Nonlinear Convex Analysis and Optimization
Vol. 1, No. 2, 2022, pp. 177–191
https://ncao.design.blog

A Halpern's Mean Subgradient Extragradient Method for Solving Variational Inequalities

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ABSTRACT

This paper deals with the solving of a variational inequality problem in a real Hilbert space. To this end, we present a Halpern's mean subgradient extragradient method. We prove the strong convergence result of the proposed method under some suitable assumptions of step-size in case of the monotone and Lipschitz continuous operator.

Article History
Received 30 May 2022
Accepted 25 Dec 2022
Keywords:
averaging matrix; subgradient extragradient method; mean value iteration; variational inequalities; strong convergence
Keywords
47H05; 65K15; 90C33

1. Introduction

Let $\mathcal{H}$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and its induced norm $\| \cdot \|$. In this paper, we present a computational iterative method to the variational inequality problem which was first introduced in [1]. Given C is a nonempty closed convex subset of $\mathcal{H}$ and a mapping $F : \mathcal{H} \to \mathcal{H}$, the variational inequality problem is to find a point $x^* \in C$ such that

$$\langle F(x^*), z - x^* \rangle \geq 0 \text{ for all } z \in C. \tag{1.1}$$

A well-known projection-type method for solving the variational inequality problems (1.1) is the extragradient method which is proposed by Korpelevich [2] in the Euclidean space. After that, it was considered in the Hilbert space by Censor et al. [3]. The extragradient method

Please cite this article as: A. Buakird et al., A Halpern's Mean Subgradient Extragradient Method for Solving Variational Inequalities, Nonlinear Convex Anal. & Optim., Vol. 1 No. 2, 177–191.

requires two projection onto the set C and two calculations of the operator F per iteration as follows:

$$\begin{cases} x_1 \in \mathcal{H}, \\ y_k := P_C(x_k - \lambda F(x_k)), \\ x_{k+1} := P_C(x_k - \lambda F(y_k)). \end{cases} \tag{1.2}$$

It was proved that, if the the solution set of the variational inequality (1.1) is nonempty and the operator F is monotone and L-Lipschitz continuous, then the sequence generated by the extragradient method (1.2) converges weakly to an element in the solution set of the variational inequality (1.1). The extragradient method has received great concentration by many authors, for instance [4–8] and references there in. Note that if the constrained set C is a general closed convex set, then one need to solve a hidden sub-problem in order to obtain the next iteration. This situation may affect the efficiency of the extragradient method. To keep away from this situation, Censor et al. [3] modified the extragradient method by replacing the second projection onto the closed and convex subset C with the one onto the subgradient half-space for updating the next iteration. This method is called subgradient-extragradient method as follows:

$$\begin{cases} x_1 \in \mathcal{H}, \\ y_k := P_C(x_k - \lambda F(x_k)), \\ T_k := \{w \in \mathcal{H} : \langle (x_k - \lambda F(x_k)) - y_k, w - y_k \rangle \leq 0\}, \\ x_{k+1} := P_{T_k}(x_k - \lambda F(y_k)). \end{cases} \tag{1.3}$$

Censor et al. [3] showed the weak convergence result of the proposed subgradient extragradient method under some appropriate condition. This method has been studied by many authors, see for instance [9–14]. In particular, Kraikaew and Saejung [10] presented the Halpern's type subgradient extragradient method for solving the variational inequality problem (1.1) by using the ideas of the classical Halpern's method [15] for the fixed-point problem and subgradient extragradient method (1.3). Their method is read as follow:

$$\begin{cases} x_1 \in \mathcal{H}, \\ y_k := P_C(x_k - \lambda F(x_k)), \\ T_k := \{w \in \mathcal{H} : \langle (x_k - \lambda F(x_k)) - y_k, w - y_k \rangle \leq 0\}, \\ x_{k+1} := \beta_k x_1 + (1-\beta_k) P_{T_k}(x_k - \lambda F(y_k)), \end{cases} \tag{1.4}$$

where $\{\beta_k\}_{k=1}^{\infty} \subset (0,1)$ satisfied $\sum_{k=1}^{\infty} \beta_k = \infty$ and $\lim_{k\to\infty} \beta_k = 0$. They showed the strong convergence result of the presented method where the operator F is monotone and L-Lipschitz continuous.

On the other hand, the fixed-point problem is the problem of finding a point $x^* \in \operatorname{Fix} T := \{x \in \mathcal{H} : x = Tx\} \neq \emptyset$, where $T : \mathcal{H} \to \mathcal{H}$ is a nonlinear operator. The classical method to solve the fixed-point problem is the Picard's iteration. This method is updated by

$$x_{k+1} = Tx_k.$$

After that, Mann [16] presented a modification of Picard's iteration so-called Mann's mean value iteration. The idea of this method is to use the mean of previous history iteration to update next iteration as follows:

$$x_{k+1} = T\overline{x}_k,$$

where $\overline{x}_k = \sum_{j=1}^{k} a_j x_j$, with $a_j \geq 0, \forall j = 1, \cdots, k$ and $\sum_{j=1}^{k} a_j = 1$. Notice that, under some appropriate condition, the Mann's mean value iteration converges weakly to a point in Fix T whereas the classical Picard's iteration may fail to converge in general, see [17] for more discussion. Further works of the Mann's mean value iteration were carried out in, for instance, [17, 18].

Motivated the result of [10] and the ideas of the Mann's mean value iteration, we present the Halpern's mean subgradient extragradient method for solving the variational inequality problem (1.1). We prove the strong convergence result of the proposed method in case of the operator F is monotone and L-Lipschitz continuous.

2. Preliminaries

In this section, we collect some basic definitions, properties, and useful tools in our work. For more details, the reader may consult the reference books [19, 20].

We denote by Id the identity operator on a real Hilbert space $\mathcal{H}$. We denote the strong convergence and weak convergence of a sequence $\{x_k\}_{k=1}^{\infty}$ to $x \in \mathcal{H}$ by $x_k \to x$ and $x_k \rightharpoonup x$, respectively.

We first recall some definitions and properties of the metric projection and half-space that will be referred to in our analysis.

Definition 2.1. [19, Definition 1.2.1] Let C be a nonempty subset of $\mathcal{H}$ and $x \in \mathcal{H}$. If there exists a point $y \in C$ such that

$$\|y - x\| \leq \|z - x\|, \qquad \forall z \in C,$$

then y is called a metric projection of x onto C and is denoted by $P_C(x)$. If $P_C(x)$ exists and is uniquely determined for all $x \in \mathcal{H}$, then the operator $P_C : \mathcal{H} \to C$ is called the metric projection onto C.

It is well-known that if C is a nonempty closed and convex subset of $\mathcal{H}$, then for any $x \in \mathcal{H}$ there exists a metric projection $P_C(x)$ and it is uniquely determined. We also note that the metric projection P_C is a nonexpansive mapping, that is, for all $x, y \in \mathcal{H}$,

$$\|P_C(x) - P_C(y)\| \leq ||x - y||.$$

The theorem below gives a characterization of the metric projection.

Theorem 2.2. [19, Theorem 1.2.4] Let $x \in \mathcal{H}$, C be a closed convex subset of $\mathcal{H}$ and $y \in C$. The following conditions are equivalent:

(i) $y = P_C(x)$

(ii) $\langle x - y, z - y \rangle \leq 0$.

Proof. See [19, Theorem 1.2.4]. ∎

The hyperplane in a Hilbert space $\mathcal{H}$ is definded as the subset

$$H(a; \gamma) := \{x \in \mathcal{H} : \langle a, x \rangle = \gamma\},$$

where $a \in \mathcal{H} \setminus \{0\}$ and $\gamma \in \mathbb{R}$ and the subset

$$H_{\leq}(a;\gamma) := \{x \in \mathcal{H} : \langle a, x\rangle \leq \gamma\}.$$

is called a half-space in $\mathcal{H}$. Notice that the hyperplane and half-space are closed and convex subsets in a Hilbert space $\mathcal{H}$. For other properties of the metric projection, hyperplane and half-space, we refer the reference books [19, 20].

An infinite lower triangular row matrix $[\alpha_{k,j}]_{k,j=1}^{\infty}$ is called *averaging* [21] if the following statements hold:

(I) $\alpha_{k,j} \geq 0, \quad$ for all $k, j \geq 1$,

(II) $\alpha_{k,j} = 0, \quad$ for all $k \geq 1$ and $j > k$,

(III) $\displaystyle\sum_{j=1}^{k} \alpha_{k,j} = 1, \quad$ for all $k \geq 1$,

(IV) $\displaystyle\lim_{k\to+\infty} \alpha_{k,j} = 0, \quad$ for all $j \geq 1$.

Next, we recall the notion of H-concentrating which play an important role in our convergence analysis.

Definition 2.3. [17, Definition 2.1] An averaging matrix $[\alpha_{k,j}]_{k,j=1}^{\infty}$ is concentrating if every nonnegative real sequence $\{x_k\}_{k=1}^{\infty}$ such that

$$x_{k+1} \leq \overline{x}_k + \tau_k \quad \forall k \in \mathbb{N}, \tag{2.1}$$

where $\{\tau_k\}_{k=1}^{\infty}$ is nonnegative real sequence with $\displaystyle\sum_{k=1}^{\infty} \tau_k < \infty$, converges.

Definition 2.4. [22, Definition 2.3] An averaging matrix $[\alpha_{k,j}]_{k,j=1}^{\infty}$ is *concentrating in the sense of Halpern*, (in short, H-concentrating) if whenever $\{\varphi_k\}_{k=1}^{\infty}$, $\{\eta_k\}_{k=1}^{\infty}$ are sequences of nonnegative real numbers such that $\displaystyle\sum_{k=1}^{\infty} \eta_k < \infty$, $\{\beta_k\}_{k=1}^{\infty}$ is a sequence in $[0, 1]$ with $\displaystyle\sum_{k=1}^{\infty} \beta_k = \infty$, $\{t_k\}_{k=1}^{\infty}$ is a sequence of real numbers with $\displaystyle\limsup_{k\to\infty} t_k \leq 0$, and

$$\begin{aligned} \overline{\varphi}_k &:= \sum_{j=1}^{k} \alpha_{k,j}\varphi_j \\ \varphi_{k+1} &\leq (1 - \beta_k)\overline{\varphi}_k + \beta_k t_k + \eta_k \end{aligned}$$

for all $k \in \mathbb{N}$, it follows that $\displaystyle\lim_{k\to\infty} \varphi_k = 0$.

For some interesting examples of H-concentrating matrices are discussed in [22].

Lemma 2.5. [17] Let $\{\varphi_k\}_{k=1}^{\infty}$ be a real sequence, $r \in \mathbb{R}$, and $[\alpha_{k,j}]_{k,j=1}^{\infty}$ be an averaging matrix. If $\varphi_k \to r$, then $\overline{\varphi}_k := \sum_{j=1}^{k} \alpha_{k,j}\varphi_j \to r$.

We end this section by recall an important technical lemma which play a key tool in our convergence result.

Lemma 2.6. [23] Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers such that there exists a subsequence $\{a_{n_j}\}_{j=1}^{\infty}$ of $\{a_n\}_{n=1}^{\infty}$ with $a_{n_j} < a_{n_{j+1}}$ for all $j \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{m_l\}_{l=1}^{\infty}$ of $\mathbb{N}$ such that $\lim_{l\to\infty} m_l = \infty$ and the following properties are satisfied by all (sufficiently large) number $l \in \mathbb{N}$:

$$a_{m_l} \leq a_{m_l+1} \quad \text{and} \quad a_n \leq a_{m_l+1}$$

In fact, m_l is the largest number n in the set $\{1, \cdots, l\}$ such that $a_n \leq a_{m_l+1}$ holds.

3. A Mean Subgradient Extragradient Method

In this section, we present a Halpern's mean subgradient extragradient method (in short, Halpern-MSEM) for solving the considered variational inequality problem (1.1).

Algorithm 1: Halpern-MSEM

Initialization: Select a starting point $x_1 \in \mathcal{H}$, a parameter $\tau > 0$, a sequence $\{\beta_k\}_{k=1}^{\infty} \subset [0, b] \subset [0, 1)$, and an averaging matrix $[\alpha_{k,j}]_{k,j=1}^{\infty}$.

Step1: Given a current iterate $x_k \in \mathcal{H}$, compute the mean iterate

$$\overline{x}_k := \sum_{j=1}^{k} \alpha_{k,j} x_j.$$

Compute

$$y_k := P_C(\overline{x}_k - \tau F(\overline{x}_k)),$$

Step2: Construct the half-space T_k

$$T_k := \{w \in \mathcal{H} : \langle \overline{x}_k - \tau F(\overline{x}_k) - y_k, w - y_k\rangle \leq 0\},$$

and calculate the next iterate

$$x_{k+1} := \beta_k x_1 + (1 - \beta_k) P_{T_k}(\overline{x}_k - \tau F(y_k)).$$

Update $k = k + 1$ and go to **Step1** .

Remark 3.1. If the parameter $\beta_k = 0$ for all $k \in \mathbb{N}$, then the sequence generated by Halpern-MSEM is the sequence of Mann-MSEM proposed by [24].

Throughout this paper, we assume the following conditions hold.

Assumption 3.1. Assume that

(i) The solution set of the variational inequality (1.1) is nonempty and denoted by VIP(F, C).

(ii) The operator $F : \mathcal{H} \to \mathcal{H}$ is monotone, that is,

$$\langle x - y, F(x) - F(y)\rangle \geq 0 \quad \text{for all} \quad x, y \in \mathcal{H}. \tag{3.1}$$

(iii) The operator $F : \mathcal{H} \to \mathcal{H}$ is L-Lipschitz continuous, that is,

$$\|F(x) - F(y)\| \leq L\|x - y\| \quad \text{for all} \quad x, y \in \mathcal{H}. \tag{3.2}$$

The following lemma states the important relation of the generated iterates.

Lemma 3.2. Let $\{x_k\}_{k=1}^{\infty}$ and $\{y_k\}_{k=1}^{\infty}$ be the sequence generated by Halpern-MSEM. For $k \geq 1$ and $u \in VIP(F, C)$, it holds

$$\begin{aligned}
\|x_{k+1} - u\|^2 &\leq (1-\beta_k)^2\|\overline{x}_k - u\|^2 - (1-\beta_k)^2(1-\tau L)\|\overline{x}_k - y_k\|^2 \\
&\quad -(1-\beta_k)^2(1-\tau L)\|P_{T_k}(\overline{x}_k - \tau F(y_k)) - y_k\|^2 \\
&\quad +2\beta_k\langle x_1 - u, x_{k+1} - u\rangle \\
&\leq (1-\beta_k)^2\sum_{j=1}^{k}\alpha_{k,j}\|x_j - u\|^2 - (1-\beta_k)^2(1-\tau L)\|\overline{x}_k - y_k\|^2 \\
&\quad -(1-\beta_k)^2(1-\tau L)\|P_{T_k}(\overline{x}_k - \tau F(y_k)) - y_k\|^2 \\
&\quad +2\beta_k\langle x_1 - u, x_{k+1} - u\rangle.
\end{aligned}$$

In particular, if $\tau L < 1$, then $\|P_{T_k}(\overline{x}_k - \tau F(y_k)) - u\| \leq \|\overline{x}_k - u\|$.

Proof. Let $u \in VIP(F, C)$ and denote $w_k := P_{T_k}(\overline{x}_k - \tau F(y_k))$. Let us note that

$$\begin{aligned}
\|x_{k+1} - u\|^2 &= \|(1-\beta_k)w_k + \beta_k x_1 - u\|^2 \\
&= \|(1-\beta_k)w_k + \beta_k x_1 - u - \beta_k u + \beta_k u\|^2 \\
&= \|(1-\beta_k)w_k + (1-\beta_k)u + \beta_k(x_1 - u)\|^2 \\
&= \|(1-\beta_k)(w_k - u) + \beta_k(x_1 - u)\|^2 \\
&\leq (1-\beta_k)^2\|w_k - u\|^2 + 2\beta_k\langle x_1 - u, (1-\beta_k)(w_k - u) + \beta_k(x_1 - u)\rangle \\
&= (1-\beta_k)^2\|w_k - u\|^2 + 2\beta_k\langle x_1 - u, x_{k+1} - u\rangle.
\end{aligned} \tag{3.3}$$

Now, we consider

$$\begin{aligned}
\|\overline{x}_k - \tau F(y_k) - u\|^2 &= \|\overline{x}_k - \tau F(y_k) - w_k + w_k - u\| \\
&= \|\overline{x}_k - \tau F(y_k) - w_k\|^2 + \|w_k - u\|^2 \\
&\quad +2\langle \overline{x}_k - \tau F(y_k) - w_k, w_k - u\rangle.
\end{aligned}$$

Since $u \in C \subset T_k$, invoking the variational characterization of the metric projection, we have $\langle \overline{x}_k - \tau F(y_k) - w_k, w_k - u\rangle \geq 0$, which implies that

$$\|\overline{x}_k - \tau F(y_k) - u\|^2 \geq \|\overline{x}_k - \tau F(y_k) - w_k\|^2 + \|w_k - u\|^2. \tag{3.4}$$

By using the monotonicity of F and the fact that $u \in VIP(F, C)$, it follows from (3.4) that

$$\begin{aligned}
\|w_k - u\|^2 &\leq \|\overline{x}_k - \tau F(y_k) - u\|^2 - \|\overline{x}_k - \tau F(y_k) - w_k\|^2 \\
&= \|\overline{x}_k - u\|^2 - 2\langle \overline{x}_k - u, \tau F(y_k)\rangle + \|\tau F(y_k)\|^2 \\
&\quad -(\|\overline{x}_k - w_k\|^2 - 2\langle \overline{x}_k - w_k, \tau F(y_k)\rangle + \|\tau F(y_k)\|^2) \\
&= \|\overline{x}_k - u\|^2 - \|\overline{x}_k - w_k\|^2 - 2\langle \overline{x}_k - u, \tau F(y_k)\rangle + 2\langle \overline{x}_k - w_k, \tau F(y_k)\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \|\overline{x}_k - u\|^2 - \|\overline{x}_k - w_k\|^2 + 2\langle u - w_k, \tau F(y_k)\rangle \\
&= \|\overline{x}_k - u\|^2 - \|\overline{x}_k - w_k\|^2 + 2\langle u - y_k, \tau F(y_k)\rangle + 2\langle y_k - w_k, \tau F(y_k)\rangle \\
&= \|\overline{x}_k - u\|^2 - \|\overline{x}_k - w_k\|^2 + 2\langle u - y_k, \tau F(y_k) - \tau F(u)\rangle \\
&\quad +2\langle u - y_k, \tau F(u)\rangle + 2\langle y_k - w_k, \tau F(y_k)\rangle \\
&\leq \|\overline{x}_k - u\|^2 - \|\overline{x}_k - w_k\|^2 + 2\langle y_k - w_k, \tau F(y_k)\rangle.
\end{aligned}
$$

By using the fact that $\|\overline{x}_k - w_k\|^2 = \|\overline{x}_k - y_k\|^2 + 2\langle \overline{x}_k - y_k, y_k - w_k\rangle + \|y_k - w_k\|^2$, we obtain

$$
\begin{aligned}
\|w_k - u\|^2 &\leq \|\overline{x}_k - u\|^2 - \|\overline{x}_k - y_k\|^2 - 2\langle \overline{x}_k - y_k, y_k - w_k\rangle - \|y_k - w_k\|^2 \\
&\quad +2\langle y_k - w_k, \tau F(y_k)\rangle \\
&= \|\overline{x}_k - u\|^2 - \|\overline{x}_k - y_k\|^2 - \|y_k - w_k\|^2 + 2\langle \overline{x}_k - \tau F(y_k) - y_k, w_k - y_k\rangle.
\end{aligned} \tag{3.5}
$$

Now, we consider the last term of the inequality (3.5). By using the Cauchy–Schwarz inequality and the L-Lipschitz continuous of the operator F, we have

$$
\begin{aligned}
\langle \overline{x}_k - \tau F(y_k) - y_k, w_k - y_k\rangle &= \langle \overline{x}_k - \tau F(\overline{x}_k) - y_k, w_k - y_k\rangle \\
&\quad +\langle \tau F(\overline{x}_k) - \tau F(y_k), w_k - y_k\rangle \\
&\leq \langle \tau F(\overline{x}_k) - \tau F(y_k), w_k - y_k\rangle \\
&\leq \tau\|F(\overline{x}_k) - \tau F(y_k)\|\|w_k - y_k\| \\
&\leq \tau L\|\overline{x}_k - y_k\|\|w_k - y_k\|,
\end{aligned} \tag{3.6}
$$

where the first inequality using the fact that $w_k \in T_k$.
It follows from the inequalities (3.5) and (3.6) that

$$
\begin{aligned}
\|w_k - u\|^2 &\leq \|\overline{x}_k - u\|^2 - \|\overline{x}_k - y_k\|^2 - \|y_k - w_k\|^2 + 2\tau L\|\overline{x}_k - y_k\|\|w_k - y_k\| \\
&= \|\overline{x}_k - u\|^2 - \|\overline{x}_k - y_k\|^2 - \|y_k - w_k\|^2 + 2\tau L\|\overline{x}_k - y_k\|\|w_k - y_k\| \\
&\quad +\tau L\|\overline{x}_k - y_k\|^2 - \tau L\|\overline{x}_k - y_k\|^2 + \tau L\|w_k - y_k\|^2 - \tau L\|w_k - y_k\|^2 \\
&= \|\overline{x}_k - u\|^2 - (1 - \tau L)\|\overline{x}_k - y_k\|^2 - (1 - \tau L)\|w_k - y_k\|^2 \\
&\quad -\tau L(\|\overline{x}_k - y_k\| - \|y_k - w_k\|)^2 \\
&\leq \|\overline{x}_k - u\|^2 - (1 - \tau L)\|\overline{x}_k - y_k\|^2 - (1 - \tau L)\|w_k - y_k\|^2,
\end{aligned} \tag{3.7}
$$

where the last inequality using the fact that $\tau L(\|\overline{x}_k - y_k\| - \|y_k - w_k\|)^2 \geq 0$.
Finally, combining the inequalities (3.3) and (3.7), we obtain

$$
\begin{aligned}
\|x_{k+1} - u\|^2 &\leq (1 - \beta_k)^2\|w_k - u\|^2 + 2\beta_k\langle x_1 - u, x_{k+1} - u\rangle \\
&\leq (1 - \beta_k)^2(\|\overline{x}_k - u\|^2 - (1 - \tau L)\|\overline{x}_k - y_k\|^2 - (1 - \tau L)\|w_k - y_k\|^2) \\
&\quad +2\beta_k\langle x_1 - u, x_{k+1} - u\rangle \\
&= (1 - \beta_k)^2\|\overline{x}_k - u\|^2 - (1 - \beta_k)^2(1 - \tau L)\|\overline{x}_k - y_k\|^2 \\
&\quad -(1 - \beta_k)^2(1 - \tau L)\|\overline{w}_k - y_k\|^2 + 2\beta_k\langle x_1 - u, x_{k+1} - u\rangle \\
&= (1 - \beta_k)^2\left\|\sum_{j=1}^{k}\alpha_{k,j}x_j - \sum_{j=1}^{k}\alpha_{k,j}u\right\|^2 - (1 - \beta_k)^2(1 - \tau L)\|\overline{x}_k - y_k\|^2 \\
&\quad -(1 - \beta_k)^2(1 - \tau L)\|w_k - y_k\|^2 + 2\beta_k\langle x_1 - u, x_{k+1} - u\rangle
\end{aligned}
$$

$$\begin{aligned} &= (1-\beta_k)^2 \left\| \sum_{j=1}^{k} \alpha_{k,j}(x_j - u) \right\|^2 - (1-\beta_k)^2(1-\tau L)\|\overline{x}_k - y_k\|^2 \\ &\quad -(1-\beta_k)^2(1-\tau L)\|w_k - y_k\|^2 + 2\beta_k\langle x_1 - u, x_{k+1} - u\rangle \\ &\le (1-\beta_k)^2 \sum_{j=1}^{k} \alpha_{k,j}\|x_j - u\|^2 - (1-\beta_k)^2(1-\tau L)\|\overline{x}_k - y_k\|^2 \\ &\quad -(1-\beta_k)^2(1-\tau L)\|w_k - y_k\|^2 + 2\beta_k\langle x_1 - u, x_{k+1} - u\rangle. \end{aligned}$$

In particular, by using the fact that $\tau L < 1$, we obtain from the inequality (3.7) that

$$\|P_{T_k}(\overline{x}_k - \tau F(y_k)) - u\|^2 \le \|\overline{x}_k - u\|^2.$$

This completes the proof. ∎

The following lemma guarantees the boundedness of the constructed sequence $\{x^n\}_{n=1}^{\infty}$.

Lemma 3.3. Let $\{x_k\}_{k=1}^{\infty}$ be the sequence generated by Halpern-MSEM and τ be a positive real number such that $\tau L < 1$. Then, we have

$$\|x_{k+1} - u\| \le \beta_k\|x_1 - u\| + (1-\beta_k)\|\overline{x}_k - u\|,$$

for all $u \in VIP(F, C)$. Moreover, $\{x_k\}_{k=1}^{\infty}$ is bounded.

Proof. Let $u \in VIP(F, C)$. We denote $w_k := P_{T_k}(\overline{x}_k - \tau F(y_k))$, which yields $x_{k+1} = \beta_k x_1 + (1-\beta_k)w_k$. We note from Lemma 3.2 that

$$\|w_k - u\| \le \|\overline{x}_k - u\|.$$

So, we have

$$\begin{aligned} \|x_{k+1} - u\| &= \|\beta_k x_1 + (1-\beta_k)w_k - u\| \\ &= \|\beta_k(x_1 - u) + (1-\beta_k)(w_k - u)\| \\ &\le \beta_k\|x_1 - u\| + (1-\beta_k)\|w_k - u\| \\ &\le \beta_k\|x_1 - u\| + (1-\beta_k)\|\overline{x}_k - u\|. \end{aligned}$$

Moreover, we have

$$\|x_{k+1} - u\| \le \max\{\|x_1 - u\|, \|\overline{x}_k - u\|\}.$$

Next, we use strong induction to prove that for all integer $k \ge 1$,

$$\|x_k - u\| \le \|x_1 - u\|.$$

Suppose that $P(n)$ is the statement

$$\|x_{n+1} - u\| \le \|x_1 - u\|.$$

We verify that $P(1)$ is true that

$$\|x_2 - u\| \le \max\{\|x_1 - u\|, \|\overline{x}_1 - u\|\}$$

$$= \|x_1 - u\|.$$

So, the claim is true when $n = 1$.
Now, let $n \in \mathbb{N}$ and assume that $P(i)$ is true for all integers i such that $1 \leq i \leq n$.
Now, we consider

$$\begin{aligned}
\|\overline{x}_n - u\| &= \|\sum_{j=1}^{n} \alpha_{n,j} x_j - \sum_{j=1}^{n} \alpha_{n,j} u\| \\
&= \|\sum_{j=1}^{n} \alpha_{n,j}(x_j - u)\| \\
&\leq \sum_{j=1}^{n} \alpha_{n,j} \|x_j - u\| \\
&\leq \sum_{j=1}^{n} \alpha_{n,j} \|x_1 - u\| \\
&= \|x_1 - u\|.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\|x_{n+1} - u\| &\leq \max\{\|x_1 - u\|, \|\overline{x}_n - u\|\} \\
&= \|x_1 - u\|.
\end{aligned}$$

It follows that for all integer $k \geq 1$,

$$\|x_k - u\| \leq \|x_1 - u\|.$$

Hence, we conclude that the sequence $\{x_k\}_{k=1}^{\infty}$ is bounded. ∎

Now, we are in a position to present our main theorem.

Theorem 3.4. Let the averaging matrix $[\alpha_{k,j}]_{k,j=1}^{\infty}$ be H-concentrating, $\tau \in (0, 1/L)$ and let $\{x_k\}_{k=1}^{\infty}$ be a sequence of Halpern-MSEM. Suppose that the sequence $\{\beta_k\}_{k=1}^{\infty}$ satisfied $\sum_{k=1}^{\infty} \beta_k = \infty$ and $\lim_{k\to\infty} \beta_k = 0$. Then $x_k \to P_{VIP(F,C)}(x_1)$.

Proof. Put $w_k = P_{T_k}(\overline{x}_k - \tau F(y_k))$ and $u = P_{VIP(F,C)}(x_1)$.
Since $\tau \in (0, 1/L)$ and $\{\beta_k\}_{k=1}^{\infty} \subset [0, b] \subset [0, 1)$ it follows from Lemma 3.2 that

$$\begin{aligned}
\|x_{k+1} - u\|^2 &\leq (1 - \beta_k)^2 \|\overline{x}_k - u\|^2 - (1 - \beta_k)^2 (1 - \tau L) \|\overline{x}_k - y_k\|^2 \\
&\quad +2\beta_k \langle x_1 - u, x_{k+1} - u\rangle \\
&\leq (1 - \beta_k) \sum_{j=1}^{k} \alpha_{k,j} \|x_j - u\|^2 - (1 - b)^2 (1 - \tau L) \|\overline{x}_k - y_k\|^2 \\
&\quad +\beta_k \langle x_1 - u, x_{k+1} - u\rangle. \qquad (3.8)
\end{aligned}$$

This implies that

$$\|x_{k+1} - u\|^2 \leq (1 - \beta_k) \|\overline{x}_k - u\|^2 + 2\beta_k \langle x_1 - u, x_{k+1} - u\rangle \qquad (3.9)$$

Let us consider the following two cases.

Case1: There exists a $k_1 \in \mathbb{N}$ such that $\|x_{k+1} - u\|^2 \leq \|x_k - u\|^2$ for all integer $k \geq k_1$. Then $\lim_{k\to\infty} \|x_k - u\|^2$ exists, say $r(u)$.

Thus, we obtain from Lemma 2.5 that $\lim_{k\to\infty} \sum_{j=1}^{k} \alpha_{k,j}\|x_j - u\|^2 = r(u)$.
By using the boundedness of $\{x_k\}_{k=1}^{\infty}$ and $\lim_{k\to\infty} \beta_k = 0$, we have from the inequality (3.8),

$$\lim_{k\to\infty} \|\overline{x}_k - y_k\| = 0.$$

Now, let $\{x_{k_r}\}_{r=1}^{\infty}$ be a sequence of $\{x_k\}_{k=1}^{\infty}$ such that

$$\limsup_{k\to\infty} \langle x_1 - u, x_{k+1} - u\rangle = \lim_{r\to\infty} \langle x_1 - u, x_{k_r} - u\rangle.$$

Note that for all $r \in \mathbb{N}$,

$$\langle \overline{x}_{k_r} - F(\overline{x}_{k_r}) - y_{k_r}, y_{k_r} - u\rangle \geq 0.$$

Next, by using the Cauchy–Schwarz inequality, we have

$$\begin{aligned} \langle \tau F(\overline{x}_{k_r}), \overline{x}_{k_r} - u\rangle &= \langle \tau F(\overline{x}_{k_r}), \overline{x}_{k_r} - y_{k_r}\rangle + \langle \tau F(\overline{x}_{k_r}), y_{k_r} - u\rangle \\ &= \langle \tau F(\overline{x}_{k_r}), \overline{x}_{k_r} - y_{k_r}\rangle + \langle \overline{x}_{k_r} - y_{k_r}, y_{k_r} - u\rangle \\ &\quad -\langle \overline{x}_{k_r} - F(\overline{x}_{k_r}) - y_{k_r}, y_{k_r} - u\rangle \\ &\leq \langle \tau F(\overline{x}_{k_r}), \overline{x}_{k_r} - y_{k_r}\rangle + \langle \overline{x}_{k_r} - y_{k_r}, y_{k_r} - u\rangle \\ &\leq \tau\|F(\overline{x}_{k_r})\|\|\overline{x}_{k_r} - y_{k_r}\| + \|\overline{x}_{k_r} - y_{k_r}\|\|y_{k_r} - u\|. \end{aligned} \tag{3.10}$$

By using definition of y_{k_r} and nonexpansivity of the metric projection, we get

$$\begin{aligned} \|y_{k_r} - u\| &= \|P_C(\overline{x}_{k_r} - \tau F(\overline{x}_{k_r})) - u\| \\ &\leq \|\overline{x}_{k_r} - \tau F(\overline{x}_{k_r}) - u\| \\ &\leq \|\overline{x}_{k_r} - u\| + \tau\|F(\overline{x}_{k_r})\|. \end{aligned}$$

Combining this and the inequality (3.10), we obtain

$$\begin{aligned} \langle \tau F(\overline{x}_{k_r}), \overline{x}_{k_r} - u\rangle &\leq \tau\|F(\overline{x}_{k_r})\|\|\overline{x}_{k_r} - y_{k_r}\| + \|\overline{x}_{k_r} - y_{k_r}\|\|\overline{x}_{k_r} - u\| \\ &\quad +\tau\|\overline{x}_{k_r} - y_{k_r}\|\|F(\overline{x}_{k_r})\|. \end{aligned} \tag{3.11}$$

Note that the boundedness of $\{x_k\}_{k=1}^{\infty}$ implies the boundedness of $\{F(\overline{x}_{k_r})\}_{r=1}^{\infty}$.
It follows from inequality (3.11) and the fact $\lim_{k\to\infty} \|\overline{x}_k - y_k\| = 0$ that

$$\limsup_{r\to\infty} \langle \tau F(\overline{x}_{k_r}), \overline{x}_{k_r} - u\rangle \leq 0.$$

Now, let $\{\overline{x}_{k_{r_i}}\}_{i=1}^{\infty}$ be a subsequence of $\{\overline{x}_{k_r}\}_{r=1}^{\infty}$ such that

$$\limsup_{r\to\infty} \langle \tau F(\overline{x}_{k_r}), \overline{x}_{k_r} - u\rangle = \lim_{i\to\infty} \langle \tau F(\overline{x}_{k_{r_i}}), \overline{x}_{k_{r_i}} - u\rangle.$$

Since $\{\overline{x}_{k_{r_i}}\}_{i=1}^{\infty}$ is bounded, there exists a point $z \in \mathcal{H}$ and a subsequence $\{\overline{x}_{k_{r_{i_p}}}\}_{p=1}^{\infty}$ of $\{\overline{x}_{k_{r_i}}\}_{i=1}^{\infty}$ such that $\overline{x}_{k_{r_{i_p}}} \rightharpoonup z \in \mathcal{H}$. By using monotonicity of F that for all $k \in \mathbb{N}$,

$$\tau\langle F(\overline{x}_k) - F(u), \overline{x}_k - u\rangle \geq 0,$$

we have

$$\langle \tau F(\overline{x}_k), \overline{x}_k - u\rangle \geq \langle \tau F(u), \overline{x}_k - u\rangle.$$

Note that

$$\begin{aligned}
\langle F(u), z-u\rangle &= \frac{1}{\tau}\lim_{p\to\infty}\langle \tau F(u), \overline{x}_{k_{r_{i_p}}} - u\rangle \\
&= \frac{1}{\tau}\lim_{i\to\infty}\langle \tau F(u), \overline{x}_{k_{r_i}} - u\rangle \\
&\leq \frac{1}{\tau}\lim_{i\to\infty}\langle \tau F(\overline{x}_{k_{r_i}}), \overline{x}_{k_{r_i}} - u\rangle \\
&= \frac{1}{\tau}\limsup_{r\to\infty}\langle \tau F(\overline{x}_{k_r}), \overline{x}_{k_r} - u\rangle \\
&\leq 0. \qquad (3.12)
\end{aligned}$$

Since $u \in VIP(F, C)$, we conclude that $z \in VIP(F, C)$.
Next, we consider

$$\begin{aligned}
\langle x_1 - u, \overline{x}_{k_{r_{i_p}}} - u\rangle &= \langle x_1 - u, \sum_{j=1}^{k_{r_{i_p}}} \alpha_{k_{r_{i_p}},j} x_j - u\rangle \\
&= \langle x_1 - u, \sum_{j=1}^{k_{r_{i_p}}} \alpha_{k_{r_{i_p}},j} x_j - \sum_{j=1}^{k_{r_{i_p}}} \alpha_{k_{r_{i_p}},j} u\rangle \\
&= \sum_{j=1}^{k_{r_{i_p}}} \alpha_{k_{r_{i_p}},j}\langle x_1 - u, x_j - u\rangle. \qquad (3.13)
\end{aligned}$$

It follows form the inequality (3.13) that

$$\begin{aligned}
\limsup_{k\to\infty}\langle x_1 - u, x_{k+1} - u\rangle &= \lim_{r\to\infty}\langle x_1 - u, x_{k_r} - u\rangle \\
&= \lim_{i\to\infty}\langle x_1 - u, x_{k_{r_i}} - u\rangle \\
&= \lim_{p\to\infty}\langle x_1 - u, x_{k_{r_{i_p}}} - u\rangle \\
&= \lim_{p\to\infty}\sum_{j=1}^{k_{r_{i_p}}} \alpha_{k_{r_{i_p}},j}\langle x_1 - u, x_j - u\rangle \\
&= \lim_{p\to\infty}\langle x_1 - u, \overline{x}_{k_{r_{i_p}}} - u\rangle \\
&= \langle x_1 - u, z - u\rangle \\
&\leq 0,
\end{aligned}$$

where the last inequality using the fact that $z \in VIP(F, C)$ and $u = P_{VIP(F,C)}(x_1)$. From the inequality (3.9) and using the definition of concentrating matrices in the sense of Halpern's, we obtain $\lim_{k\to\infty}\|x_k - u\|^2 = 0$. Hence, we can conclude that $x_k \to u$.

Case2: Suppose that there exists a subsequence $\{x_{m_j}\}_{j=1}^{\infty}$ of $\{x_k\}_{k=1}^{\infty}$ such that for all $j \in \mathbb{N}$

$$\|x_{m_j} - u\| < \|x_{m_j+1} - u\|.$$

From Lemma 2.6, there exists a nondecreasing sequence $\{k_p\}_{k=1}^{\infty}$ of $\mathbb{N}$ such that $\lim_{p\to\infty} k_p = \infty$ and the following inequalities hold for all $p \in \mathbb{N}$:

$$\|x_{k_p} - u\| \leq \|x_{k_p+1} - u\| \quad \text{and} \quad \|x_p - u\| \leq \|x_{k_p+1} - u\|. \tag{3.14}$$

From the inequality (3.8), we obtain

$$\begin{aligned}
\|x_{k_p} - u\|^2 &\leq \|x_{k_p+1} - u\|^2 \\
&\leq (1-\beta_{k_p})\|\overline{x}_{k_p} - u\|^2 - (1-\beta_{k_p})^2(1-\tau L)\|\overline{x}_{k_p} - y_{k_p}\|^2 \\
&\quad +2\beta_{k_p}\langle x_1 - u, x_{k_p+1} - u\rangle \\
&\leq (1-\beta_{k_p})\sum_{j=1}^{k_p}\alpha_{k_p,j}\|x_j - u\|^2 - (1-b)^2(1-\tau L)\|\overline{x}_{k_p} - y_{k_p}\|^2 \\
&\quad +\beta_k\langle x_1 - u, x_{k_p+1} - u\rangle.
\end{aligned} \tag{3.15}$$

It follows from the inequality (3.9) that

$$\|x_{k_p+1} - u\|^2 \leq (1-\beta_k)\|\overline{x}_{k_p} - u\|^2 + 2\beta_{k_p}\langle x_1 - u, x_{k_p+1} - u\rangle. \tag{3.16}$$

By using the boundedness of $\{x_k\}_{k=1}^{\infty}$ and inequality (3.14), we have $\lim_{p\to\infty}\|x_{k_p} - u\|$ exists, say $z(u)$.

Thus, we obtain from Lemma 2.5 that $\lim_{p\to\infty}\sum_{j=1}^{k_p}\alpha_{k_p,j}\|x_j - u\| = z(u)$.

By using the boundedness of $\{x_k\}_{k=1}^{\infty}$ again and $\lim_{k\to\infty}\beta_k = 0$, we have from the inequality (3.15) that

$$\lim_{p\to\infty}\|\overline{x}_{k_p} - y_{k_p}\| = 0.$$

Now, let $\{x_{k_{p_i}}\}_{i=1}^{\infty}$ be a subsequence of $\{x_{k_p}\}_{p=1}^{\infty}$ such that

$$\limsup_{p\to\infty}\langle x_1 - u, x_{k_p+1} - u\rangle = \lim_{i\to\infty}\langle x_1 - u, x_{k_{p_i}} - u\rangle.$$

Note that for all $i \in \mathbb{N}$,

$$\langle\overline{x}_{k_{p_i}} - F(\overline{x}_{k_{p_i}}) - y_{k_{p_i}}, y_{k_{p_i}} - u\rangle \leq 0.$$

Next, using definition of $y_{k_{p_i}}$ and the nonexpansivity of the metric projection, we have

$$\begin{aligned}
\langle\tau F(\overline{x}_{k_{p_i}}), \overline{x}_{k_{p_i}} - u\rangle &= \langle\tau F(\overline{x}_{k_{p_i}}), \overline{x}_{k_{p_i}} - y_{k_{p_i}}\rangle + \langle\tau F(\overline{x}_{k_{p_i}}), y_{k_{p_i}} - u\rangle \\
&= \langle\tau F(\overline{x}_{k_{p_i}}), \overline{x}_{k_{p_i}} - y_{k_{p_i}}\rangle + \langle\overline{x}_{k_{p_i}} - y_{k_{p_i}}, y_{k_{p_i}} - u\rangle \\
&\quad -\langle\overline{x}_{k_{p_i}} - F(\overline{x}_{k_{p_i}}) - y_{k_{p_i}}, y_{k_{p_i}} - u\rangle \\
&\leq \langle\tau F(\overline{x}_{k_{p_i}}), \overline{x}_{k_{p_i}}q - y_{k_{p_i}}\rangle + \langle\overline{x}_{k_{p_i}} - y_{k_{p_i}}, y_{k_{p_i}} - u\rangle \\
&\leq \tau\|F(\overline{x}_{k_{p_i}})\|\|\overline{x}_{k_{p_i}} - y_{k_{p_i}}\| + \|\overline{x}_{k_{p_i}} - y_{k_{p_i}}\|\|y_{k_{p_i}} - u\| \\
&= \tau\|F(\overline{x}_{k_{p_i}})\|\|\overline{x}_{k_{p_i}} - y_{k_{p_i}}\| + \|\overline{x}_{k_{p_i}} - y_{k_{p_i}}\|\|P_C(\overline{x}_{k_{p_i}} - \tau F(\overline{x}_{k_{p_i}}) - u\| \\
&\leq \tau\|F(\overline{x}_{k_{p_i}})\|\|\overline{x}_{k_{p_i}} - y_{k_{p_i}}\| + \|\overline{x}_{k_{p_i}} - y_{k_{p_i}}\|\|\overline{x}_{k_{p_i}} - \tau F(\overline{x}_{k_{p_i}}) - u\| \\
&\leq \tau\|F(\overline{x}_{k_{p_i}})\|\|\overline{x}_{k_{p_i}} - y_{k_{p_i}}\| + \|\overline{x}_{k_{p_i}} - y_{k_{p_i}}\|\|\overline{x}_{k_{p_i}} - u\|
\end{aligned}$$

$$+\tau\|\overline{x}_{k_{p_i}} - y_{k_{p_i}}\|\|F(\overline{x}_{k_{p_i}})\|.$$

Combining this inequality and using the boundedness of $\{x_k\}_{k=1}^{\infty}$ and the fact that $\lim_{p\to\infty}\|\overline{x}_{k_p} - y_{k_p}\| = 0$, we obtain

$$\limsup_{i\to\infty}\langle \tau F(\overline{x}_{k_{p_i}}), \overline{x}_{k_{p_i}} - u\rangle \leq 0.$$

Now, let $\{\overline{x}_{k_{p_{i_r}}}\}_{r=1}^{\infty}$ be a subsequence of $\{\overline{x}_{k_{p_i}}\}_{i=1}^{\infty}$ such that

$$\limsup_{i\to\infty}\langle \tau F(\overline{x}_{k_{p_i}}), \overline{x}_{k_{p_i}} - u\rangle = \lim_{r\to\infty}\langle \tau F(\overline{x}_{k_{p_{i_r}}}), \overline{x}_{k_{p_{i_r}}} - u\rangle.$$

Since $\{\overline{x}_{k_{p_{i_r}}}\}_{r=1}^{\infty}$ is bounded, there exists a point $v \in \mathcal{H}$ and a subsequence $\{\overline{x}_{k_{p_{i_r}}}\}_{r=1}^{\infty}$ of $\{\overline{x}_{k_{p_{i_{r_t}}}}\}_{t=1}^{\infty}$ such that $\overline{x}_{k_{p_{i_{r_t}}}} \rightharpoonup v \in \mathcal{H}$.
By using the monotonicity of F, for all integer $k \geq 1$, we have

$$\tau\langle F(\overline{x}_k) - F(u), \overline{x}_k - u\rangle \geq 0$$

or equivalent

$$\langle \tau F(\overline{x}_k), \overline{x}_k - u\rangle \geq \langle \tau F(u), \overline{x}_k - u\rangle.$$

Thus

$$\begin{aligned}
\langle F(u), v - u\rangle &= \frac{1}{\tau}\lim_{t\to\infty}\langle \tau F(u), \overline{x}_{k_{p_{i_{r_t}}}} - u\rangle \\
&= \frac{1}{\tau}\lim_{r\to\infty}\langle \tau F(u), \overline{x}_{k_{p_{i_r}}} - u\rangle \\
&\leq \frac{1}{\tau}\lim_{r\to\infty}\langle \tau F(\overline{x}_{k_{p_{i_r}}}), \overline{x}_{k_{p_{i_r}}} - u\rangle \\
&= \frac{1}{\tau}\limsup_{r\to\infty}\langle \tau F(\overline{x}_{k_{p_i}}), \overline{x}_{k_{p_{i_r}}} - u\rangle \\
&\leq 0. \qquad (3.17)
\end{aligned}$$

This implies that $v \in VIP(F, C)$.
Next, we consider

$$\begin{aligned}
\langle x_1 - u, \overline{x}_{k_{p_{i_{r_t}}}} - u\rangle &= \langle x_1 - u, \sum_{j=1}^{k_{p_{i_{r_t}}}}\alpha_{k_{p_{i_{r_t}}},j}x_j - u\rangle \\
&= \langle x_1 - u, \sum_{j=1}^{k_{p_{i_{r_t}}}}\alpha_{k_{p_{i_{r_t}}},j}x_j - \sum_{j=1}^{k_{p_{i_{r_t}}}}\alpha_{k_{p_{i_{r_t}}},j}u\rangle \\
&= \sum_{j=1}^{k_{p_{i_{r_t}}}}\alpha_{k_{p_{i_{r_t}}},j}\langle x_1 - u, x_j - u\rangle. \qquad (3.18)
\end{aligned}$$

It follows form (3.17) and (3.18) that

$$\limsup_{p\to\infty}\langle x_1 - u, x_{k_p+1} - u\rangle = \lim_{i\to\infty}\langle x_1 - u, x_{k_{p_i}} - u\rangle$$

$$
\begin{aligned}
&= \lim_{r\to\infty} \langle x_1 - u, x_{k_{p_{i_r}}} - u\rangle \\
&= \lim_{t\to\infty} \langle x_1 - u, x_{k_{p_{i_{r_t}}}} - u\rangle \\
&= \lim_{t\to\infty} \sum_{j=1}^{k_{p_{i_{r_t}}}} \alpha_{k_{p_{i_{r_t}}},j} \langle x_1 - u, x_j - u\rangle \\
&= \lim_{t\to\infty} \langle x_1 - u, \overline{x}_{k_{p_{i_{r_t}}}} - u\rangle \\
&= \langle x_1 - u, v - u\rangle \\
&\leq 0,
\end{aligned}
$$

where the last inequality using the fact that $v \in VIP(F, C)$ and $u = P_{VIP(F,C)}(x_1)$.
From the inequality (3.16) and using the definition of concentrating matrices in the sense of Halpern's, we obtain $\lim\limits_{p\to\infty} \|x_{k_p} - u\|^2 = 0$.
Finally, using the inequality (3.14), we obtain

$$\lim_{p\to\infty} \|x_p - u\| \leq \lim_{p\to\infty} \|x_{k_p+1} - u\| = 0.$$

Hence, we conclude that $x_p \to u$.
This completes the proof. ∎

4. Conclusions

The objective of this work was the solving of a variational inequality problem of a monotone and Lipschitz continuous operator in Hilbert spaces. We presented an iterative method so-called a Halpern's mean subgradient extragradient method. Moreover, we proved strong convergence of the generated sequence of iterates to a solution of the considered problem.

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