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A Note on Endpoints of Generalized Osilike-Berinde-nonexpansive Mappings

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ABSTRACT

The semiclosed principle for generalized Osilike-Berinde-nonexpansive mappings in 2-uniformly convex geodesic spaces is proved. The existence of endpoints and common endpoints for generalized Osilike-Berinde-nonexpansive mappings in this setting is also established. Our results extend and improve many results in the literature.

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1. Introduction

Let (M,d) be a metric space, $x \in M$, and C be a nonempty subset of M. We define

$$\text{dist}(x,C) := \inf\{d(x,y) : y \in C\}, \quad R(x,C) := \sup\{d(x,y) : y \in C\},$$

and

$$\text{diam}(C) := \sup\{d(y,z) : y,z \in C\}.$$

We denote the family of nonempty closed bounded subsets of C by $\mathcal{CB}(C)$. The Pompeiu-Hausdorff distance on $\mathcal{CB}(C)$ is defined by

$$H(A,B) := \max\left\{\sup_{a\in A}\text{dist}(a,B),\ \sup_{b\in B}\text{dist}(b,A)\right\} \ \text{for all}\ \ A,B \in \mathcal{CB}(C).$$

A mapping T from C into $\mathcal{CB}(C)$ is called a multi-valued mapping. In particular, if Tx is a singleton for every x in C, then T is called a single-valued mapping. A point x in C is

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called a fixed point of T if $x \in Tx$. We denote the set of all fixed points of T by $Fix(T)$. A multi-valued mapping $T : C \to \mathcal{CB}(C)$ is called a contraction if there exists a constant $\alpha \in [0, 1)$ such that

$$H(Tx, Ty) \leq \alpha\, d(x, y) \text{ for all } x, y \in C. \tag{1.1}$$

If (1.1) is valid when α is equal to 1, then T is called nonexpansive.

Fixed point theory is an important tool for finding solutions of problems in the form of equations or inequalities. One of the fundamental and celebrated results in metric fixed point theory is the so-called Banach contraction principle which stated that every single-valued contraction on a complete metric space always has a fixed point (see [1]). The principle was extended to multi-valued mappings by Nadler [2] in 1969.

In 1995, Osilike [3] generalized the concept of a single-valued contraction to a mapping $f : C \to C$ such that there exist $\alpha \in [0, 1)$ and $L \in [0, \infty)$ for which

$$d(f(x), f(y)) \leq \alpha\, d(x, y) + L\, d(x, f(x)) \text{ for all } x,\ y \in C.$$

In 2007, Berinde and Berinde [4] extended the Osilike's concept to a multi-valued mapping $T : C \to \mathcal{CB}(C)$ such that there exist $\alpha \in [0, 1)$ and $L \in [0, \infty)$ for which

$$H(Tx, Ty) \leq \alpha\, d(x, y) + L\,\mathrm{dist}(x, Tx) \text{ for all } x, y \in C. \tag{1.2}$$

By combining the ideas of [3] and [4], Bunlue and Suantai [5] introduced a general class of multi-valued mappings in the following manner: a mapping $T : C \to \mathcal{CB}(C)$ is said to be Osilike-Berinde-nonexpansive if there exists $\mu \geq 0$ such that

$$H(Tx, Ty) \leq d(x, y) + \mu\,\mathrm{dist}(x, Tx) \text{ for all } x,\ y \in C.$$

In [5], the authors also obtained fixed point theorems and convergence theorems for Osilike-Berinde-nonexpansive mappings in uniformly convex Banach spaces and Banach spaces which satisfy the Opial's condition.

The concept of endpoints (or strict fixed point) for multi-valued mappings is an important concept which lies between the concept of fixed points for single-valued mappings and the concept of fixed points for multi-valued mappings. In 1986, Corley [6] proved that a maximization with respect to a cone is equivalent to the problem of finding an endpoint of a certain multi-valued mapping. In 2010, Amini-Harandi [7] proved the existence of endpoints for multi-valued contractions in complete metric spaces. After that, Ahmad et al. [8] applied his result to guarantee the existence of solutions of the mixed Hadamard and Riemann-Liouville fractional inclusion problems. For more details and further applications of the endpoint theory, the reader is referred to [9–14].

Recently, Panyanak [15] introduced the concept of generalized Osilike-Berinde-nonexpansive mappings in metric spaces and showed that it was weaker than the concept of Osilike-Berinde-nonexpansive mapping. In [15], the author also proved the semiclosed principle and applied it to obtain endpoint and common endpoint theorems for generalized Osilike-Berinde-nonexpansive mappings with nonempty compact values in 2-uniformly convex hyperbolic spaces.

In this paper, motivated by the above results, we prove the semiclosed principle for generalized Osilike-Berinde-nonexpansive mappings with nonempty closed and bounded values in the general setting of 2-uniformly convex geodesic spaces. We also obtain common endpoint theorems for a family of generalized Osilike-Berinde-nonexpansive mappings in this setting. Our results extend and improve the results of Bunlue and Suantai [5], Panyanak [15] and many others.

2. Preliminaries

Throughout this paper, $\mathbb{N}$ stands for the set of natural numbers and $\mathbb{R}$ stands for the set of real numbers. Let C be a nonempty subset of a metric space (M, d) and $T : C \to \mathcal{CB}(C)$ be a multi-valued mapping. A point x in C is called an endpoint of T if $Tx = \{x\}$. We denote the set of all endpoints of T by $End(T)$. Notice that the following statements hold:

- If x is an endpoint of T, then x is a fixed point of T.
- $x \in Fix(T)$ if and only if $\operatorname{dist}(x, Tx) = 0$.
- $x \in End(T)$ if and only if $R(x, Tx) = 0$.

A sequence $\{x_n\}$ in C is called an approximate endpoint sequence of T [7] if

$$\lim_{n\to\infty} R(x_n, Tx_n) = 0.$$

Moreover, if $\{T_\alpha : \alpha \in \Omega\}$ is a family of multi-valued mappings from C into $\mathcal{CB}(C)$, then $\{x_n\}$ is called an approximate common endpoint sequence of $\{T_\alpha : \alpha \in \Omega\}$ [16] if $\lim\limits_{n\to\infty} R(x_n, T_\alpha x_n) = 0$ for all $\alpha \in \Omega$.

Definition 2.1. A multi-valued mapping $T : C \to \mathcal{CB}(C)$ is said to be generalized Osilike-Berinde-nonexpansive if there exists $\mu \geq 0$ such that

$$H(Tx, Ty) \leq d(x, y) + \mu R(x, Tx) \text{ for all } x, y \in C. \tag{2.1}$$

Let (M, d) be a metric space and $x, y \in M$. A continuous mapping $\gamma : [0, 1] \to M$ is called a geodesic joining x and y if $\gamma(0) = x, \gamma(1) = y$ and

$$d(\gamma(t), \gamma(t')) = |t - t'| d(x, y) \text{ for all } t, t' \in [0, 1].$$

A metric space (M, d) is said to be a geodesic space if for any two points in M there exists a geodesic joining them. Moreover, if any two points in M are joined by a unique geodesic, then we say that M is a uniquely geodesic space.

A geodesic space (M, d) is called 2-uniformly convex [17] if there exists a constant $c_M \in (0, 1]$ such that for any $x, y, z \in M$ and for any geodesic $\gamma : [0, 1] \to M$ joining x and y, the following inequality holds:

$$d^2(\gamma(t), z) \leq (1 - t)d^2(x, z) + td^2(y, z) - c_M t(1 - t)d^2(x, y) \text{ for all } t \in [0, 1].$$

It is known from [18] that every 2-uniformly convex geodesic space is uniquely geodesic.

Example 2.2. (1) Every uniformly convex Banach space is a 2-uniformly convex geodesic space (see [19]).

(2) If M is a CAT(0) space, then it is a 2-uniformly convex geodesic space (see [20]).

(3) If $\kappa > 0$ and M is a CAT(κ) space with $\operatorname{diam}(M) \leq \frac{\pi/2-\varepsilon}{\sqrt{\kappa}}$ for some $\varepsilon \in (0, \pi/2)$, then by Lemma 2.3 of [21] we can conclude that

$$d^2(\gamma(t), z) \leq (1 - t)d^2(x, z) + td^2(y, z) - \frac{R}{2} t(1 - t)d^2(x, y) \text{ for all } t \in [0, 1],$$

where $R = (\pi - 2\varepsilon)\tan(\varepsilon)$. This clearly implies that M is a 2-uniformly convex geodesic space.

From now on, M stands for a complete 2-uniformly convex geodesic space. Let C be a nonempty subset of M and $\{x_n\}$ be a bounded sequence in M. The asymptotic radius of $\{x_n\}$ relative to C is defined by

$$r(C, \{x_n\}) := \inf\left\{ \limsup_{n\to\infty} d(x_n, x) : x \in C\right\}.$$

The asymptotic center of $\{x_n\}$ relative to C is the set

$$A(C, \{x_n\}) := \left\{x \in C : \limsup_{n\to\infty} d(x_n, x) = r(C, \{x_n\})\right\}.$$

It is known from [22] that if C is a nonempty closed convex subset of M, then $A(C, \{x_n\})$ consists of exactly one point. Now, we give the definition of Δ-convergence.

Definition 2.3. Let C be a nonempty closed convex subset of M and $x \in C$. Let $\{x_n\}$ be a bounded sequence in M. We say that $\{x_n\}$ Δ-converges to x if $A(C, \{x_{n_k}\}) = \{x\}$ for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$. In this case, we write $x_n \xrightarrow{\Delta} x$.

It is known from [23] that every bounded sequence in M has a Δ-convergent subsequence.

Definition 2.4. Let C be a nonempty closed convex subset of M and $T : C \to \mathcal{CB}(C)$. Let I be the identity mapping on C. We say that $I - T$ is semiclosed if for any sequence $\{x_n\}$ in C such that $x_n \xrightarrow{\Delta} x$ and $R(x_n, Tx_n) \to 0$, one has $Tx = \{x\}$.

3. Main Results

We start this section by proving the semiclosed principle for generalized Osilike-Berinde-nonexpansive mappings in 2-uniformly convex geodesic spaces. Notice that this is an extension of Theorem 3.1 in [15].

Theorem 3.1. *Let C be a nonempty closed convex subset of M and I the identity mapping on C. Let $T : C \to \mathcal{CB}(C)$ be a generalized Osilike-Berinde-nonexpansive mapping with $\mu \geq 0$. Then $I - T$ is semiclosed.*

Proof. Let $\{x_n\}$ be a sequence in C such that $x_n \xrightarrow{\Delta} x$ and $R(x_n, Tx_n) \to 0$. Let y be an arbitrary point in Tx. For each $n \in \mathbb{N}$, we can choose y_n in Tx_n such that

$$d(y, y_n) \leq \text{dist}(y, Tx_n) + \frac{1}{n}.$$

It follows from (2.1) that

$$\begin{aligned} d(x_n, y) &\leq d(x_n, y_n) + d(y_n, y) \\ &\leq R(x_n, Tx_n) + H(Tx_n, Tx) + \frac{1}{n} \\ &\leq (1 + \mu)R(x_n, Tx_n) + d(x_n, x) + \frac{1}{n}, \end{aligned}$$

which implies $\limsup_{n\to\infty} d(x_n, y) \leq \limsup_{n\to\infty} d(x_n, x)$. Since $x_n \xrightarrow{\Delta} x$, it must be the case that $y = x$. Hence, $Tx = \{x\}$ and the proof is completed. ∎

By applying Theorem 3.1, we prove a common endpoint theorem for a family of generalized Osilike-Berinde-nonexpansive mappings.

Theorem 3.2. *Let C be a nonempty closed convex subset of M and $\{T_\alpha : \alpha \in \Omega\}$ a family of generalized Osilike-Berinde-nonexpansive mappings from C into $\mathcal{CB}(C)$. If $\{T_\alpha : \alpha \in \Omega\}$ has a bounded approximate common endpoint sequence in C, then it has a common endpoint in C.*

Proof. Let $\{x_n\}$ be a bounded approximate common endpoint sequence of $\{T_\alpha : \alpha \in \Omega\}$. As we have observed, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \xrightarrow{\Delta} x$. It follows from Theorem 3.1 that $T_\alpha x = \{x\}$ for all $\alpha \in \Omega$. This completes the proof. ∎

As a consequence of Theorem 3.2, we can obtain the following result.

Corollary 3.3. *Let C be a nonempty closed convex subset of M and $T : C \to \mathcal{CB}(C)$ a generalized Osilike-Berinde-nonexpansive mapping. Then T has an endpoint if and only if T has a bounded approximate endpoint sequence in C.*

Now, we prove another common endpoint theorem which can be viewed as an extension of Theorem 3.3 in [15].

Theorem 3.4. *Let C be a nonempty closed convex subset of M and $\{T_\alpha : \alpha \in \Omega\}$ a family of generalized Osilike-Berinde-nonexpansive mappings from C into $\mathcal{CB}(C)$. Suppose there exist two disjoint subsets $\mathcal{A}$ and $\mathcal{B}$ of Ω such that $\mathcal{A} \cup \mathcal{B} = \Omega$. Also, suppose each $\alpha \in \mathcal{A}$, the mapping T_α has a bounded approximate endpoint sequence in $\cap_{\beta\in\mathcal{B}} End(T_\beta)$. Then $\{T_\alpha : \alpha \in \Omega\}$ has a common endpoint in C.*

Proof. Fix $\alpha \in \mathcal{A}$ and let $\{x_n\}$ be a bounded approximate endpoint sequence of T_α in $\cap_{\beta\in\mathcal{B}} End(T_\beta)$. Without loss of generality, we may assume that $x_n \xrightarrow{\Delta} x$. According to Theorem 3.1, $x \in End(T_\alpha)$. Fix $\beta \in \mathcal{B}$ and let $y \in T_\beta x$. Since T_β is generalized Osilike-Berinde-nonexpansive, by (2.1) we get

$$\begin{aligned} d(y, x_n) &= \operatorname{dist}(y, T_\beta x_n) \\ &\leq H(T_\beta x, T_\beta x_n) \\ &\leq d(x, x_n) + \mu R(x_n, T_\beta x_n). \end{aligned}$$

This implies that $\limsup_{n\to\infty} d(y, x_n) \leq \limsup_{n\to\infty} d(x, x_n)$. Since $x_n \xrightarrow{\Delta} x$, we have $y = x$ for all $y \in T_\beta x$ and hence $T_\beta x = \{x\}$. This shows that x is a common endpoint of $\{T_\alpha : \alpha \in \Omega\}$. ∎

As a consequence of Theorem 3.4, we also obtain the following result.

Corollary 3.5. *Let C be a nonempty closed convex subset of M and $T, S : C \to \mathcal{CB}(C)$ be generalized Osilike-Berinde-nonexpansive mappings. Suppose that T has a bounded approximate endpoint sequence in $End(S)$. Then T and S has a common endpoint in C.*

Finally, we finish the paper with a numerical example.

Example 3.6. [24, 25] Let (M, d) be the Euclidean space $\mathbb{R}^2$ and $C = [0, 1] \times [0, 1]$. Let $T : C \to \mathcal{CB}(C)$ be defined by

$$T(a, b) := \text{the closed convex hull of } \{(0, 0), (a, 0), (0, b)\}.$$

Then T is a generalized Osilike-Berinde-nonexpansive mapping with $End(T) = \{(0, 0)\}$. Let $\{x_n\}$ be the sequence of Mann iteration defined by $x_1 \in C$ and

$$x_{n+1} = \frac{1}{n}x_n + (1 - \frac{1}{n})y_n, \ \ n \in \mathbb{N},$$

where $y_n \in Tx_n$ such that $d(x_n, y_n) = R(x_n, Tx_n)$. We see that, in any case of x_1, the sequence $\{x_n\}$ converges to $(0, 0)$ as n tends to ∞.

n	$x_1 = (1, 0.4)$ boundary case	$x_1 = (0.3, 0.5)$ interior case	$x_1 = (0.6, 0.6)$ diagonal case	$x_1 = (0, 1)$ corner case
	x_n	x_n	x_n	x_n
1	(1.00000,0.40000)	(0.30000,0.50000)	(0.60000,0.60000)	(0,1.00000)
2	(1.00000,0.40000)	(0.30000,0.50000)	(0.60000,0.60000)	(0,1.00000)
3	(0.50000,0.20000)	(0.15000,0.25000)	(0.30000,0.30000)	(0,0.50000)
4	(0.16667,0.06667)	(0.05000,0.08333)	(0.10000,0.10000)	(0,0.16667)
5	(0.04167,0.01667)	(0.01250,0.02083)	(0.02500,0.02500)	(0,0.04167)
6	(0.00833,0.00333)	(0.00250,0.00417)	(0.00500,0.00500)	(0,0.00833)
7	(0.00139,0.00056)	(0.00042,0.00069)	(0.00083,0.00083)	(0,0.00139)
8	(0.00020,0.00008)	(0.00006,0.00010)	(0.00012,0.00012)	(0,0.00020)
9	(0.00002,0.00001)	(0.00001,0.00001)	(0.00001,0.00001)	(0,0.00002)
10	(0.00000,0.00000)	(0.00000,0.00000)	(0.00000,0.00000)	(0,0.00000)

Table 1. Numerical experiments

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