Nonlinear Convex Analysis and Optimization
Vol. 1, No. 2, 2022, pp. 211–225
https://ncao.design.blog

NONLINEAR CONVEX ANALYSIS AND OPTIMIZATION
An International Journal on Numerical, Computation and Applications

Implicit Midpoint Scheme for Enriched Nonexpansive Mappings

Sani Salisu [a,b,1,*], Lukman Hashim [c,2], Adamu Yusuf Inuwa [a,d,3], Abbas Umar Saje [e,4]

[a] Center of Excellence in Theoretical and Computational Science (TaCS-CoE) & KMUTT Fixed Point Research Laboratory, Room SCL 802, Fixed Point Laboratory, Science Laboratory Building, Departments of Mathematics, Faculty of Science, King Mongkut's University of Technology Thonburi (KMUTT), 126 Pracha-Uthit Road, Bang Mod, Thung Khru, Bangkok 10140, Thailand
[b] Department of Mathematics, Faculty of Natural and Applied Sciences, Sule Lamido University Kafin Hausa, Jigawa, Nigeria
[c] Department of Mathematics, Faculty of Science, Umaru Ali Shinkafi Polytechnic Sokoto. Sokoto, Nigeria
[d] Department of Mathematical Sciences, Faculty of Physical Sciences, Bayero University, Kano. Kano, Nigeria.
[e] Department of Science Education, Faculty of Education, Sule Lamido University Kafin Hausa, Jigawa, Nigeria
[1] sani.salisu@slu.edu.ng; [2] lukmanhashimu@gmail.com; [3] ayinuwa.mth@buk.edu.ng; [4] abbasumarsaje40@gmail.com
* Corresponding Author

ABSTRACT

In this article, an implicit scheme for approximating fixed points of enriched nonexpansive mappings is proposed and analyzed. The scheme is structured based on the implicit midpoint rule of certain ordinary differential equation due to stiffness. Convergence properties of the scheme are analyzed, and the scheme is shown to iteratively approach a fixed point of the underlined mapping. Numerical illustrations are given to show the implementation of the scheme with respect to certain enriched mappings.

Article History
Received 11 Oct 2022
Revised 23 Nov 2022
Accepted 20 Dec 2022

Keywords:
Enriched nonexpansive mapping; Implicit midpoint scheme; Fixed point; Hilbert space; Stiff equation

MSC
47H09; 47H10; 47J25; 47N20; 65J15

1. Introduction

Physical phenomena are mostly modeled as differential equations or inclusions problems. Such differential problems may have exact solutions that are tedious to obtain or may not even have an exact solution. Thus the need for numerical methods that yield results exhibiting the structure of the solutions. The implicit midpoint scheme is very promising for handling such differential equations, especially when stiffness is involved [1–5].

Please cite this article as: S. Salisu et al., Implicit Midpoint Scheme for Enriched Nonexpansive Mappings, Nonlinear Convex Anal. & Optim., Vol. 1 No. 2, 211–225.

Given a boundary value problem of the form

$$u' = g(u), \quad u(0) = u_1, \tag{1.1}$$

the implicit midpoint scheme generates a sequence $\{u_n\}$ by solving the recursive form

$$u_{n+1} = u_n + \eta g\left(\frac{u_n + u_{n+1}}{2}\right), \quad n \geq 1, \tag{1.2}$$

to find the updated iterate, where η is known as step-size. If $g : \mathbb{R}^m \to \mathbb{R}^m$ is sufficiently smooth and lipschitz, then $\{u_n\}$ converges uniformly to a solution of (1.1) over $t \in [0, s]$ for any fixed $s > 0$ with $\eta \to 0$.

Let $f \equiv I_d - g$, that is, $f(u) = u - g(u)$ for all $u \in \mathbb{R}^m$. Then (1.1) reduces to

$$u' = u - f(u), \quad u(0) = u_1, \tag{1.3}$$

which can be handled numerically as in (1.2) by

$$u_{n+1} = u_n + \eta\left[\frac{u_n + u_{n+1}}{2} - g\left(\frac{u_n + u_{n+1}}{2}\right)\right], \quad n \geq 1. \tag{1.4}$$

The equilibrium stage associated with the problem in (1.3) seeks u such that

$$u = f(u). \tag{1.5}$$

When a point u satisfies (1.5) then is called a fixed point of f. This motivates the approximation of fixed point of a nonlinear mapping using the implicit midpoint scheme, which can be traced to [6].

Given a linear space $\mathcal{H}$ and $u_1 \in \mathcal{H}$, the scheme

$$u_{n+1} = u_n + \eta_n\left[\frac{u_n + u_{n+1}}{2} - T\left(\frac{u_n + u_{n+1}}{2}\right)\right], \quad n \geq 1$$

where $\eta_n \in (0, 1)$, is equivalent to

$$u_{n+1} = (1 - \beta_n)u_n + \beta_n T\left(\frac{u_n + u_{n+1}}{2}\right), \quad n \geq 1, \tag{1.6}$$

where $\beta_n = \dfrac{2\eta_n}{2 + \eta_n}$. When $\mathcal{H}$ is a Hilbert space and $T : \mathcal{H} \to \mathcal{H}$ is nonexpansive mapping, that is,

$$\|Tu - Tw| \leq \|u - w\|, \quad \forall\, u, w \in \mathcal{H},$$

then the sequence generated by (1.6) is shown in [7] to converge to a fixed point of T provided such a point exists and $\{\beta_n\}$ satisfied the following conditions:

1. $\liminf\limits_{n\to\infty} \beta_n > 0$,
2. $\beta_{n+1} \leq \alpha\beta_n$ for all $n \geq 1$ and some $\alpha > 0$.

On the other hands, Berinde introduced the class of enriched nonexpansive mappings as a superclass of the class of nonexpansive mappings in [8]. Let $(\mathcal{H}, \|\cdot\|)$ be a normed linear space and a mapping $T : \mathcal{H} \to \mathcal{H}$ is said to be an *α-enriched nonexpansive* if there exist $\alpha \geq 0$ such that

$$\|\alpha(u - w) + Tu - Tw\| \leq (\alpha + 1)\|u - w\|, \; \forall u, w \in \mathcal{H}. \tag{1.7}$$

For recent development of enriched nonexpansive mappings, see for example, [9, 10] and the references therein. Following this development, we can deduce the Mann scheme involving α-enriched nonexpansive mapping in linear cases as follows:

$$u_{n+1} = \left(1 - \frac{\beta_n}{\alpha + 1}\right) u_n + \frac{\beta_n}{\alpha + 1} Gu_n, \quad n \geq 1. \tag{1.8}$$

Motivated by the aforementioned discoveries, the purpose of this paper is to propose an implicit midpoint scheme for approximating fixed points of enriched nonexpansive mappings and to analyze the convergence properties of the proposed scheme. It is worth noting that fixed points of enriched nonexpansive mappings have applications in many practical problems as they incorporate certain Lipschitz mappings with constants greater than 1. We shall give two numerical examples of this type Lipschitz mappings and use them to show the explicit reduction of the scheme and the numerical implementations.

2. Preliminaries

In the sequel, unless otherwise stated, $\mathcal{E}$ stands for a nonempty closed convex subset of a real Hilbert space $\mathcal{H}$. Given a mapping $G : \mathcal{E} \to \mathcal{H}$, we call a sequence $\{u_n\}$ an approximate fixed point sequence for G if

$$\|u_n - Gu_n\| \to 0 \quad \text{as} \quad n \to \infty.$$

Recall that Hilbert spaces possess Opial property, that is, for a sequence $\{u_n\} \subset \mathcal{H}$ that converges weakly to u^*,

$$\liminf_{n\to\infty} \|u_n - u^*\| < \liminf_{n\to\infty} \|u_n - y\|, \quad \forall\, y \in \mathcal{H}\backslash\{u^*\}.$$

Equivalently, we have

$$\limsup_{n\to\infty} \|u_n - u^*\| < \limsup_{n\to\infty} \|u_n - y\|, \quad \forall\, y \in \mathcal{H}\backslash\{u^*\}.$$

Following the Opial property, we can easily deduce the demiclosedness principle of some generalised nonexpansive mappings as follows.

Lemma 2.1. *Let $G : \mathcal{E} \to \mathcal{E}$ be an α-enriched nonexpansive mapping. Suppose that $\{u_n\}$ is an approximate fixed point sequence for G and also $\{u_n\}$ weakly converges to u^*. Then u^* is a fixed point of G.*

Proof. Consider a mapping from $\mathcal{E}$ into itself defined by $G_\alpha u = \dfrac{\alpha}{\alpha+1} u + \dfrac{1}{\alpha+1} Gu$ for all $u \in \mathcal{E}$. Then observe that

$$\|u_n - G_\alpha u^*\| \leq \|u_n - G_\alpha u_n\| + \|G_\alpha u_n - G_\alpha u^*\|$$

$$
\begin{aligned}
&= \frac{1}{\alpha+1}\|u_n - Gu_n\| + \frac{1}{\alpha+1}\|\alpha(u_n-u^*) + Gu_n - Gu^*\| \\
&\le \|u_n - Gu_n\| + \frac{1}{\alpha+1}\|\alpha(u_n-u^*) + Gu_n - Gu^*\|.
\end{aligned}
$$

This and the fact that G is α-enriched nonexpansive mapping imply

$$\|u_n - G_\alpha u^*\| \le \|u_n - Gu_n\| + \|u_n - u^*\|.$$

Consequently we get

$$\limsup_{n\to\infty}\|u_n - G_\alpha u^*\| \le \limsup_{n\to\infty}\|u_n - u^*\|,$$

which by Opial property implies that $G_\alpha u^* = u^*$. Therefore, we have $Gu^* = u^*$ ∎

Some identities involving two points in real Hilbert spaces are very crucial in obtaining our main results.

Lemma 2.2. *Let $u, w \in \mathcal{H}$ and $a \in \mathbb{R}$. Then*

(i) $\|u+w\|^2 = \|u\|^2 + \|w\|^2 + 2\langle u, w\rangle$;

(ii) $\|u-w\|^2 = \|u\|^2 + \|w\|^2 - 2\langle u, w\rangle$;

(iii) $\|au+(1-a)w\|^2 = a\|u\|^2 + (1-a)\|w\|^2 - a(1-a)\|u-w\|^2$.

3. The Implicit Midpoint Scheme and Its Convergence

In this section, we state the scheme and analyzed its convergence properties.

Algorithm 3.1. Initialize $u_1 \in \mathcal{H}$ arbitrary and find u_{n+1} such that

$$u_{n+1} = \left(1 - \frac{2\beta_n}{\alpha(2-\beta_n)+2}\right)u_n + \frac{2\beta_n}{\alpha(2-\beta_n)+2}G\left(\frac{u_n+u_{n+1}}{2}\right), \quad n \ge 1, \tag{3.1}$$

where $\beta_n \in (0,1)$ for all natural number n, $\alpha \ge 0$ and $G : \mathcal{H} \to \mathcal{H}$ is a mapping.

Remark 3.2. It is easy to see that setting $\alpha = 0$, Algorithm 3.1 reduces to (1.6). This is interesting since (1.7) implies that every nonexpansive mapping is 0-enriched nonexpansive.

Remark 3.3. observe that

$$u_{n+1} = \left(1 - \frac{2\beta_n}{\alpha(2-\beta_n)+2}\right)u_n + \frac{2\beta_n}{\alpha(2-\beta_n)+2}G\left(\frac{u_n+u_{n+1}}{2}\right)$$

if and only if

$$[\alpha(2-\beta_n)+2]\,u_{n+1} = [\alpha(2-\beta_n)+2-2\beta_n]\,u_n + 2\beta_n G\left(\frac{u_n+u_{n+1}}{2}\right)$$

if and only if

$$[2(\alpha+1)-\alpha\beta_n]\,u_{n+1} = [2(\alpha+1)(1-\beta_n)+\alpha\beta_n]\,u_n + 2\beta_n G\left(\frac{u_n+u_{n+1}}{2}\right)$$

if and only if

$$2(\alpha+1)u_{n+1} = 2(\alpha+1)(1-\beta_n)u_n + \alpha\beta_n(u_n+u_{n+1}) + 2\beta_n G\left(\frac{u_n+u_{n+1}}{2}\right)$$

if and only if

$$u_{n+1} = (1-\beta_n)u_n + \frac{\alpha\beta_n}{\alpha+1}\left(\frac{u_n+u_{n+1}}{2}\right) + \frac{\beta_n}{\alpha+1}G\left(\frac{u_n+u_{n+1}}{2}\right) \tag{3.2}$$

Lemma 3.4. *Let $\{u_n\}$ be a sequence generated by Algorithm 3.1. Then the sequence possesses the following properties:*

(P1) $\|u_{n+1}-u^*\| \le \|u_n-u^*\|$ *for all* $u^* \in \mathcal{F}(G)$ *and* $n \ge 1$.

(P2) $\sum_{n=1}^{\infty} \beta_n \|u_n-u_{n+1}\|^2 < \infty$.

(P3) $\sum_{n=1}^{\infty} \beta_n(1-\beta_n)\|u_n-G_\alpha(w_n)\|^2 < \infty$,

where $w_n = \dfrac{u_n+u_{n+1}}{2}$ *and* G_α *is* α*-relaxation mapping of* G *defined by*

$$G_\alpha : x \to \frac{\alpha}{\alpha+1}x + \frac{1}{\alpha+1}Gx \quad \forall\, x \in Dom(G). \tag{3.3}$$

Proof. Let $u^* \in \mathcal{F}(T)$ and set $w_n = \dfrac{u_n+u_{n+1}}{2}$. Then it follows from (3.2) and Lemma 2.2(iii) that

$$\begin{aligned}
\|u_{n+1}-u^*\|^2 &= \left\|(1-\beta_n)u_n + \frac{\alpha\beta_n}{\alpha+1}\left(\frac{u_n+u_{n+1}}{2}\right) + \frac{\beta_n}{\alpha+1}G\left(\frac{u_n+u_{n+1}}{2}\right) - u^*\right\|^2 \\
&= \left\|(1-\beta_n)u_n + \frac{\alpha\beta_n}{\alpha+1}w_n + \frac{\beta_n}{\alpha+1}G(w_n) - u^*\right\|^2 \\
&= \left\|(1-\beta_n)(u_n-u^*) + \beta_n\left(\frac{\alpha}{\alpha+1}w_n + \frac{1}{\alpha+1}G(w_n) - u^*\right)\right\|^2 \\
&= (1-\beta_n)\|u_n-u^*\|^2 + \beta_n\left\|\frac{\alpha}{\alpha+1}w_n + \frac{1}{\alpha+1}G(w_n) - u^*\right\|^2 \\
&\quad - \beta_n(1-\beta_n)\left\|\frac{\alpha}{\alpha+1}w_n + \frac{1}{\alpha+1}G(w_n) - u_n\right\|^2 \\
&= (1-\beta_n)\|u_n-u^*\|^2 + \frac{\beta_n}{(\alpha+1)^2}\|\alpha(w_n-u^*) + G(w_n) - G(u^*)\|^2 \\
&\quad - \beta_n(1-\beta_n)\|u_n - G_\alpha(w_n)\|^2.
\end{aligned}$$

Since G is enriched nonexpansive mapping then

$$\|\alpha(w_n-u^*) + G(w_n) - G(u^*)\| \le \|u_n-u^*\|.$$

Consequently, we get

$$\|u_{n+1}-u^*\|^2 \le (1-\beta_n)\|u_n-u^*\|^2+\beta_n\|w_n-u^*\|^2-\beta_n(1-\beta_n)\|u_n-G_\alpha(w_n)\|^2.$$

Applying Lemma 2.2(iii), we get

$$\begin{aligned}\|u_{n+1}-u^*\|^2 &\le (1-\beta_n)\|u_n-u^*\|^2+\beta_n\left\|\frac{1}{2}(u_n-u^*)+\frac{1}{2}(u_{n+1}-u^*)\right\|^2\\ &\quad -\beta_n(1-\beta_n)\|u_n-G_\alpha(w_n)\|^2\\ &\le (1-\beta_n)\|u_n-u^*\|^2+\frac{\beta_n}{2}\|u_n-u^*\|^2+\frac{\beta_n}{2}\|u_{n+1}-u^*\|^2\\ &\quad -\frac{\beta_n}{4}\|u_{n+1}-u_n\|^2-\beta_n(1-\beta_n)\|u_n-G_\alpha(w_n)\|^2.\end{aligned}$$

This implies that

$$\begin{aligned}\left(1-\frac{\beta_n}{2}\right)\|u_{n+1}-u^*\|^2 &\le \left(1-\frac{\beta_n}{2}\right)\|u_n-u^*\|^2-\frac{\beta_n}{4}\|u_{n+1}-u_n\|^2\\ &\quad -\beta_n(1-\beta_n)\|u_n-G_\alpha(w_n)\|^2.\end{aligned}$$

Thus

$$\begin{aligned}\|u_{n+1}-u^*\|^2 &\le \|u_n-u^*\|^2-\frac{\beta_n}{2(2-\beta_n)}\|u_{n+1}-u_n\|^2\\ &\quad -\frac{\beta_n(1-\beta_n)}{2(2-\beta_n)}\|u_n-G_\alpha(w_n)\|^2. \qquad (3.4)\end{aligned}$$

Clearly, $(P1)$ holds. Moreover, $\{\|u_n-u^*\|\}$ converges. Furthermore, (3.4) implies that

$$\|u_{n+1}-u^*\|^2 \le \|u_n-u^*\|^2-\frac{\beta_n}{2(2-\beta_n)}\|u_{n+1}-u_n\|^2.$$

So,

$$\begin{aligned}\beta_n\|u_{n+1}-u_n\|^2 &\le 2(2-\beta_n)\left(\|u_n-u^*\|^2-\|u_{n+1}-u^*\|^2\right)\\ &\le 4\left(\|u_n-u^*\|^2-\|u_{n+1}-u^*\|^2\right).\end{aligned}$$

Taking sum up to $m>1$, we have

$$\sum_{n=1}^{m}\beta_n\|u_{n+1}-u_n\|^2 \le 4\left(\|u_1-u^*\|^2-\|u_{n+1}-u^*\|^2\right)\le 4\|u_1-u^*\|^2.$$

Taking limit as $m\to\infty$, we get $(P2)$. From (3.4), we get

$$\|u_{n+1}-u^*\|^2 \le \|u_n-u^*\|^2-\frac{\beta_n(1-\beta_n)}{2(2-\beta_n)}\|u_n-G_\alpha(w_n)\|^2$$

which consequently resulted to

$$\beta_n(1-\beta_n)\|u_n-G_\alpha(w_n)\|^2 \le 2(2-\beta_n)\left(\|u_n-u^*\|^2-\|u_{n+1}-u^*\|^2\right)$$

$$\leq 4\left(\|u_n - u^*\|^2 - \|u_{n+1} - u^*\|^2\right).$$

It turns out that

$$\begin{aligned}\sum_{n=1}^{m} \beta_n (1-\beta_n) \|u_n - G_\alpha(w_n)\|^2 &\leq 2(2-\beta_n)\left(\|u_n - u^*\|^2 - \|u_{n+1} - u^*\|^2\right)\\ &\leq 4\|u_1 - u^*\|^2.\end{aligned}$$

Consequently $(P3)$ holds and the proof is achieved. ∎

Lemma 3.5. *Suppose that $\{u_n\}$ is a sequence generated through Algorithm 3.1 with $\{\beta_n\}$ such that $\beta_{n+1} \leq \eta\beta_n$ for all $n \geq 1$ and some $\eta > 0$. Then*

$$\|u_{n+1} - u_n\| \to 0 \quad as \quad n \to \infty.$$

Proof. Set $w_n = \dfrac{u_n + u_{n+1}}{2}$ and G_α be the mapping defined in (3.3). Then Algorithm 3.1 and (3.2) yield that

$$\begin{aligned}\|u_{n+2} - u_{n+1}\| &= \beta_{n+1}\|u_{n+1} - G_\alpha w_{n+1}\|\\ &\leq \beta_{n+1}\|u_{n+1} - G_\alpha w_n\| + \beta_{n+1}\|G_\alpha w_n - G_\alpha w_{n+1}\|\\ &\leq \beta_{n+1}(1-\beta_n)\|u_n - G_\alpha w_n\| + \beta_{n+1}\|G_\alpha w_n - G_\alpha w_{n+1}\|\\ &= \beta_{n+1}(1-\beta_n)\|u_n - G_\alpha w_n\|\\ &\quad + \frac{\beta_n}{\alpha+1}\|\alpha(w_n - w_{n+1}) + Gw_n - Gw_{n+1}\|.\end{aligned}$$

This and the asssumption that G is α-enriched nonexpansive mapping yield

$$\begin{aligned}\|u_{n+2} - u_{n+1}\| &= \beta_{n+1}(1-\beta_n)\|u_n - G_\alpha w_n\| + \beta_{n+1}\|w_n - w_{n+1}\|\\ &= \beta_{n+1}(1-\beta_n)\|u_n - G_\alpha w_n\| + \beta_{n+1}\left\|\frac{u_n + u_{n+1}}{2} - \frac{u_{n+1} + u_{n+2}}{2}\right\|\\ &= \beta_{n+1}(1-\beta_n)\|u_n - G_\alpha w_n\| + \frac{\beta_{n+1}}{2}\|u_n - u_{n+1}\|\\ &\quad + \frac{\beta_{n+1}}{2}\|u_{n+1} - u_{n+2}\|.\end{aligned}$$

This yields

$$\begin{aligned}\|u_{n+2} - u_{n+1}\| &\leq \frac{2\beta_{n+1}(1-\beta_n)}{2-\beta_{n+1}}\|u_n - G_\alpha w_n\| + \frac{\beta_{n+1}}{2-\beta_{n+1}}\|u_n - u_{n+1}\|\\ &\leq 2\beta_{n+1}(1-\beta_n)\|u_n - G_\alpha w_n\| + \beta_{n+1}\|u_n - u_{n+1}\|.\end{aligned}$$

Using the fact that $(a+b)^2 \leq 2a^2 + 2b^2$ for all $a, b \geq 0$, we have

$$\|u_{n+2} - u_{n+1}\|^2 \leq 4\beta_{n+1}^2(1-\beta_n)^2\|u_n - G_\alpha w_n\|^2 + 2\beta_{n+1}^2\|u_n - u_{n+1}\|^2.$$

Since $\beta_{n+1} \leq \eta\beta_n$ then we have

$$\|u_{n+2} - u_{n+1}\|^2 \leq 4\eta\beta_n(1-\beta_n)\|u_n - G_\alpha w_n\|^2 + 2\eta\beta_n\|u_n - u_{n+1}\|^2.$$

This and Lemma 3.4 complete the proof. ∎

Lemma 3.6. *Let $\{u_n\}$ be as in Lemma 3.5. Suppose in addition $\liminf_{n\to\infty} \beta_n > 0$. Then $\{u_n\}$ is an approximate fixed point sequence for G.*

Proof. From (3.2), we get that

$$\|u_{n+1} - u_n\| = \beta_n \|u_n - G_\alpha w_n\|,$$

where $w_n = \dfrac{u_n + u_{n+1}}{2}$. It follows from the assumption that $\liminf_{n\to\infty} \beta_n > 0$ and Lemma 3.5 that

$$\|u_n - G_\alpha w_n\| \to 0. \tag{3.5}$$

We obtain the following from the hypothesis that G is α-enriched nonexpansive mapping.

$$\begin{aligned}\|u_n - Gu_n\| &= (\alpha+1)\|u_n - G_\alpha u_n\| \\ &\le (\alpha+1)(\|u_n - G_\alpha w_n\| + \|G_\alpha w_n - G_\alpha u_n\|) \\ &= (\alpha+1)\|u_n - G_\alpha w_n\| + \|\alpha(w_n - u_n) + Gw_n - Gu_n\| \\ &\le (\alpha+1)\|u_n - G_\alpha w_n\| + \|u_n - w_n\| \\ &= (\alpha+1)\|u_n - G_\alpha w_n\| + \frac{1}{2}\|u_{n+1} - u_n\|.\end{aligned}$$

This, (3.5) and Lemma 3.5 imply that

$$\|u_n - Gu_n\| \to 0 \quad \text{as} \quad n \to \infty.$$

∎

Theorem 3.7. *Let $G : \mathcal{E} \to \mathcal{E}$ be an α-enriched nonexpansive mapping with a fixed point. Suppose that $\{u_n\}$ is a sequence generated through Algorithm 3.1 with $\{\beta_n\}$ satisfying*

1. $\liminf_{n\to\infty} \beta_n > 0$;
2. $\beta_{n+1} \le \eta\beta_n$ *for all* $n \ge 1$ *and some* $\eta > 0$.

Then $\{u_n\}$ converges weakly to a fixed point of G.

Proof. It follows from Lemma 3.4 (P1) that $\{u_n\}$ is bounded. Then there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ that converges weakly to $u^* \in \mathcal{E}$. Consequently, we obtain from Lemma 3.6 and Lemma 2.1 that $u^* \in \mathcal{F}(G)$. Suppose there exists another subsequent of $\{u_n\}$, say $\{u_{n_j}\}$ that converges weakly to u^o. Then, similarly to u^*, we have that $u^o \in \mathcal{F}(G)$. Lemma 3.4 (P1) implies that $\lim_{n\to\infty} \|u_n - u^*\|$ and $\lim_{n\to\infty} \|u_n - u^o\|$ exist. Now, following Lemma 2.2(ii), we have that

$$\begin{aligned}\lim_{n\to\infty} \|u_n - u^*\|^2 &= \lim_{j\to\infty} \left\|u_{n_j} - u^*\right\|^2 \\ &= \lim_{j\to\infty} \left\|\left(u_{n_j} - u^o\right) + \left(u^o - u^*\right)\right\|^2 \\ &= \|u^* - u^o\|^2 + \lim_{j\to\infty} \left\|u_{n_j} - u^o\right\|^2\end{aligned}$$

$$
\begin{aligned}
&= \|u^* - u^o\|^2 + \lim_{k\to\infty} \|u_{n_k} - u^o\|^2 \\
&= 2\|u^* - u^o\|^2 + \lim_{k\to\infty} \|u_{n_k} - u^*\|^2 \\
&= 2\|u^* - u^o\|^2 + \lim_{n\to\infty} \|u_n - u^*\|^2 .
\end{aligned}
$$

Thus we have $2\|u^* - u^o\|^2 = 0$ which implies that $u^* = u^o$. Therefore $\{u_n\}$ converges weakly to a fixed point of G. ∎

4. Numerical Illustrations

Our aim in this part is to show that the implementation of the implicit methods can be achieved explicitly just by solving for the next iteration once. Furthermore, we show the impact of the methods on solving a stiff equation involving enriched nonexpansive mapping.

Example 4.1. Let $\mathcal{H} = \mathbb{R}$ be endowed with the usual norm and take $\mathcal{E} = \left[\frac{1}{2}, 2\right]$. Let $G : \mathcal{E} \to \mathcal{E}$ be defined by $Gu = \frac{1}{u}$, for all $u \in \mathcal{E}$. Then G is $\frac{3}{2}$-enriched nonexpansive mapping with 1 as fixed point but not nonexpansive [8]. For this example, Algorithm 3.1 gives

$$
u_{n+1} = \left(1 - \frac{2\beta_n}{\alpha(2-\beta_n)+2}\right) u_n + \frac{2\beta_n}{\alpha(2-\beta_n)+2}\left(\frac{2}{u_n + u_{n+1}}\right), \quad n \geq 1.
$$

Solving for u_{n+1}, we get

$$
u_{n+1} = \frac{-t_n u_n + \sqrt{t_n^2 u_n^2 - 4(-2t_n + t_n u_n^2 - u_n^2)}}{2}, \quad n \geq 1, \tag{4.1}
$$

where $t_n = \dfrac{2\beta_n}{\alpha(2-\beta_n)+2}$.

To show the numerical patterns of the scheme for this example, we set $\beta_n = \frac{n+1}{n+2}$ and generate the results up to $n = 15$ as given in Table 1. The table is generated using (4.1) for six distinct initial points.

Table 1. Numerical results for Algorithm 3.1

n	u_n						
1	2	1.9	1.8	1.65	0.8	0.75	0.5
2	1.522588	1.464797	1.407825	1.324153	0.920759	0.945322	0.833333
3	1.215907	1.189459	1.163901	1.127378	0.973123	0.988545	0.945884
4	1.072012	1.062566	1.053593	1.041042	0.991928	0.997775	0.983973
5	1.020381	1.017626	1.01503	1.011437	0.997808	0.999596	0.995667
6	1.005156	1.004452	1.00379	1.002878	0.999453	0.999931	0.99892
7	1.001205	1.00104	1.000885	1.000672	0.999873	0.999989	0.999749
8	1.000265	1.000228	1.000194	1.000147	0.999972	0.999998	0.999945
9	1.000055	1.000048	1.000041	1.000031	0.999994	1	0.999988
10	1.000011	1.00001	1.000008	1.000006	0.999999	1	0.999998
11	1.000002	1.000002	1.000002	1.000001	1	1	1
12	1	1	1	1	1	1	1
13	1	1	1	1	1	1	1
14	1	1	1	1	1	1	1
15	1	1	1	1	1	1	1

Remark 4.2. Note that the table shows the numbers obtained by the scheme up to six digits for six different initial points. Moreover, the iteration converges to 1, which is the only fixed point of the underline mapping.

Example 4.3. Consider the initial value problem

$$\frac{\mathrm{du}}{\mathrm{dt}} = -5u(t), \quad u(0) = u_1 = 1, \quad t \geq 0.$$

This problem usually arise from disease and population model, logistic model and many physical phenomena due to science and engineering. This problem has the solution

$$u(t) = e^{-5t}, \quad u(t) \to 0, \text{ as } t \to \infty.$$

However, finding exact solution of differential equations is very tedious in most cases or even impossible. So engineers may want to handle problem discretely by seeking for a numerical pattern that exhibits the same behaviour with the exact solution. Now, let G be a mapping such that $u \mapsto -6u$. Then G is obviously not nonexpansive mapping. However G is 5/2-enriched nonexpansive mapping since

$$\begin{aligned}\left\|\frac{5}{2}(u-w) + Gu - Gw\right\| &= \left\|\frac{5}{2}(u-w) - 6u + 6w\right\| \\ &= \left\|\left(\frac{5}{2} - 6\right)(u-w)\right\|\end{aligned}$$

$$= \frac{7}{2}\|u - w\|$$
$$= \left(\frac{5}{2} + 1\right)\|u - w\|.$$

For this example, Algorithm 3.1 gives

$$u_{n+1} = \left(1 - \frac{2\beta_n}{\alpha(2-\beta_n)+2}\right)u_n + \frac{2\beta_n}{\alpha(2-\beta_n)+2}\left(-3\left(u_n + u_{n+1}\right)\right), \quad n \geq 1.$$

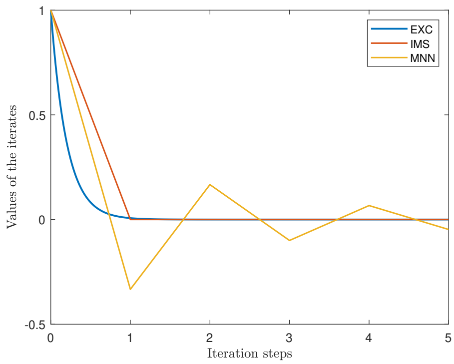
Solving for u_{n+1}, we get

$$u_{n+1} = \frac{(\alpha(2-\beta_n)+2-8\beta_n)\,u_n}{\alpha(2-\beta_n)+2+6\beta_n}, \quad n \geq 1. \tag{4.2}$$

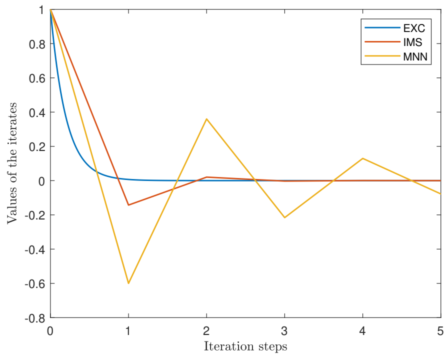
To investigate the numerical stability of our method using the given stiff equation, we consider 4 cases. Case1 is for for $\beta_n = \frac{n+1}{n+2}$, Case 2 for $\beta_n = \frac{4}{5}$, Case 3 for $\beta_n = \frac{n(n+2)}{(n+1)^2}$ and Case 4 for $\beta_n = \frac{2}{3}$. The generated results are shown in Table 2 and Figure 1. For the representations, EXC stands for the exact solution, IMS stands for our proposed method (Implicit Midpoint Scheme) and MNN is for the Mann method in (1.8). The table shows, at each n, how far the value of the iterate u_n is from the value of the exact solution e^{-5n}.

Table 2. Numerical values for $\left|u_n - e^{-5n}\right|$

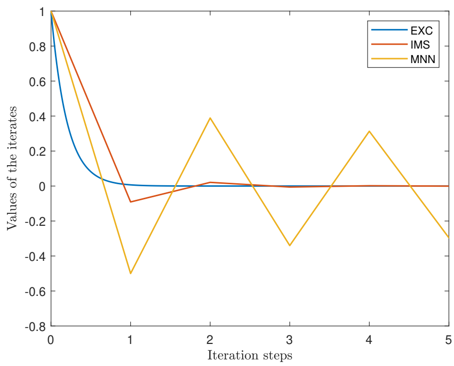
	Case 1		Case 2		Case 3		Case 4	
n	IMS	MNN	IMS	MNN	IMS	MNN	IMS	MNN
1	0.993262	0.993262	0.993262	0.993262	0.993262	0.993262	0.993262	0.993262
2	4.54E-05	0.333379	0.142903	0.600045	0.090954	0.500045	4.54E-05	0.333379
3	3.06E-07	0.166666	0.020408	0.36	0.020979	0.388889	3.06E-07	0.111111
4	2.06E-09	0.1	0.002915	0.216	0.005803	0.340278	2.06E-09	0.037037
5	1.39E-11	0.066667	0.000416	0.1296	0.001725	0.313056	1.39E-11	0.012346
6	9.36E-14	0.047619	5.95E-05	0.07776	0.000532	0.295664	9.36E-14	0.004115
7	6.31E-16	0.035714	8.50E-06	0.046656	0.000168	0.283596	6.31E-16	0.001372
8	4.25E-18	0.027778	1.21E-06	0.027994	5.35E-05	0.274733	4.25E-18	0.000457
9	2.92E-20	0.022222	1.73E-07	0.016796	1.73E-05	0.26795	2.86E-20	0.000152
10	6.08E-23	0.018182	2.48E-08	0.010078	5.60E-06	0.262591	1.93E-22	5.08E-05
11	3.53E-23	0.015152	3.54E-09	0.006047	1.82E-06	0.25825	1.30E-24	1.69E-05
12	8.93E-24	0.012821	5.06E-10	0.003628	5.97E-07	0.254664	8.76E-27	5.65E-06
13	2.40E-24	0.010989	7.22E-11	0.002177	1.96E-07	0.25165	5.90E-29	1.88E-06
14	6.54E-25	0.009524	1.03E-11	0.001306	6.44E-08	0.249082	3.98E-31	6.27E-07
15	1.81E-25	0.008333	1.47E-12	0.000784	2.12E-08	0.246868	2.68E-33	2.09E-07
16	5.06E-26	0.007353	2.11E-13	0.00047	7.00E-09	0.244939	1.80E-35	6.97E-08
17	1.43E-26	0.006536	3.01E-14	0.000282	2.31E-09	0.243244	1.22E-37	2.32E-08
18	4.09E-27	0.005848	4.30E-15	0.000169	7.64E-10	0.241743	8.19E-40	7.74E-09
19	1.18E-27	0.005263	6.14E-16	0.000102	2.53E-10	0.240403	5.52E-42	2.58E-09
20	3.43E-28	0.004762	8.77E-17	6.09E-05	8.37E-11	0.239201	3.72E-44	8.60E-10



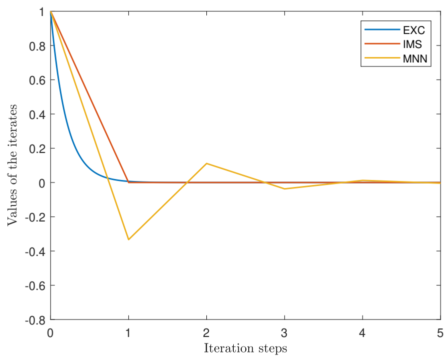
(a) Numerical stability due to Case 1



(b) Numerical stability due to Case 2



(c) Numerical stability due to Case 3



(d) Numerical stability due to Case 4

Fig. 1. Numerical Stability of Implicit Midpoint Rule

5. Conclusion Remarks

In this work, we studied the implicit midpoint scheme for getting a fixed point of an enriched nonexpansive mapping in the framework of Hilbert spaces. We established that the sequence generated converges weakly to a fixed point of the underlined mapping. Note that in finite-dimensional spaces, the convergence is strong (thanks to the Bolzano-Weierstrass theorem). We gave two examples where the mappings are not nonexpansive and solved for the explicit form of the proposed scheme. The numerical results due to the scheme are shown, and the distance between the iterate of the proposed scheme and that of the exact solution is shown in comparison to the well-known Krasnoselskii-Mann iteration. Despite the computational expansiveness and tediousness of the implementation of implicit methods, our numerical data shows that, for the example considered, the proposed scheme achieves a significant stage of numerical stability earlier than the Krasnoselskii-Mann scheme.

For further studies, this method can also be considered in the framework of geodesically connected spaces since most of these spaces appear to be like nonlinear versions of Hilbert spaces [11, 12].

Acknowledgements

The authors are grateful to the anonymous referees for their helpful comments and suggestions, which improved the presentation of the manuscript.

References

[1] W. Auzinger, R. Frank, Asymptotic error expansions for stiff equations: an analysis for the implicit midpoint and trapezoidal rules in the strongly stiff case. Numer Math. 56 (1989), 469–99. Available from: https://doi.org/10.1007/BF01396649.

[2] G. Bader, P. Deuflhard, A semi-implicit mid-point rule for stiff systems of ordinary differential equations, Numer Math. 41 (1983), 373–98. Available from: https://doi.org/10.1007/BF01418331.

[3] S. Somali, S. Davulcu, Implicit midpoint rule and extrapolation to singularly perturbed boundary value problems, Int. J. Comput Math. 75 (2000), 117–27. Available from: https://doi.org/10.1080/00207160008804969.

[4] S. Somali, Implicit midpoint rule to the nonlinear degenerate boundary value problems, Int. J. Comput Math. 79 (2002), 327–32. Available from: https://doi.org/10.1080/00207160211930.

[5] C. Schneider, Analysis of the linearly implicit mid-point rule for differential-algebraic equations, Electron Trans Numer Anal. 1 (1993), 1–10. Available from: http://emis.math.tifr.res.in/journals/ETNA/vol.1.1993/pp1-10.dir/pp1-10.pdf.

[6] H-K. Xu, R.G. Ori, An implicit iteration process for nonexpansive mappings. Numer. Funct. Anal. Optim. 22 (2001), 767–73. Available from: https://doi.org/10.1081/NFA-100105317.

[7] M.A. Alghamdi, M.A. Alghamdi, N. Shahzad, H-K. Xu, The implicit midpoint rule for nonexpansive mappings in Hilbert spaces. Fixed Point Theory Appl. 96 (2014), 1–9. Available from: https://doi.org/10.1186/1687-1812-2014-96.

[8] V. Berinde, Approximating fixed points of enriched nonexpansive mappings by Krasnoselskij iteration in Hilbert spaces. Carpathian J. Math. 35 (2019), 293–304. Available from: https://www.jstor.org/stable/26905206

[9] V. Berinde, Approximating fixed points of enriched nonexpansive mappings in Banach spaces by using a retraction-displacement condition, Carpathian J. Math. 36 (2020), 27–34. Available from: https://www.jstor.org/stable/26898779.

[10] S. Salisu, P. Kumam, S. Sriwongsa, On fixed points of enriched contractions and enriched nonexpansive mappings, Carpathian J. Math. 39 (2023), 237-254. Available from: https://doi.org/10.1007/s12190-022-01763-6.

[11] S. Salisu, M.S. Minjibir, P. Kumam, S. Sriwongsa, Convergence theorems for fixed points in $\mathrm{CAT}_p(0)$ spaces, J. Appl. Math. Comput. 69 (2023), 631–650. Available from: https://doi.org/10.1007/s12190-022-01763-6.

[12] S. Salisu, P. Kumam, S. Sriwongsa, Strong convergence theorems for fixed point of multi-valued mappings in Hadamard spaces, J. Inequal. Appl. 2022:143 (2022), 28 pages. Available from: https://doi.org/10.1186/s13660-022-02870-5