0$ is a given small constant. We choose the parameters $\sigma_t = \frac{1}{10t+1}$, $\rho_t = 0.5$, $\beta = 0.3$. In our numerical experiments, for $u =$ ones $([N, 1])$ and $u_0 =$ ones $([N, 1])$, we consider the following four choices.

Data 1: $K = 20, N = 2^{14}, M = 2^{12}$;

Data 2: $K = 40, N = 2^{14}, M = 2^{12}$;

Data 3: $K = 20, N = 2^{16}, M = 2^{14}$;

Data 4: $K = 40, N = 2^{16}, M = 2^{14}$.

We use $MSE < \epsilon = 10^{-4}$ as stopping criterion for all methods. The results of the numerical experiments interms of number of iterations (Iter. (t)) and the cpu-run time in seconds (cpu(s)) are reported in Table 3 and Figures 3-6.

Table 3. The experiments of compressed sensing via Algorithm 3, Algorithm (4.3), and Algorithm (4.4)

	Algorithm 3			Algorithm (4.3)			Algorithm (4.4)		
	Iter. (t)	cpu(s)	MSE	Iter. (t)	cpu(s)	MSE	Iter. (t)	cpu(s)	MSE
Data 1	45	4.1563	9.8312e-05	77	7.2441	9.9181e-05	58	5.4076	9.8756e-05
Data 2	74	6.9091	9.8402e-05	96	9.0256	9.8874e-05	83	7.8555	9.6997e-05
Data 3	26	31.3975	9.4105e-05	43	53.1184	9.5889e-05	34	42.1191	9.9678e-05
Data 4	34	41.0025	9.7336e-05	52	63.3655	9.7688e-05	40	48.7032	9.9927e-05

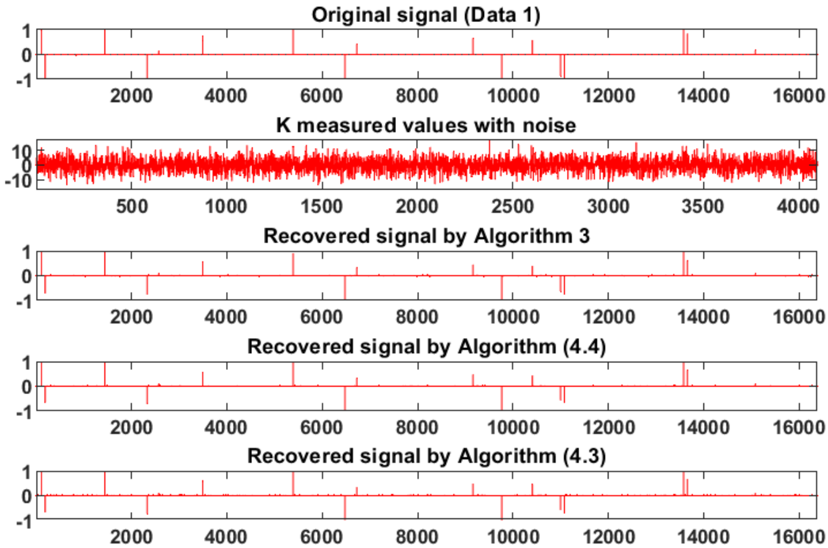


Fig. 3. Original K-sparse vs recovered sparse signals by compared methods for Data 1

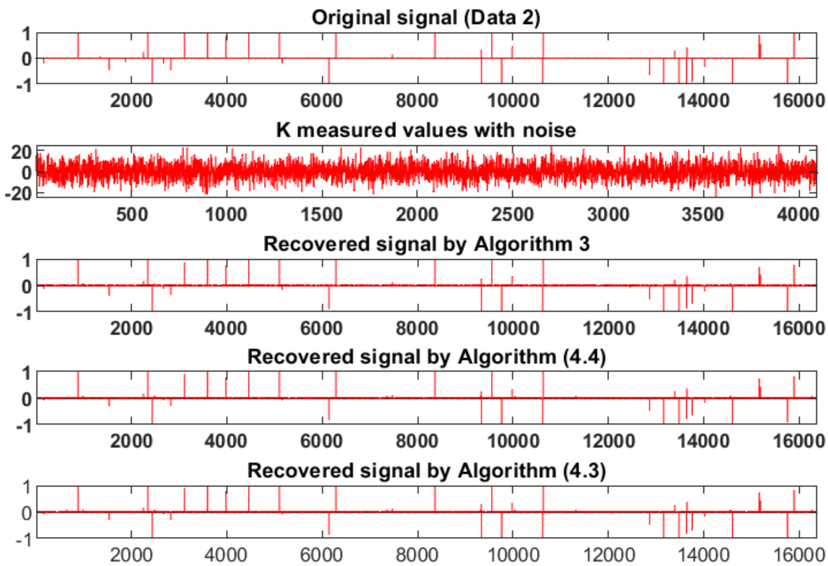


Fig. 4. Original K-sparse vs recovered sparse signals by compared methods for Data 2

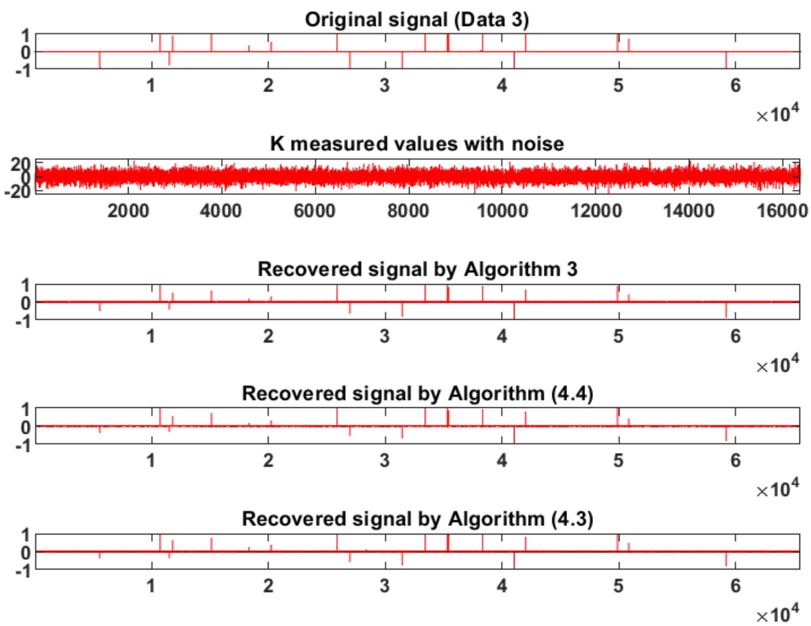


Fig. 5. Original K-sparse vs recovered sparse signals by compared methods for Data 3

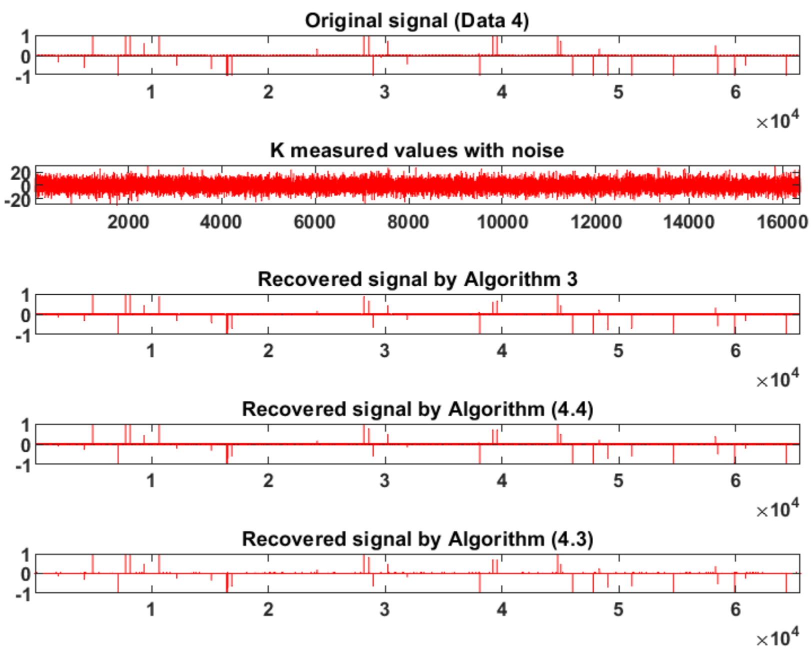


Fig. 6. Original K-sparse vs recovered sparse signals by compared methods for Data 4

It can be observed from Table 3 and Figures 3-6 that the recovered signal by the proposed method has less number of iterations and small cpu(s) time to converge than by the compared methods.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the manuscript: checked, read, and approved the final manuscript.

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