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Weighted Lavrentiev Regularization Method for Ill-posed Equations: Finite Dimensional Realization

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ABSTRACT

In this paper, we study weighted Lavrentiev regularization method for ill-posed operator equations in the finite dimensional subspaces of a Hilbert space. Using general Holder type source condition we obtain an optimal order error estimate. Adaptive parameter choice strategy proposed by Pereverzev and Schock (2005) is used for choosing the regularization parameter. We applied the proposed method to an academic example to test the validity of theoretical result.

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1. Introduction

The aim of this paper is to study finite dimensional fractional or weighted Laurentiev regularization method (WLR) (cf. [1–6]) to approximate the solution $\widehat{x}$ of the linear ill-posed equation

$$Ax = y, \tag{1.1}$$

where $A : H \to H$ is a positive self-adjoint operator defined on a Hilbert space H. In general the problem of solving operator equation (1.1) is ill-posed ([7–9]). Lavrentiev regularization (LR) method is used to approximate the solution $\widehat{x}$ (assumed to be exist) of the equation (1.1). In LR method the minimizer w_α^δ of the functional

$$J_\alpha(x) = \langle Ax, x\rangle - 2\langle y, x\rangle + \alpha\langle x, x\rangle, \quad \alpha > 0, \tag{1.2}$$

is taken as an approximation for $\widehat{x}$. One can see [10](in Hilbert scale) (c.f [11] (for Tikhonov regularization)) that the solution of (1.2) over smoothen the solution $\widehat{x}$, to overcome this, WLR was studied in [4, 10] (also see [2, 5–8].)

In WLR method the minimizer $w_{\alpha,\beta}^\delta$ of the functional

$$J_\alpha^\beta(x) = \langle Ax, x\rangle - 2\langle y, x\rangle + \alpha\langle A^\beta x, x\rangle, \quad \alpha > 0, \tag{1.3}$$

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where $\beta \in [0,1)$, is taken as an approximation for $\widehat{x}$. The minimizer $w_{\alpha,\beta}$ of the above functional satisfies the operator equation

$$A^{\beta}\left(A^{1-\beta}+\alpha I\right)x = y. \tag{1.4}$$

Note that, in practice the available data $y^{\delta} \in H$ with

$$\|y - y^{\delta}\| \leq \delta. \tag{1.5}$$

So, (1.4) with, y^{δ} in place of y, has a minimizer, then $y^{\delta} \in R(A^{\beta})$. This, is a severe restriction, but notice that this restriction can be overcome by considering finite dimentional realization of (1.3). So, we consider the finite dimensional realization of (1.4), namely, we consider the $w^{\delta}_{\alpha,\beta,h}$, the solution of

$$(A_h^{1-\beta}+\alpha I)x = A_h^{-\beta}P_h y^{\delta}, \tag{1.6}$$

where P_h is the orthogonal projection onto $R(P_h)$ and $A_h = P_h A P_h$.

Remark 1.1. Note that, $P_h y^{\delta} \in R(A_h)$, i.e., $P_h y^{\delta} \in R(A_h^{\beta})$ for $\beta \in [0,1)$. So, $w^{\delta}_{\alpha,\beta,h}$ is well defined.

One of the main constrain in regularization methods is the choice of the regularization parameter α. In this paper, we consider the finite dimensional version of the adaptive parameter choice method considered by Pereverzev and Schock in [12] for choosing the regularization parameter α in (1.6). Throughout this paper c, c_1, c_2, etc., denote generic constants which may take different values at different occasions.

The rest of the paper is organized as follows. In Section 2 we provide error estimates for $\|w^{\delta}_{\alpha,\beta,h} - w_{\alpha,\beta,h}\|$, $\|w_{\alpha,\beta,h} - w_{\alpha,\beta}\|$ and $\|w_{\alpha,\beta} - \widehat{x}\|$, where $w_{\alpha,\beta,h}$ is the solution of (1.6) with y in the place of y^{δ}. In Section 3 we consider the finite dimensional version of the adaptive parameter choice strategy for weighted simplified regularization method. Numerical example is given in Section 4 and finally the paper ends with conclusion in Section 5.

2. Error Estimates

In this Section we obtain the error estimates for $\|w^{\delta}_{\alpha,\beta,h} - w_{\alpha,\beta,h}\|$ and $\|w_{\alpha,\beta,h} - w_{\alpha,\beta}\|$ under the assumption (1.5) and the source condition;

$$\widehat{x} \in \{x \in H : x = A^{\nu}z, \|z\| \leq \rho\}, 0 < \nu \leq 1-\beta\}. \tag{2.1}$$

If $\widehat{x}$ satisfies (2.1), then we have [4, Lemma 2.1]

$$\|w_{\alpha,\beta} - \widehat{x}\| = O(\alpha^{\frac{\nu}{1-\beta}}). \tag{2.2}$$

For the results that follow, we impose the following conditions (cf. [13]). Let

$$\epsilon_h := \|A(I - P_h)\|$$

and assume that $\lim_{h\to 0}\epsilon_h = 0$. The above assumption is satisfied if $P_h \to I$ point-wise and if A is a compact operator. Let $A_h := P_h A P_h$. Then

$$\|A_h - A\| \leq \|P_h A(P_h - I)\| + \|(P_h - I)A\| \leq 2\epsilon_h.$$

In order to obtain an estimate for $\|w_{\alpha,\beta,h} - w_{\alpha,\beta}\|$, we shall make use of the following formula ([14, Page 287]);

$$B^z x \quad = \quad \frac{\sin \pi z}{\pi}\int_0^{\infty} t^z \left[(B+tI)^{-1}x - \frac{\theta(t)}{t}x + \ldots + (-)^n \frac{\theta(t)}{t^n}B^{n-1}x\right] dt$$

$$+\frac{\sin \pi z}{\pi}\left[\frac{x}{z}-\frac{Bx}{z-1}+\ldots+(-1)^{n-1}\frac{B^{n-1}x}{z-n+1}\right],\ x\in H,$$

where

$$\theta(t)=\begin{cases}0 & \text{if } \ 0\le t\le 1\\ 1 & \text{if } \ 1<t<\infty\end{cases}$$

for any positive self-adjoint operator B and for any complex number z such that $0 < Rez < n$. Taking $z = 1-\beta,\ 0\le\beta<1$, we have

$$B^{1-\beta}x=\frac{\sin \pi(1-\beta)}{\pi}\left[\frac{x}{1-\beta}+\int_0^\infty t^{1-\beta}(B+tI)^{-1}xdt-\int_1^\infty \frac{x}{t^\beta}dt\right].$$

Using the above formula, for any $Z\in H$, we have,

$$[A_h^{1-\beta}-A^{1-\beta}]Z=\frac{\sin \pi(1-\beta)}{\pi}\int_0^\infty t^{1-\beta}(A_h+tI)^{-1}(A-A_h)(A+tI)^{-1}Zdt. \tag{2.3}$$

Proposition 2.1. *Suppose y^δ satisfies (1.5) and $w_{\alpha,\beta,h}$ satisfies (1.6) with y in place of y^δ. Then, for $\beta\le\nu\le 1$, the following estimates hold.*

(i) $\|w^\delta_{\alpha,\beta,h}-w_{\alpha,\beta,h}\|=O\left(\frac{\delta}{\alpha^{\frac{1}{1-\beta}}}\right)$.

and

(ii) $\|w_{\alpha,\beta,h}-w_{\alpha,\beta}\|=O\left(\frac{\epsilon_h}{\alpha^{\frac{1}{1-\beta}}}\right)$.

In particular,

(iii) $\|w^\delta_{\alpha,\beta,h}-\widehat{x}\|\le c_1\frac{\delta+\epsilon_h}{\alpha^{\frac{1}{1-\beta}}}+c_2\,\alpha^{\frac{\nu}{1-\beta}}$.

Proof. From (1.6), we have

$$\begin{aligned}\|w_{\alpha,\beta,h}-w^\delta_{\alpha,\beta,h}\| &= \|(A_h^{1-\beta}+\alpha I)^{-1}A_h^{-\beta}P_h(y-y^\delta)\|\\ &\le \sup_{\lambda>0}\left|\frac{\lambda^{-\beta}}{(\lambda^{1-\beta}+\alpha I)}\right|\|y-y^\delta\|\\ &= O\left(\frac{\delta}{\alpha^{\frac{1}{1-\beta}}}\right),\end{aligned}$$

proving (i). To Prove (ii), notice that

$$\begin{aligned}w_{\alpha,\beta,h} &= (A_h^{1-\beta}+\alpha I)^{-1}A_h^{-\beta}P_hy\\ &= (A_h^{1-\beta}+\alpha I)^{-1}A_h^{-\beta}P_hA\widehat{x}\\ &= (A_h^{1-\beta}+\alpha I)^{-1}A_h^{1-\beta}\widehat{x}+(A_h^{1-\beta}+\alpha I)^{-1}A_h^{-\beta}P_hA(I-P_h)\widehat{x},\\ w_{\alpha,\beta} &= (A^{1-\beta}+\alpha I)^{-1}A^{-\beta}y\\ &= (A^{1-\beta}+\alpha I)^{-1}A^{1-\beta}\widehat{x}\end{aligned}$$

and hence

$$w_{\alpha,\beta,h}-w_{\alpha,\beta} = [(A_h^{1-\beta}+\alpha I)^{-1}A_h^{1-\beta}-(A^{1-\beta}+\alpha I)^{-1}A^{1-\beta}]\widehat{x}$$

$$+(A_h^{1-\beta}+\alpha I)^{-1}A_h^{-\beta}P_hA(I-P_h)\widehat{x}.$$

So

$$\|w_{\alpha,\beta,h}-w_{\alpha,\beta}\| \leq \|\| + \|(A_h^{1-\beta}+\alpha I)^{-1}A_h^{-\beta}P_hA(I-P_h)\widehat{x}\|, \tag{2.4}$$

where $= [(A_h^{1-\beta}+\alpha I)^{-1}A_h^{1-\beta}-(A^{1-\beta}+\alpha I)^{-1}A^{1-\beta}]\widehat{x}$.

Further, we have

$$\|(A_h^{1-\beta}+\alpha I)^{-1}A_h^{-\beta}P_hA(I-P_h)\widehat{x}\| \leq \frac{\epsilon_h}{\alpha^{\frac{1}{1-\beta}}}\|\widehat{x}\| \tag{2.5}$$

and

$$\begin{aligned}
&= [(A_h^{1-\beta}+\alpha I)^{-1}A_h^{1-\beta}-(A^{1-\beta}+\alpha I)^{-1}A^{1-\beta}]\widehat{x}\\
&= (A_h^{1-\beta}+\alpha I)^{-1}[A_h^{1-\beta}(A^{1-\beta}+\alpha I)-(A_h^{1-\beta}+\alpha I)A^{1-\beta}](A^{1-\beta}+\alpha I)^{-1}\widehat{x}\\
&= (A_h^{1-\beta}+\alpha I)^{-1}\alpha[A_h^{1-\beta}-A^{1-\beta}](A^{1-\beta}+\alpha I)^{-1}\widehat{x}.
\end{aligned}$$

So by (2.3), we have

$$\begin{aligned}
\|\| &= \|\frac{\sin\pi(1-\beta)}{\pi}\alpha(A_h^{1-\beta}+\alpha I)^{-1}\\
&\quad\times\int_0^\infty t^{1-\beta}(A_h+tI)^{-1}(A-A_h)(A+tI)^{-1}(A^{1-\beta}+\alpha I)^{-1}\widehat{x}dt\|.\\
&\leq \frac{\sin\pi(1-\beta)}{\pi}\|\alpha(A_h^{1-\beta}+\alpha I)^{-1}\|\\
&\quad\times\int_0^\infty t^{1-\beta}\|(A_h+tI)^{-1}(A-A_h)(A+tI)^{-1}(A^{1-\beta}+\alpha I)^{-1}\widehat{x}\|dt\\
&\leq \frac{\sin\pi(1-\beta)}{\pi}\|\alpha(A_h^{1-\beta}+\alpha I)^{-1}\|\\
&\quad\times[\int_0^1 t^{1-\beta}\|(A_h+tI)^{-1}(A-A_h)(A+tI)^{-1}(A^{1-\beta}+\alpha I)^{-1}\widehat{x}\|dt\\
&\quad+\int_1^\infty t^{1-\beta}\|(A_h+tI)^{-1}(A-A_h)(A+tI)^{-1}(A^{1-\beta}+\alpha I)^{-1}\widehat{x}\|dt]\\
&\leq \frac{\sin\pi(1-\beta)}{\pi}[\int_0^1 t^{1-\beta}\|(A_h+tI)^{-1}\|\|A-A_h\|\|(A+tI)^{-1}(A^{1-\beta}+\alpha I)^{-1}A^\nu z\|dt\\
&\quad+\int_1^\infty t^{1-\beta}\|(A_h+tI)^{-1}\|\|A-A_h\|\|(A+tI)^{-1}\|\|(A^{1-\beta}+\alpha I)^{-1}\widehat{x}\|dt]\\
&\leq \frac{\sin\pi(1-\beta)}{\pi}[\int_0^1 t^{1-\beta}\frac{2\epsilon_h}{t}\|(A+tI)^{-1}A^\nu(A^{1-\beta}+\alpha I)^{-1}z\|dt\\
&\quad+\int_1^\infty\frac{t^{1-\beta}2\epsilon_h}{t^2}\|(A^{1-\beta}+\alpha I)^{-1}A^\nu z\|dt].
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|\| &\leq \frac{\sin\pi(1-\beta)}{\pi}\left[\int_0^1 t^{1-\beta}\frac{2\epsilon_h}{t^{2-\nu}}\|(A^{1-\beta}+\alpha I)^{-1}z\|dt\right.\\
&\quad\left.+2\epsilon_h\|z\|\frac{\nu^\nu}{(1-\beta)(1-\beta-\nu)^{1-\nu}}\frac{1}{\alpha^{1-\frac{\nu}{1-\beta}}}\int_1^\infty\frac{1}{t^{1+\beta}}dt\right]\\
&\leq \frac{\sin\pi(1-\beta)}{\pi}\left[\frac{\|z\|}{\alpha}\int_0^1\frac{2\epsilon_h}{t^{1-\beta-\nu}}dt\right.
\end{aligned}$$

$$
\begin{aligned}
& + 2\epsilon_h\|z\|\frac{\nu^\nu}{\beta(1-\beta)(1-\beta-\nu)^{1-\nu}}\frac{1}{\alpha^{1-\frac{\nu}{1-\beta}}}\Bigg] \\
\leq & \frac{\sin\pi(1-\beta)}{\pi}\left[\frac{\|z\|}{\alpha}\frac{2\epsilon_h}{\beta+\nu} + 2\epsilon_h\|z\|\frac{\nu^\nu}{\beta(1-\beta)(1-\beta-\nu)^{1-\nu}}\frac{1}{\alpha^{1-\frac{\nu}{1-\beta}}}\right] \\
\leq & 2\frac{\sin\pi(1-\beta)\|z\|}{\pi}\left[\frac{1}{\nu} + \frac{\nu^\nu}{\beta(1-\beta)(1-\beta-\nu)^{1-\nu}}\right]\frac{\epsilon_h}{\alpha^{\frac{1}{1-\beta}}}. \qquad (2.6)
\end{aligned}
$$

Hence (ii) follows from (2.4), (2.5), (2.6) and the fact that $\max\{\frac{1}{\alpha}, \frac{1}{\alpha^{1-\frac{\nu}{1-\beta}}}\} \leq \frac{1}{\alpha^{\frac{1}{1-\beta}}}$.

The result (iii) now follows from (i), (ii) and (2.2).

□

3. Adaptive Selection of the Parameter

Note that by (iii) of Proposition 2.1, we have

$$\|w^\delta_{\alpha,\beta,h} - \widehat{x}\| \leq C\left(\frac{\delta+\epsilon_h}{\alpha^{\frac{1}{1-\beta}}} + \alpha^{\frac{\nu}{1-\beta}}\right), \qquad (3.1)$$

where

$$C = \max\{c_1, c_2\}.$$

Further, observe that the error $\frac{\delta+\epsilon_h}{\alpha^{\frac{1}{1-\beta}}} + \alpha^{\frac{\nu}{1-\beta}}$ in (3.1) is of optimal order if $\alpha_\delta := \alpha(\delta)$ satisfies $\frac{\delta+\epsilon_h}{\alpha^{\frac{1}{1-\beta}}} = \alpha^{\frac{\nu}{1-\beta}}$. That is $\alpha_\delta = (\delta+\epsilon_h)^{\frac{1-\beta}{\nu+1}}$. Pereverzev and Schock in [12], introduced the adaptive selection of the parameter strategy, we modified adaptive method suitably for the situation for choosing the parameter α to obtain the optimal order in (3.1). Let $i \in \{0, 1, 2, \cdots, N\}$ and $\alpha_i = \mu^i\alpha_0$ where $\mu > 1$ and $\alpha_0 > \delta$.

Let

$$l := max\left\{i : \alpha_i^{\frac{1+\nu}{1-\beta}} \leq \delta+\epsilon_h\right\} < N \quad \text{and} \qquad (3.2)$$

$$k := max\left\{i : \|w^\delta_{\alpha_i,\beta,h} - w^\delta_{\alpha_j,\beta,h}\| \leq 4C\,\frac{\delta+\epsilon_h}{\alpha_j^{\frac{1}{1-\beta}}}, j = 0, 1, 2, \cdots, i-1\right\} \qquad (3.3)$$

where C is as in (3.1). We have the following Theorem.

Theorem 3.1. [15] *Assume that there exists $i \in \{0, 1, \cdots, N\}$ such that $\alpha_i^{\frac{1+\nu}{1-\beta}} \leq \delta+\epsilon_h$. Let assumptions of Proposition 2.1 be fulfilled, and let l and k be as in (3.2) and (3.3) respectively. Then $l \leq k$; and*

$$\|w^\delta_{\alpha_k,\beta,h} - \widehat{x}\| \leq 6\,C\mu^{\frac{\nu+1}{1-\beta}}(\delta+\epsilon_h)^{\frac{\nu}{\nu+1}}.$$

Proof. To prove $l \leq k$, it is enough to show that, for each $i \in \{1, 2, \ldots N\}$, $\alpha_i^{\frac{\nu}{1-\beta}} \leq \frac{\delta+\epsilon_h}{\alpha_i^{\frac{1}{1-\beta}}} \Longrightarrow \|w^\delta_{\alpha_i,\beta,h} - w^\delta_{\alpha_j,\beta,h}\| \leq 4\,C\,\frac{\delta+\epsilon_h}{\alpha_j^{\frac{1}{1-\beta}}}$, $\forall j = 0, 1, 2, \ldots i-1$. For $j < i$, we have

$$
\begin{aligned}
\|\, w^\delta_{\alpha_i,\beta,h} - w^\delta_{\alpha_j,\beta,h}\,\| &\leq \|\, w^\delta_{\alpha_i,\beta,h} - \widehat{x}\,\| + \|\,\widehat{x} - w^\delta_{\alpha_j,\beta,h}\,\| \\
&\leq C\left(\alpha_i^{\frac{\nu}{1-\beta}} + \frac{\delta+\epsilon_h}{\alpha_i^{\frac{1}{1-\beta}}}\right) + C\left(\alpha_j^{\frac{\nu}{1-\beta}} + \frac{\delta+\epsilon_h}{\alpha_j^{\frac{1}{1-\beta}}}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\leq \; 2\,C\,\alpha_i^{\frac{\nu}{1-\beta}} + 2\,C\,\frac{\delta+\epsilon_h}{\alpha_j^{\frac{1}{1-\beta}}} \\
&\leq \; 4\,C\,\frac{\delta+\epsilon_h}{\alpha_j^{\frac{1}{1-\beta}}}.
\end{aligned}
$$

Thus the relation $l \leq k$ is proved. Further note that

$$
\| \widehat{x} - w^{\delta}_{\alpha_k,\beta,h} \| \leq \| \widehat{x} - w^{\delta}_{\alpha_l,\beta,h} \| + \| w^{\delta}_{\alpha_l,\beta,h} - w^{\delta}_{\alpha_k,\beta,h} \|
$$

where

$$
\| \widehat{x} - w^{\delta}_{\alpha_l,\beta,h} \| \leq C \left(\alpha_l^{\frac{\nu}{1-\beta}} + \frac{\delta+\epsilon_h}{\alpha_l^{\frac{1}{1-\beta}}} \right) \leq 2\,C\,\frac{\delta+\epsilon_h}{\alpha_l^{\frac{1}{1-\beta}}}.
$$

Now since $l \leq k$, we have

$$
\| w^{\delta}_{\alpha_k,\beta,h} - w^{\delta}_{\alpha_l,\beta,h} \| \;\; \leq \;\; 4\,C\,\frac{\delta+\epsilon_h}{\alpha_l^{\frac{1}{1-\beta}}}.
$$

Hence

$$
\| \widehat{x} - w^{\delta}_{\alpha_k,\beta,h} \| \leq 6\,C\,\frac{\delta+\epsilon_h}{\alpha_l^{\frac{1}{1-\beta}}}
$$

Again, since $\alpha_\delta^{\frac{\nu+1}{1-\beta}} = \delta + \epsilon_h \leq \alpha_{l+1}^{\frac{\nu+1}{1-\beta}} \leq \mu^{\frac{\nu+1}{1-\beta}} \alpha_l{}^{\frac{\nu+1}{1-\beta}}$, and $\alpha_l \leq \alpha_\delta \leq \alpha_{l+1}$, it follows that

$$
\frac{\delta+\epsilon_h}{\alpha_\delta^{\frac{1}{1-\beta}}} \leq \frac{\delta+\epsilon_h}{\alpha_l^{\frac{1}{1-\beta}}} \leq \mu^{\frac{\nu+1}{1-\beta}} \alpha_l{}^{\frac{\nu}{(1-\beta)}} \leq \mu^{\frac{\nu+1}{1-\beta}} \alpha_\delta{}^{\frac{\nu}{(1-\beta)}} \leq \mu^{\frac{\nu+1}{1-\beta}} (\delta+\epsilon_h)^{\frac{\nu}{\nu+1}}.
$$

This completes the proof.

3.1. Implementation of adaptive choice rule

Finally the balancing algorithm associated with the choice of the parameter specified in Theorem 3.1 involves the following steps:

- Choose $\alpha_0 > 0$ such that $\delta < \alpha_0$ and $\mu > 1$.
- Choose $\alpha_i := \mu^i \alpha_0$, $i = 0, 1, 2, \cdots, N$.

3.2. Algorithm

1. Set $i = 0$.
2. Solve $w^{\delta}_{\alpha_i,\beta,h}$ by using (1.6).
3. If $\|w^{\delta}_{\alpha_i,\beta,h} - w^{\delta}_{\alpha_j,\beta,h}\| > 4\,C\,\frac{\delta+\epsilon_h}{\alpha_j^{\frac{1}{1-\beta}}}$, $j = 0, 1, 2, \dots i-1$, then take $k = i-1$ and return $w_{\alpha_k,\beta,h}$.
4. Else set $i = i + 1$ and go to 2.

4. Numerical Examples

In this section, we consider an academic example for the numerical discussion to validate our theoretical results. The discrete version of the operator A is taken from the Regularization Toolbox by Hansen [16].

Relative errors $E_{\alpha,\beta,h} := \left(\frac{\|w^{\delta}_{\alpha,\beta,h}-\widehat{x}\|}{\|\widehat{x}\|}\right)$, and α are presented in the tables for different values of β, n (size of the mesh) and noise level δ.

Example 4.1 [17] Define the function

$$\phi(x) = \begin{cases} 1+\cos\left(\frac{x\pi}{3}\right) & |x|<3 \\ 0 & |x|\geq 3. \end{cases}$$

Consider the problem of solving integral equation

$$[Tx](s) := \int_{-6}^{6} k(s,t)\, x(t)dt = g(s), \quad -6 \leq s \leq 6, \tag{4.1}$$

where $k(s,t) = \phi(s-t)$, $g(s) = (6-|s|)\left(1+\frac{1}{2}\cos\left(\frac{s\pi}{3}\right)\right) + \frac{9}{2\pi}\sin\left(\frac{|s|\pi}{3}\right)$. We take $A = T^*T$. The solution of this problem $\widehat{x}(t)$ is given by $\widehat{x}(t) = \phi(t)$. We have introduced the random noise level $\delta = 0.05$ and 0.01 in the exact data. Relative errors and α values are showcased in Tables 1 obtained using adaptive method for different values of β, n and δ. In Fig:1, Fig:3, Fig:5 and Fig:7 (plot (a)), contains the computed solution (C.S) and exact solution (exact sol.) and in Fig:2, Fig:4, Fig:6 and Fig:8 (plot (b)), contains the exact data and noise data.

Table 1. Relative errors obtained from Adaptive method

β		$n=100$		$n=500$		$n=1000$	
		$\delta=0.05$	$\delta=0.01$	$\delta=0.05$	$\delta=0.01$	$\delta=0.05$	$\delta=0.01$
	α	$1.393509e+00$	$4.884165e-01$	$1.266827e+00$	$3.335950e-01$	$1.151661e+00$	$3.032682e-01$
0	$E_{\alpha,\beta,h}$	$1.114813e-01$	$5.507954e-02$	$1.063084e-01$	$4.492278e-02$	$1.040632e-01$	$4.212422e-02$
	α	$1.686146e+00$	$4.440150e-01$	$1.532860e+00$	$3.032682e-01$	$1.393509e+00$	$2.756984e-01$
0.15	$E_{\alpha,\beta,h}$	$1.059385e-01$	$4.664234e-02$	$1.004385e-01$	$4.396330e-02$	$1.002486e-01$	$4.037034e-02$
	α	$1.854761e+00$	$4.036500e-01$	$1.686146e+00$	$2.756984e-01$	$1.266827e+00$	$2.506349e-01$
0.25	$E_{\alpha,\beta,h}$	$1.057641e-01$	$4.397211e-02$	$1.008134e-01$	$4.237829e-02$	$8.230723e-02$	$3.856134e-02$
	α	$2.040237e+00$	$3.669545e-01$	$1.854761e+00$	$2.506349e-01$	$1.151661e+00$	$2.278499e-01$
0.35	$E_{\alpha,\beta,h}$	$1.047137e-01$	$4.313658e-02$	$1.000185e-01$	$3.810415e-02$	$6.896331e-02$	$3.495328e-02$

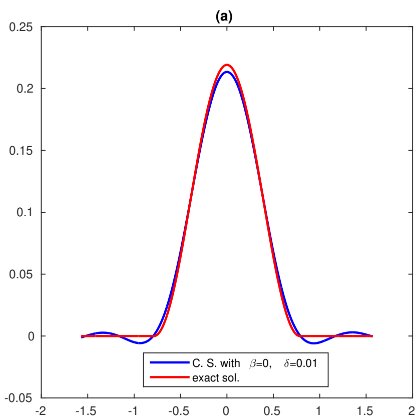

Fig:1 Solution of *Phillips* example with $\delta=0.01$, $\beta=0$ and $n=1000$.

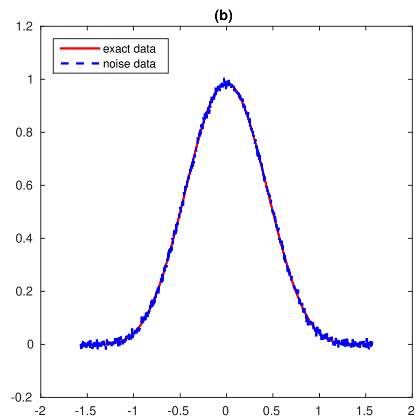

Fig:2 Data of *Phillips* example with $\delta=0.01$, $\beta=0$ and $n=1000$.

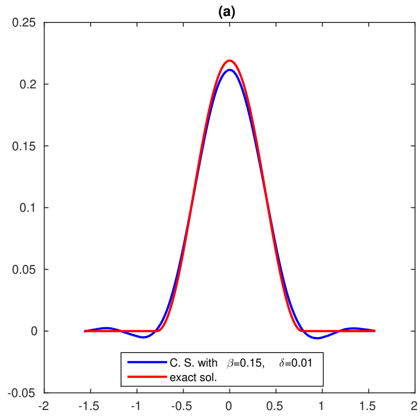

Fig:3 Solution of *Phillips* example with $\delta = 0.01$, $\beta = 0.15$ and $n = 1000$.

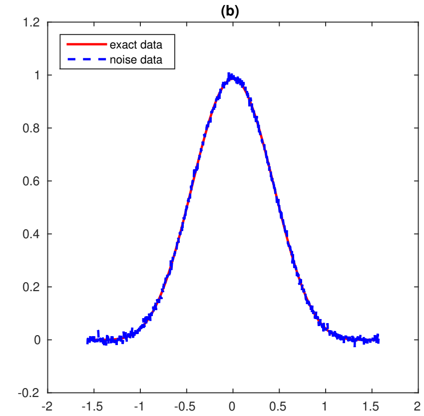

Fig:4 Data of *Phillips* example with $\delta = 0.01$, $\beta = 0.15$ and $n = 1000$.

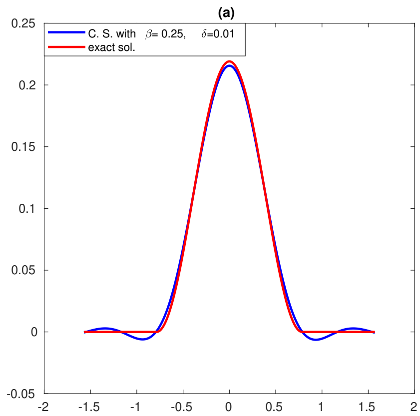

Fig:5 Solution of *Phillips* example with $\delta = 0.01$, $\beta = 0.25$ and $n = 1000$.

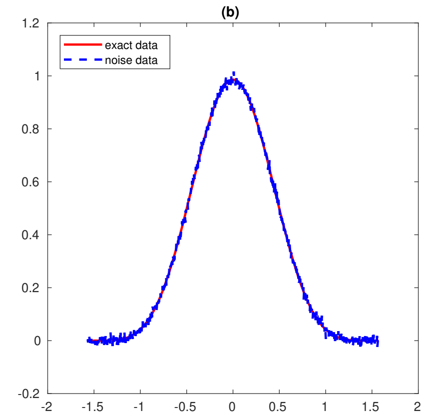

Fig:6 Data of *Phillips* example with $\delta = 0.01$, $\beta = 0.25$ and $n = 1000$.

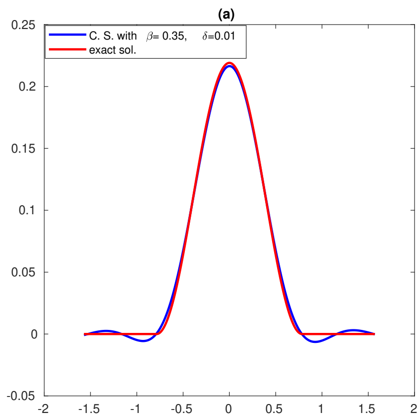

Fig:7 Solution of *Phillips* example with $\delta = 0.01$, $\beta = 0.35$ and $n = 1000$.

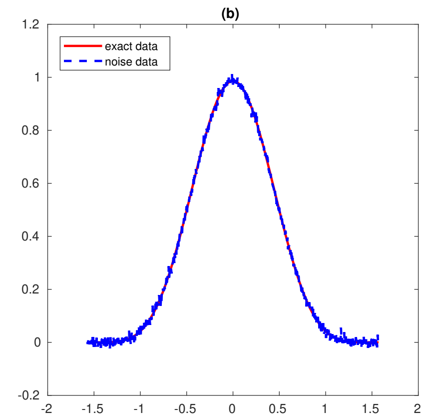

Fig:8 Data of *Phillips* example with $\delta = 0.01$, $\beta = 0.35$ and $n = 1000$.

5. Conclusion

In this paper we considered weighted simplified regularization method for ill-posed equations in the finite dimensional subspaces of a Hilbert space involving positive self-adjoint operator. We obtained an optimal order error estimate under a general Holder type source condition and the regularization parameter is chosen using adaptive parameter choice strategy introduced by Pereverzev and Schock [17]. Numerical experiments confirms the theoretical results.

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