Nonlinear Convex Analysis and Optimization
Vol. 1, No. 2, 2022, pp. 161–175
https://ncao.design.blog

Generalizations of the Tarski Type Fixed Point Theorems

Sehie Park[a,b,1,*]

[a] The National Academy of Sciences, Republic of Korea, Seoul 06579;
[b] Department of Mathematical Sciences, Seoul National University, Seoul 08826, Republic of Korea
[1] park35@snu.ac.kr; sehiepark@gmail.com
[*] Corresponding Author

ABSTRACT

In the present article, by applying our new 2023 Metatheorem and the Brøndsted-Jachymski Principle, we recall and improve fixed point theorems related to Alfred Tarski (1901-1983). First, we obtain various forms of generalizations of the Knaster-Tarski fixed point theorem. Actually, their nature is that, for a chain P with an upper bound $v \in P$ in a partially ordered set $(X, \preccurlyeq)$, a progressive map $f : P \to P$ (that is, $x \preccurlyeq f(x)$ for all $x \in P$) has a maximal fixed element $v \in P$. Further, we obtain several improved versions of the Tarski-Kantorovitch type fixed point theorems.

Article History
Received 30 Aug 2022
Accepted 05 Dec 2022
Keywords:
Partially ordered set; preorder; metric space; fixed point; stationary point; maximal element
MSC
03E04; 03E25; 06A06; 06A75; 47H10; 54E35; 54H25; 58E30; 65K10

1. Introduction

There are several fields in the fixed point theory. *Analytical* fixed point theory is originated from Brouwer in 1912 and concerns mainly with topological vector spaces. *Metric* fixed point theory is originated from Banach in 1922 and deals with generalizations of contractions and nonexpansive maps. *Topological* fixed point theory relates mainly originated works of Lefschetz, Nielsen, and Reidemeister.

The *Ordered* fixed point theory began by Zermelo [1] (1908) implicitly and developed mainly by Knaster [2] (1928), Zorn [3] (1935), Bourbaki [4] (1949-50), Tarski [5],[6] (1949, 1955), Ekeland [7],[8] (1973, 1974), Caristi [9] (1976), Brézis-Browder [10] (1976), Takahashi [11] (1991), and many others. Moreover, in 1985-86 [12],[13], we discovered a Metatheorem stating that any maximum elements in ordered sets can be fixed points, stationary points, collectively fixed points, collectively stationary points, and conversely. Consequently, Ordered fixed point theory is a rich source of information on fixed points of families of multimaps on ordered sets.

Recently in 2022 [14],[15],[16],[17],[18], we obtained an extended form of Metatheorem and applied it to a large number of known or new results. Moreover in 2022 [19], we found the Brøndsted-Jachymski Principle on partially ordered sets showing the equalities of maximal element sets, fixed point sets and periodic point sets of progressive selfmaps. Recently we obtained the new 2023 Metatheorem in [20] which expands the old versions in 1984-85 and

Please cite this article as: S. Park, Generalizations of the Tarski Type Fixed Point Theorems, Nonlinear Convex Anal. & Optim., Vol. 1 No. 2, 161–175.

2022. See [21]. For the recent related research, see [20], [22], [23], [24].

In the present article, by applying the 2023 Metatheorem and its particular Brøndsted-Jachymski Principle, we recall and improve fixed point theorems related to Tarski. First, we obtain various forms of generalizations of the Knaster-Tarski fixed point theorem. Actually, their nature is that, for a chain P with an upper bound $v \in P$ in a partially ordered set $(X, \preccurlyeq)$, a progressive map $f : P \to P$ (that is, $x \preccurlyeq f(x)$ for all $x \in P$) has a maximal fixed element $v \in P$. Further, we obtain several improved versions of the Tarski-Kantorovitch type fixed point theorems.

This article is organized as follows: Section 2 is based on Jinlu Li [25] and we recall some properties of partially ordered sets (posets) and lattices. In Section 3, we recall our new 2023 Metatheorem and introduce its two useful consequences, that is, the Brøndsted Principle and the Brøndsted-Jachymski Principle. Section 4 devotes to show certain consequences of our Metatheorem and their previously known applications. In Section 5, we collect a large number of the Tarski type fixed point theorems and show that they can be improved by our Metatheorem. Section 6 devotes to improve a recent work of Espínola-Wiśnicki [26] by applying our method. In Section 7, we obtain far reaching generalizations of the well-known Tarski-Kantorovitch fixed point theorem and their applications to some known previous works. Finally, Section 8 is for some conclusion and epilogue.

2. Preliminaries

A preordered set is the one having reflexivity and transivity; and a partially ordered set (poset) is the one having additional anti-symmetry.

For the reader's convenience, in this section, based on Jinlu Li [25], we recall some properties of partially ordered sets and lattices.

We say that a poset $(P, \preccurlyeq)$ is chain-complete iff every chain in P has an upper bound in P. The poset $(P, \preccurlyeq)$ is called a complete lattice iff both $\vee S$ and $\wedge S$ exist in P, for any nonempty subset S of P.

For a chain-complete lattice $(P, \preccurlyeq)$, $\vee S$ exists in P for every non-empty subset S of $(P, \preccurlyeq)$.

Given posets $(X, \preccurlyeq^X)$ and $(U, \preccurlyeq^U)$, we say that a map $F : X \multimap U$ is order-increasing upward if $x \preccurlyeq^X y$ in X and $z \in F(x)$, then there is $w \in F(y)$ such that $z \preccurlyeq^U w$. F is order-increasing downward if $x \preccurlyeq^X y$ in X and $w \in F(y)$, then there is $z \in F(x)$ such that $z \preccurlyeq^U w$. F is said to be order-increasing whenever F is both (order) increasing upward and downward.

As a special case, a single-valued map f from a poset $(X, \preccurlyeq^X)$ to another poset $(U, \preccurlyeq^U)$ is said to be order-increasing (or order-preserving) whenever $x \preccurlyeq^X y$ implies $f(x) \preccurlyeq^U f(y)$. An order-increasing map $f : X \to U$ is said to be strictly order-increasing whenever $x \prec^X y$ implies $f(x) \prec^U f(y)$. A map f from $(X, \preccurlyeq^X)$ to $(U, \preccurlyeq^U)$ is said to be order-decreasing (or order-reversing) whenever $x \preccurlyeq^X y$ implies $f(x) \succcurlyeq^U f(y)$.

In this article, every multimap is non-empty valued.

3. Our Metatheorem and the Brøndsted-Jachymski Principle

In order to give some equivalents of the Ekeland variational principle, we introduced a Metatheorem in 1985-86 [12],[13] on equivalent statements in Ordered fixed point theory.

Later we found some more additional equivalent statements and, consequently, we obtain an extended version of the Metatheorem in 2022 [14],[15],[16],[18],[19],[21]. Later this is rearranged in [20] and called the new 2023 Metatheorem.

Recently, motivated by Brøndsted [27], we adopted the following in Park [19]:

Brøndsted Principle. *Let $(X, \preccurlyeq)$ be a preordered set and $f : X \to X$ be a progressive map (that is, $x \preccurlyeq f(x)$ for all $x \in X$). Then a maximal element $v \in X$ is a fixed point of f.*

For a preordered set $(X, \preccurlyeq)$ and a map $f : X \to X$, we define

Max$(\preccurlyeq)$: the set of maximal elements;

Min$(\preccurlyeq)$: the set of minimal elements;

Fix(f) : the set of fixed points of f;

Per(f) : the set of periodic points $x \in X$; that is, $x = f^n(x)$ for some $n \in \mathbb{N}$.

In our previous work [19], we established the following based on Brøndsted [27] in 1976 and Jachymski [28] in 2003:

Brøndsted-Jachymski Principle. *Let $(X, \preccurlyeq)$ be a poset and $f : X \to X$ be a progressive map. Then X admits a maximal element $v \in X \iff v$ is a fixed point of $f \iff v$ is a periodic point, that is,*

$$\mathrm{Max}(\preccurlyeq) \subset \mathrm{Fix}(f) = \mathrm{Per}(f).$$

If $f : X \to X$ is a anti-progressive (that is, $f(x) \preccurlyeq x$ for all $x \in X$), then we have

$$\mathrm{Min}(\preccurlyeq) \subset \mathrm{Fix}(f) = \mathrm{Per}(f).$$

This can be derived from Metatheorem and is not claiming the non-emptiness of three sets. We noticed that, in most applications of this principle, the existence of a maximal element or a fixed point is achieved by an upper bound of a chain in $(X, \preccurlyeq)$ as we can see examples in the next section.

4. Main Theorems

In this section we introduce our new 2023 Metatheorem in [20] and obtain an useful consequence for the present article:

Metatheorem. *Let X be a set, A its nonempty subset, and $G(x, y)$ a sentence formula for $x, y \in X$. Then the following are equivalent:*

(α) *There exists an element $v \in A$ such that $G(v, w)$ for any $w \in X \backslash \{v\}$.*

($\beta 1$) *If $f : A \to X$ is a map such that for any $x \in A$ with $x \neq f(x)$, there exists a $y \in X \backslash \{x\}$ satisfying $\neg G(x, y)$, then f has a fixed element $v \in A$, that is, $v = f(v)$.*

($\beta 2$) *If $\mathfrak{F}$ is a family of maps $f : A \to X$ such that for any $x \in A$ with $x \neq f(x)$, there exists a $y \in X \backslash \{x\}$ satisfying $\neg G(x, y)$, then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v = f(v)$ for all $f \in \mathfrak{F}$.*

(γ1) *If $f : A \to X$ is a map such that $\neg G(x, f(x))$ for any $x \in A$, then f has a fixed element $v \in A$, that is, $v = f(v)$.*

(γ2) *If $\mathfrak{F}$ is a family of maps $f : A \to X$ satisfying $\neg G(x, f(x))$ for all $x \in A$ with $x \neq f(x)$, then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v = f(v)$ for all $f \in \mathfrak{F}$.*

(δ1) *If $F : A \multimap X$ is a multimap such that, for any $x \in A \backslash F(x)$ there exists $y \in X \backslash \{x\}$ satisfying $\neg G(x, y)$, then F has a fixed element $v \in A$, that is, $v \in F(v)$.*

(δ2) *Let $\mathfrak{F}$ be a family of multimaps $F : A \multimap X$ such that, for any $x \in A \backslash F(x)$ there exists $y \in X \backslash \{x\}$ satisfying $\neg G(x, y)$. Then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v \in F(v)$ for all $F \in \mathfrak{F}$.*

(ϵ1) *If $F : A \multimap X$ is a multimap satisfying $\neg G(x, y)$ for any $x \in A$ and any $y \in F(x) \backslash \{x\}$, then F has a stationary element $v \in A$, that is, $\{v\} = F(v)$.*

(ϵ2) *If $\mathfrak{F}$ is a family of multimaps $F : A \multimap X$ such that $\neg G(x, y)$ holds for any $x \in A$ and any $y \in F(x) \backslash \{x\}$, then $\mathfrak{F}$ has a common stationary element $v \in A$, that is, $\{v\} = F(v)$ for all $F \in \mathfrak{F}$.*

(η) *If Y is a subset of X such that for each $x \in A \backslash Y$ there exists a $z \in X \backslash \{x\}$ satisfying $\neg G(x, z)$, then there exists a $v \in A \cap Y$.*

Here, $\neg$ denotes the negation. We give the proof for completeness.

Proof. Note that each of $(\beta), (\delta), (\epsilon)$ implies (γ) and that $(\beta 2) - (\zeta 2)$ imply $(\beta 1) - (\epsilon 1)$, respectively. We adopt our previous proof for $(\alpha) \Longrightarrow (\gamma 1)$ as follows:

- $(\alpha) \Longrightarrow (\delta 1)$: Suppose $v \notin F(v)$ in $(\delta 1)$. Then there exists a $y \in X \backslash \{v\}$ satisfying $\neg G(v, y)$. This contradicts (α).
- $(\delta 1) \Longrightarrow (\beta 1)$: Clear.
- $(\beta 1) \Longrightarrow (\gamma 1)$: Clear.

We prove $(\gamma 1) \Longrightarrow (\alpha)$ as follows:

- $(\gamma 1) \Longrightarrow (\epsilon 1)$: Suppose F has no stationary element, that is, $F(x) \backslash \{x\} \neq \emptyset$ for any $x \in A$. Choose a choice function f on $\{F(x) \backslash \{x\} : x \in A\}$. Then f has no fixed element by its definition. However, $\neg G(x, f(x))$ for any $x \in A$. Therefore, by $(\gamma 1)$, f has a fixed element, a contradiction.
- $(\epsilon 1) \Longrightarrow (\gamma 2)$: Define a multimap $F : A \multimap X$ by $F(x) := \{f(x) : f \in \mathfrak{F}\} \neq \emptyset$ for all $x \in A$. Since $\neg G(x, f(x))$ for any $x \in A$ and any $f \in \mathfrak{F}$, by $(\epsilon 1)$, F has a stationary element $v \in A$, which is a common fixed element of $\mathfrak{F}$.
- $(\gamma 2) \Longrightarrow (\alpha)$: Suppose that for any $x \in A$, there exists a $y \in X \backslash \{x\}$ satisfying $\neg G(x, y)$. Choose $f(x)$ to be one of such y. Then $f : A \to X$ has no fixed element by its definition. However, $\neg G(x, f(x))$ for all $x \in A$. Let $\mathfrak{F} = \{f\}$. By $(\gamma 2)$, f has a fixed element, a contradiction.

- Consequently, we showed equivalency of $(\alpha) - (\gamma 2)$.
  We show that $(\alpha) \Longleftrightarrow (\epsilon 2)$ as follows:

- $(\alpha)+(\epsilon 1) \Longrightarrow (\epsilon 2)$: By (α), there exists a $v \in A$ such that $G(v,w)$ for all $w \in X\backslash\{v\}$. For each $F \in \mathfrak{F}$, by $(\epsilon 1)$, we have a $v_F \in A$ such that $\{v_F\} = F(v_F)$. Suppose $v \neq v_F$. Then $G(v, v_F)$ holds by (α) and $\neg G(v, v_F)$ holds by assumption on $(\epsilon 2)$. This is a contradiction. Therefore $v = v_F$ for all $F \in \mathfrak{F}$.

- $(\epsilon 2) \Longrightarrow (\epsilon 1) \Longrightarrow (\alpha)$: Already shown.

- $(\alpha) \Longrightarrow (\eta)$: By (α), there exists a $v \in A$ such that $G(v,w)$ for all $w \neq v$. Then by the hypothesis, we have $v \in Y$. Therefore, $v \in A \cap Y$.

- $(\eta) \Longrightarrow (\alpha)$: For all $x \in A$, let $A(x) := \{y \in X : x \neq y, \neg G(x,y)\}$. Choose $Y = \{x \in X : A(x) = \emptyset\}$. If $x \notin Y$, then there exists a $z \in A(x)$. Hence the hypothesis of (η) is satisfied. Therefore, by (η), there exists a $v \in A \cap Y$. Hence $A(v) = \emptyset$; that is, $G(v,w)$ for all $w \neq v$. Hence (α) holds.

This completes our proof. ∎

Remark 4.1. (1) All of the elements v's in Metatheorem are same as we have seen in the proof.

(2) We adopted the Axiom of Choice in $(\gamma 1) \Longrightarrow (\epsilon 1)$.

The following is a useful consequence of Metatheorem without listing $(\beta 1) - (\epsilon 1)$.

Theorem 4.2. *Let $(X, \preceq)$ be a preordered set and A be a nonempty subset of X. Then the following statements are equivalent:*

(α) *There exists a maximal (resp. minimal) element $v \in A$ such that $v \npreceq w$ (resp. $w \npreceq v$) for any $w \in X\backslash\{v\}$.*

(β) *If $\mathfrak{F}$ is a family of maps $f : A \to X$ such that for any $x \in A$ with $x \neq f(x)$, there exists a $y \in X\backslash\{x\}$ satisfying $x \preceq y$ (resp. $y \preceq x$), then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v = f(v)$ for all $f \in \mathfrak{F}$.*

(γ) *If $\mathfrak{F}$ is a family of maps $f : A \to X$ satisfying $x \preceq f(x)$ (resp. $f(x) \preceq x$) for all $x \in A$ with $x \neq f(x)$, then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v = f(v)$ for all $f \in \mathfrak{F}$.*

(δ) *Let $\mathfrak{F}$ be a family of multimaps $F : A \multimap X$ such that, for any $x \in A\backslash F(x)$ there exists $y \in X\backslash\{x\}$ satisfying $x \preceq y$ (resp. $y \preceq x$). Then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v \in F(v)$ for all $F \in \mathfrak{F}$.*

(ϵ) *If $\mathfrak{F}$ is a family of multimaps $F : A \multimap X$ such that $x \preceq y$ (resp. $y \preceq x$) holds for any $x \in A$ and any $y \in F(x)\backslash\{x\}$, then $\mathfrak{F}$ has a common stationary element $v \in A$, that is, $\{v\} = F(v)$ for all $F \in \mathfrak{F}$.*

(η) *If Y is a subset of X such that for each $x \in A\backslash Y$ there exists a $z \in X\backslash\{x\}$ satisfying $x \preceq z$ (resp. $z \preceq x$), then there exists a $v \in A \cap Y$.*

Remark 4.3. Note that we claimed that $(\alpha)-(\eta)$ are equivalent in Theorem 4.2 and did not claim that they are true. For a counter-example, consider the real line with the usual order. However, we gave many examples that they are true based on the original sources; see the articles mentioned in [34].

Let $(X, \preccurlyeq)$ be a preordered set and $F : X \multimap X$ a multimap with the set $\mathrm{Fix}(F)$ of its fixed points. For every $x \in X$, we denote

$$SF(x) := \{z \in X : u \preccurlyeq z \text{ for some } u \in F(x)\}.$$

Theorem 4.4. *Let $(X, \preccurlyeq)$ be a poset, $F : X \multimap X$ be a multimap, $x_0 \in X$ such that $A = (SF(x_0), \preccurlyeq)$ has an upper bound $v \in A$. Then*

- (α) *$v \in A$ is a maximal element of $(X, \preccurlyeq)$ (that is, $v \not\preccurlyeq w$ for any $w \in X \backslash \{v\}$) and its equivalent statements $(\beta)-(\eta)$ in Theorem 4.2 hold.*

Proof. (α) Since $z \preccurlyeq v$ for any $z \in A = SF(x_0)$, $u \preccurlyeq z$ for some $u \in F(x_0)$. Since $u \preccurlyeq z \preccurlyeq v$, we have $v \in SF(x_0)$. Hence v is a maximal element of A. The equivalency $(\beta)-(\eta)$ follows from Theorem 4.2 or Metatheorem. ∎

Note that Theorem 4.2 (γ) implies the following:

Corollary 4.5. *Let $(X, \preccurlyeq)$ be a poset, $F : X \multimap X$ be a map, $x_0 \in X$ such that $A = (SF(x_0), \preccurlyeq)$ has an upper bound $v \in A$. Then, for any progressive map $f : A \to A$, we have*

$$\mathrm{Max}(\preccurlyeq) = \mathrm{Fix}(f) = \mathrm{Per}(f) = \{v\}.$$

If $F = 1_X$ is the identity map, then $SF(x_0) = S(x_0) = \{y \in X : x_0 \preccurlyeq y\}$.

Corollary 4.6. *Let $(X, \preccurlyeq)$ be a poset, $x_0 \in X$ such that $A = (S(x_0), \preccurlyeq)$ has an upper bound $v \in A$. Then Theorem 4.4 $(\alpha)-(\eta)$ hold.*

Example 4.7. We obtained some particular cases of Corollary 4.6 and their applications as follows:

(1) Such particular cases appear in Theorem 1.1 [14], Theorem 3.1 and 4.2 [16], Theorem 4.2] [19], Theorem 3.1 [22], and others.

(2) In [22], we applied Corollary 4.6 to theorems due to Knaster-Tarski, Nadler, Zermello, Zorn, Tarski-Kantorovitch, and Edelstein.

(3) In [20], particular forms of Corollary 4.6 are stated to improve the Knaster-Tarski theorem, the Abian-Brown theorem, the Tarski-Kantorovitch theorem, Zorn's lemma, and many others in Howard-Rubin [32].

(4) In [24], we stated Corollary 4.6 for a particular case and (α), (γ) improve Zorn's Lemma, Zermelo fixed point theorem, resp. Moreover, (γ) implies the Brøndsted-Jachymski principle.

(5) Our forth-coming [34] contains various applications of the 2023 Metatheorem which is an extended version of several previous ones in 1980's and 2022.

5. Known Knaster-Tarski Type Fixed Point Theorems

In [29],[30], Jinlu Li collected some Tarski type fixed point theorems and others as follows. In this section, we show that most of them are consequences of our new Theorem 4.2.

In 1955, Tarski [6] proved a fixed point theorem on chain-complete lattices for single-valued maps:

Tarski fixed point theorem [6]. *Let $(P, \preccurlyeq)$ be a chain-complete lattice and let $f : P \to P$ be an order-increasing map. If there is an x^* in P with $x^* \preccurlyeq f(x^*)$, then f has a fixed point.*

It was extended to chain-complete posets in 1961 [31]. Let $A = \{f^n(x^*) : n \in \mathbb{N}\} \cup \{\text{its upper bounds}\} \subset S(x^*)$. Since A is a chain with an upper bound $v \in A$ and $f|_A : A \to A$ is order increasing, $x \preccurlyeq f(x)$ for all $x \in A$, that is f is progressive on A. Hence the Tarski theorem follows from Theorem 4.2 $(\gamma 1)$. Moreover, this should have the conclusion of the Brøndsted-Jachymski principle.

Abian-Brown fixed point theorem [31]. [*Let $(P, \preccurlyeq)$ be a chain-complete poset and let $f : P \to P$ be an order-increasing map. If there is an $x^* \in P$ with $x^* \preccurlyeq f(x^*)$, then f has an $\preccurlyeq$-maximum fixed point.*

This follows from Theorem 4.4 $(\alpha), (\gamma 1)$. Moreover, this should have the conclusion of the Brøndsted-Jachymski principle.

Extension of Tarski fixed point theorem [23]. *Let $(P, \preccurlyeq)$ be a complete lattice and $F : P \multimap P$ a multimap. If F satisfies the following two conditions:*

T1. *F is order-increasing upward (isotone).*

T2. *$(SF(x), \preccurlyeq)$ is an inductively ordered set for each $x \in P$.*

Then F has a fixed point, that is, there exists $x^* \in P$ such that $x^* \in F(x^*)$.

This is reformulation of Theorem 4.4 $(\delta 1)$ or $(\epsilon 1)$ with $F = T$.

The following form of Theorem 4.2 $(\gamma 1)$ is given by Jachymski [33, Theorem 2.1]:

Knaster-Tarski fixed point theorem. *Let $(P, \preccurlyeq)$ be a poset in which every chain has a supremum. Assume that $f : P \to P$ is isotone and there is an element $p_0 \in P$ such that $p_0 \preccurlyeq f(p_0)$. Then f has a fixed point.*

Let $A = \{f^n(p_0) : n \in \mathbb{N}\} \cup \{\text{its supremum}\}$. Then $f : A \to P$ is progressive, and the conclusion can be improved to $\mathrm{Fix}(f) = \mathrm{Per}(f) \supset \mathrm{Max}(\preccurlyeq) \neq \emptyset$.

From this the 'if' part of the following holds:

Knaster-Tarski-Davis fixed point theorem [34]. *Let $(P, \preccurlyeq)$ be a lattice. Then every order-increasing self-map on P has a fixed point if, and only if, $(P, \preccurlyeq)$ is a complete lattice.*

In 1984, Fujimoto [35] extended the Tarski fixed point theorem from single-valued maps to multimaps:

Fujimoto fixed point theorem. *Let $(P, \preccurlyeq)$ be a complete lattice and let $F : P \multimap P$ be a multimap satisfying the following two conditions:*

F1. *F is isotone (F is order-increasing upward), that is, if $x \preccurlyeq y$ in P, then for any $z \in F(x)$ there is a $w \in F(y)$ such that $z \preccurlyeq w$.*

F2. *The set $SF(x) = \{z \in P : u \preccurlyeq z$ for some $u \in F(x)\}$ is an inductively ordered set for each $x \in P$.*

Then F has a fixed point, that is, there exists $x^* \in P$ such that $x^* \in F(x^*)$.

This is reformulation of Theorem 4.4 ($\delta 1$) with $F = T$.

In 2015, Jinlu Li [15, Theorem 2.2] generalized the Fujimoto fixed point theorem to chain-complete posets as follows.

Li's Theorem. *Let $(P, \preccurlyeq)$ be a chain-complete poset and let $F : P \multimap P$ satisfy the following:*

L1. *F is order-increasing upward;*

L2. *$(SF(x), \preccurlyeq)$ is inductive with a finite number of maximal elements, for every $x \in P$.*

L3. *There is an element y in P with $y \preccurlyeq u$, for some $u \in F(y)$.*

Then F has a fixed point.

Note that L2 implies (α) in Theorem 4.4 and L3 implies the assumption of (δ) with $F = T$. Therefore, this theorem follows from Theorem 4.4 (δ).

In [29] and [30], several fixed point theorems are proved by Jinlu Li for multimaps on chain-complete posets and some applications have been provided. In many works, he obtained order theoretic fixed point theorems on posets and their applications.

In the following, we apply our results to two recent articles [26] and [36].

6. Espínola-Wiśnicki Fixed Point Theorem

In 2018, Espínola-Wiśnicki [26] stated in their Abstract as follows:

"Let X be a poset with the property that each family of order intervals of the form $[a, b]$, $[a, \rightarrow)$ with the finite intersection property has a nonempty intersection. We show that every directed subset of X has a supremum. Then we apply the above result to prove that if X is a topological space with a partial order $\preccurlyeq$ for which the order intervals are compact, $\mathcal{F}$ a nonempty commutative family of monotone maps from X into X and there exists $c \in X$ such that $c \preccurlyeq f(c)$ for every $f \in \mathcal{F}$, then the set of common fixed points of F is nonempty and has a maximal element ..."

Let $(X, \preccurlyeq)$ be a poset. A map $f : X \to X$ is said to be *monotone* (or *increasing*) if $f(x) \preccurlyeq f(y)$ whenever $x \preccurlyeq y$. Consider the sets $[a, \rightarrow) = \{x \in X : a \preccurlyeq x\}$ and $(\leftarrow, b] = \{x \in X : x \preccurlyeq b\}$. Along this section the concept of order intervals in X will be restricted to the sets $[a, \rightarrow)$ and $[a, b] = [a, \rightarrow) \cap (\leftarrow, b]$. Remember that a subset J of a poset X is *directed* if each finite subset of J has an upper bound in J.

Lemma 6.1. [26] *Let X be a poset for which each family of order intervals of the form $[a, b], [a, \rightarrow)$ with the finite intersection property has a nonempty intersection. Then every directed subset of X has a supremum.*

Combining Lemma 6.1 and the Knaster-Tarski theorem, they obtain immediately.

Theorem 6.2. [26] *Let X be a topological space with a partial order $\preccurlyeq$ for which order intervals are compact and let $f : X \to X$ be monotone. If there exists $c \in X$ such that $c \preccurlyeq f(c)$, then the set of all fixed points of f is nonempty and has a maximal element.*

But the application of Lemma 6.1 is wider. Having it they obtain a short and independent proof of the following strengthening of Theorem 6.2.

Theorem 6.3. [26] *Let X be a topological space with a partial order $\preccurlyeq$ for which order intervals are compact and let $\mathcal{F}$ be a nonempty commutative family of monotone maps from X into X. If there exists $c \in X$ such that $c \preccurlyeq f(c)$ for every $f \in \mathcal{F}$, then the set of common fixed points of $\mathcal{F}$ is nonempty and has a maximal element.*

However, from Lemma 6.1, the present author derive the following:

Theorem 6.4. *Let $(X, \preccurlyeq)$ be a poset for which each family of order intervals of the form $[a, b], [a, \rightarrow)$ with the finite intersection property has a nonempty intersection. Let $x_0 \in X$ and $\varphi : X \to X$ a progressive map. Then*

(0) $B = \{\varphi^n(x_0) \in X : n \in \mathbb{N}\}$ *has an upper bound $v \in X$,*

and the equivalent statements $(\alpha) - (\eta)$ in Theorem 4.2 hold for

$$A = B \cup \{\text{its upper bounds}\},$$

where

(α) *There exists a maximal element $v \in A$ such that $v \not\preccurlyeq w$ for any $w \in X \backslash \{v\}$.*

($\gamma 1$) *If $f : A \to X$ is a map such that $x \preceq f(x)$ for any $x \in A$, then f has a fixed element $v \in A$, that is, $v = f(v)$.*

($\gamma 2$) *If $\mathfrak{F}$ is a family of maps $f : A \to X$ satisfying $x \preccurlyeq f(x)$ for all $x \in A$ with $x \neq f(x)$, then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v = f(v)$ for all $f \in \mathfrak{F}$.*

($\delta 1$) *If $T : A \multimap X$ is a multimap such that for any $x \in A \backslash T(x)$ there exists a $y \in X \backslash \{x\}$ satisfying $x \preceq y$, then T has a fixed element $v \in A$, that is, $v \in T(v)$.*

Proof. (0) Since $\varphi : X \to X$ is progressive, B is a directive subset of X. By Lemma 6.1, it has a supremum.

(α) This follows from Theorem 6.4 (α).

(α)–(η) are equivalent as in Theorem 4.2.

∎

Note that Theorem 6.4 (α) and $(\gamma 1)$ improve Theorem 6.2, and Theorem 6.4 $(\delta 1)$ improves Theorem 6.3.

Note that, in [26], a large number of particular cases and some applications of their Theorem 6.3 were given. All of such results can be improved or extended according to our new Theorem 6.4.

Theorem 6.4 can be rewritten as follows:

Theorem 6.5. *Let $(X, \preccurlyeq)$ be a partially ordered set, $x_0 \in X$, $\varphi : X \to X$ a map, $B = \{\varphi^n(x_0) \in X : n \in \mathbb{N}\}$, $A = B \cup \{\text{its infimum}\}$, and $\varphi(x) \preccurlyeq x$ for all $x \in A$. Then the following equivalent conditions $(\alpha) - (\eta)$ hold.*

- (α) *There exists a minimal element $v \in A$ such that $w \not\preccurlyeq v$ for any $w \in X \backslash \{v\}$.*
- (β) *If $\mathfrak{F}$ is a family of maps $f : A \to X$ satisfying $f(x) \preccurlyeq x$ for all $x \in A$ with $x \neq f(x)$, then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v = f(v)$ for all $f \in \mathfrak{F}$.*
- (γ) *If $\mathfrak{F}$ is a family of maps $f : A \multimap X$ such that $f(x) \preccurlyeq x$ holds for any $x \in A$ with $x \neq f(x)$, then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v = f(v)$ for all $f \in \mathfrak{F}$.*
- (δ) *If $\mathfrak{F}$ is a family of multimaps $T : A \multimap X$ such that, for any $x \in A$ there exists $y \in X \backslash \{x\}$ satisfying $y \preccurlyeq x$, then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v = T(v)$ for all $T \in \mathfrak{F}$.*
- (ϵ) *If $\mathfrak{F}$ is a family of multimaps $T : A \multimap X$ such that $y \preccurlyeq x$ holds for any $x \in A$ and any $y \in T(x) \backslash \{x\}$, then $\mathfrak{F}$ has a common stationary element $v \in A$, that is, $\{v\} = T(v)$ for all $T \in \mathfrak{F}$.*
- (η) *If Y is a subset of X such that for each $x \in A \backslash Y$ there exists a $z \in X \backslash \{x\}$ satisfying $z \preccurlyeq x$, then there exists a $v \in A \cap Y$.*

Proof. (α) Let $v \in X$ be an infimum of A. Then $v \in A$ and $\varphi(v) \preccurlyeq v$. Since v is an infimum of B, $v \preccurlyeq \varphi^n(x_0)$ and $\varphi(v) \preccurlyeq v \preccurlyeq \varphi^n(x_0)$ for all $n \in \mathbb{N}$. Hence $\varphi(v)$ is a lower bound of B and belongs to A. Therefore, $v \preccurlyeq \varphi(v)$. Consequently, $v = \varphi(v)$ and v is the minimal element by the Brøndsted-Jachymski Principle.

The equivalence of $(\alpha) - (\eta)$ is routine. ∎

Theorem 6.6. *Let $(X, \preccurlyeq)$ be a partially ordered set, $x_0 \in X$, $\varphi : X \to X$ a map, $B = \{\varphi^n(x_0) \in X : n \in \mathbb{N}\}$, $A = B \cup \{\text{its infimum}\}$ and $\varphi(x) \preccurlyeq x$ for all $x \in A$. Then*

$$\mathrm{Fix}(\varphi) = \mathrm{Per}(\varphi) \supset \mathrm{Min}(\preccurlyeq) \neq \emptyset.$$

7. Tarski-Kantorovitch Type Theorems

A selfmap f of a partially ordered set $(P, \preccurlyeq)$ is said to be $\preccurlyeq$-*continuous* if for every countable chain C having a supremum, the image $f(C)$ has a supremum and $\sup f(C) = f(\sup C)$. It is easily seen that a $\preccurlyeq$-continuous map is isotone. $(P, \preccurlyeq)$ is said to be $\preccurlyeq$-*complete* if every countable chain has a supremum.

Following Dugundji and Granas [37, p.15], Jachymski [33] and Jachymski et al. [38, 39] introduced the following:

Theorem 7.1 (Tarski-Kantorovitch). *Let $(P, \preccurlyeq)$ be a $\preccurlyeq$-complete poset and a map $f : P \to P$ be $\preccurlyeq$-continuous. If there exists $p_0 \in P$ such that $p_0 \preccurlyeq f(p_0)$, then f has a fixed point $p_* := \sup\{f^n(p_0) : n \in \mathbb{N}\}$.*

Without using the terminology like $\preccurlyeq$-completeness and $\preccurlyeq$-continuity, this theorem can be written as follows:

Theorem 7.2. *Let $(P, \preccurlyeq)$ be a poset and $f : P \to P$ be a map such that*

(a) there exists $p_0 \in P$ such that $p_0 \preccurlyeq f(p_0)$,

(b) $B = \{f^n(p_0) : n \in \mathbb{N}\}$ has a supremum, and

(c) $\sup f(B) = f(\sup B)$.

Then f has a fixed point $p^* := \sup B$ and

$$\mathrm{Fix}(f) = \mathrm{Per}(f) \supset \mathrm{Max}(\preccurlyeq) = \{p^*\}.$$

Whenever f is isotone, this follows from the Knaster-Tarski theorem without assuming (c) and can be equivalently reformulated as follows:

Theorem 7.3. *Let $(P, \preccurlyeq)$ be a poset and $f : P \to P$ be a map such that*

(a) there exists $p_0 \in P$ such that $f(p_0) \preccurlyeq p_0$,

(b) $B = \{f^n(p_0) : n \in \mathbb{N}\}$ has an infimum, and

(c) $\inf f(B) = f(\inf B)$.

Then f has a fixed point $p^* := \inf B$ and

$$\mathrm{Fix}(f) = \mathrm{Per}(f) \supset \mathrm{Min}(\preccurlyeq) = \{p^*\}.$$

In 2015, Shahzad-Valero-Alghamdi-Alghamdi [36] stated "we present a few fixed point results in the framework of topological posets. To this end, we introduce an appropriate notion of completeness and order-continuity. Special attention is paid to the case that the topology of the topological poset is induced by an extended quasi-metric. Finally, the applicability of the exposed results is illustrated providing a methodology to determine the asymptotic upper bound of the complexity of those algorithms whose running time of computing is the solution to a special type of recurrence equation."

From now, we follow [36]: "Let $(X, \preccurlyeq)$ be a poset. In the sequel, we will denote by $\downarrow_{\preccurlyeq} x$, with $x \in X$, the set $\{y \in X : y \preccurlyeq x\}$. As usual, a sequence $(x_n)_{n\in\mathbb{N}}$ in $(X, \preccurlyeq)$ is decreasing if $x_{n+1} \preccurlyeq x_n$ for all $n \in \mathbb{N}$. A map $f : X \to X$ is said to be *monotone* if $f(x) \preccurlyeq f(y)$ whenever $x \preccurlyeq y$.

In the following, we will say that a topological poset $(X, \tau, \preccurlyeq)$ is $\preccurlyeq$-τ-complete provided that every decreasing sequence $(x_n)_{n\in\mathbb{N}}$ in $(X, \preccurlyeq)$ has a lower bound x to which it converges with respect to τ. From now on, a map from a topological poset $(X, \tau, \preccurlyeq)$ into itself is said to be monotone-$\preccurlyeq$-continuous if, it is monotone and, in addition, given $z \in X$, then the sequence $(f^{n+1}(z))_{n\in\mathbb{N}}$ converges to $f(x)$ with respect to τ whenever the sequence $(f^n(z))_{n\in\mathbb{N}}$ is decreasing and x is a lower bound of it such that $(f^n(z))_{n\in\mathbb{N}}$ converges to x with respect to τ."

Now we can prove the main theorem of [36] as follows:

Theorem 7.4. [36] *Let $(X, \tau, \preccurlyeq)$ be a $\preccurlyeq$-τ-complete topological poset and let $f : X \to X$ be a map. Assume that the following assertions hold:*

(1) *(X, τ) is Hausdorff.*

(2) *There exists $x_0 \in X$ such that $f(x_0) \preccurlyeq x_0$.*

(3) *f is monotone-$\preccurlyeq$-continuous.*

Then f has a fixed point x^* such that $x^* \in \downarrow_{\preccurlyeq} x_0$.

Proof. We use Theorem 7.3 with $\varphi = f$. Let $B = \{f^n(x_0) \in X : n \in \mathbb{N}\}$. Since f is monotone and $f(x_0) \preccurlyeq x_0$, we have $f^{n+1}(x_0) \preccurlyeq f^n(x_0)$ for all $n \in \mathbb{N}$. Since X is $\preccurlyeq$-τ-complete, B has a lower bound $x^* \in X$. That is, $x^* \preccurlyeq f^n(x_0)$ and $f(x^*) \preccurlyeq f^{n+1}(x_0) \preccurlyeq f^n(x_0)$ for all $n \in \mathbb{N}$. Note that $f^{n+1}(x_0)$ converges to $f(x^*)$ and x^*. Since X is Hausdorff, the conclusion follows. ∎

In the light of the preceding notions, we are able to improve their main result.

Theorem 7.5. *Let $(X, \preccurlyeq)$ be a poset, $x_0 \in X$, $\varphi : X \to X$ be monotone such that $\varphi(x_0) \preccurlyeq x_0$. Let $B = \{\varphi^n(x_0) \in X : n \in \mathbb{N}\}$ have an infimum $v \in X$. Then*

$$\mathrm{Fix}(\varphi) = \mathrm{Per}(\varphi) \supset \mathrm{Min}(\preccurlyeq) = \{v\}.$$

Proof. Let $A = B \cup \{v\}$. In order to apply Theorem 7.2, we have to show $\varphi(x) \preccurlyeq x$ for all $x \in A$. This holds for any $x \in B$ by the monotonicity and $\varphi(x_0) \preccurlyeq x_0$. For the infimum v of B, we have $v \preccurlyeq \varphi^n(x_0)$ for all $n \in \mathbb{N}$ and hence $\varphi(v) \preccurlyeq \varphi^{n+1}(x_0)$ by the monotonicity of φ. Then $\varphi(v) \preccurlyeq \varphi^{n+1}(x_0) \preccurlyeq \varphi^n(x_0)$ for all $n \in \mathbb{N}$. Hence $\varphi(v)$ is also a lower bound of B and $\varphi(v) \preccurlyeq v$. Now we have $\varphi(x) \preccurlyeq x$ for all $x \in A$. The conclusion follows from Theorems 7.1 and 7.2. ∎

8. Conclusion

Alfred Tarski (1901-1983) was a Polish-American mathematician, logician, and philosopher. He made important contributions in many areas in mathematics; especially, in the foundations of mathematics. One of his earlier famous result is the Banach-Tarski paradox. He is recognized as one of the four greatest logicians of all times, the other three being Aristoteles, Frege, and Gödel.

Tarski was also a pioneer of fixed point theory on lattices. In the present article, we investigate the current versions of two fixed point theorems of Tarski and their successors. Consequently, we obtained improved equivalent versions of his results and certain related results.

In 2022, we found some add-ups to the old Metatheorem in Ordered fixed point theory and its particular forms–the Brøndsted Principle and the Brøndsted-Jachymski Principle. In several of our previous works, we applied them in 2022 to many of known results of others. Later we rearranged the order of statements to the 2023 Metatheorem in [20], [21], which will be the basis of future study in various fields of mathematics. We hope we can continue to find more applications of our new 2023 Metatheorem.

References

[1] E. Zermelo, Neuer Beweis für die Möglichkeit einer Wohlordnung, Math. Ann., 65 (1908), 107–128.

[2] B. Knaster, Un theoreme sur les fonctions d'ensembles, Ann. Soc. Polon. Math., 6 (1928), 133–134.

[3] M. Zorn, A remark on method in transfinite algebra, Bull. Amer. Math. Soc., 41 (1935), 667–670.

[4] N. Bourbaki, Sur le theoreme de Zorn, Arch. Math., 2 (6) (1949), 434–437.

[5] A. Tarski, A fixpoint theorem for lattices and its applications-preliminary report, Bull. Amer. Math. Soc., 55 (1949), 1051–1052.

[6] A. Tarski, A lattice-theoretical fixpoint theorem and its applications, Pacific J. Math., 5 (1955), 285–309.

[7] I. Ekeland, Sur les problèmes variationnels, C.R. Acad. Sci. Paris, 275 (1972) 1057–1059; 276 (1973), 1347–1348.

[8] I. Ekeland, On the variational principle, J. Math. Anal. Appl., 47 (1974), 324–353.

[9] J. Caristi, Fixed point theorems for mappings satisfying inwardness conditions, Trans. Amer. Math. Soc., 215 (1976), 241–251.

[10] H. Brézis and F.E. Browder, A general principle on ordered sets in nonlinear functional analysis, Adv. Math., 21 (1976), 355–364.

[11] W. Takahashi, Existence theorems generalizing fixed point theorems for multivalued mappings, Fixed point theory and applications, (1991), 397–406.

[12] S. Park, Some applications of Ekeland's variational principle to fixed point theory, in: Approximation Theory and Applications, Pitman Res. Notes Math., 133 (1985), 159–172.

[13] S. Park, Equivalent formulations of Ekeland's variational principle for approximate solutions of minimization problems and their applications, Operator Equations and Fixed Point Theorems, 1 (1986), 55–68.

[14] S. Park, Equivalents of various maximum principles, Results in Nonlinear Analysis, 5 (2) (2022), 169–184.

[15] S. Park, Applications of various maximum principles, J. Fixed Point Theory, 3 (1) (2022): 23.

[16] S. Park, Equivalents of maximum principles for several spaces, Top. Algebra Appl., 10 (1) (2022), 68–76.

[17] S. Park, Equivalents of generalized Brøndsted principle, J. Informatics Math. Sci., to appear.

[18] S. Park, Equivalents of ordered fixed point theorems of Kirk, Caristi, Nadler, Banach, and others, Adv. Th. Nonlinear Anal. Appl., 6 (4) (2022), 420–432.

[19] S. Park, Extensions of ordered fixed point theorems, DOI: 10.13140/RG.2.2.21699.48160

[20] S. Park, Equivalents of some ordered fixed point theorems, J. Advances Math. Comp, Sci. 38(1) (2023), 52–67.

[21] S. Park, Foundations of ordered fixed point theory, J. Nat. Acad. Sci., ROK, Nat. Sci. Ser. 61 (2) (2022), 1–51.

[22] S. Park, Variations of ordered fixed point theorems, Linear Nonlinear Anal. 8 (3) (2022), 225-237.

[23] S. Park, Applications of generalized Zorn's Lemma, J. Nonlinear Anal. Optim. 13 (2) (2022), 75 -84. ISSN : 1906-9605

[24] S. Park, Equivalents of certain minimum principles, Letters Nonlinear Anal. Appl. 1(1) (2023), 1–11.

[25] J. Li, Inductive properties of fixed point sets of mappings on posets and on partially ordered topological spaces, Fixed Point Theory Appl., 2015 (1) (2015), 1–14.

[26] R. Espínola and A. Wiśnicki, The Knaster-Tarski theorem versus monotone nonexpansive mappings, Bulletin Polish Acad. Sci. Math., 66 (2018), 1–7.

[27] A. Brøndsted, Fixed point and partial orders, Proc. Amer. Math. Soc., 60 (1976), 365–366.

[28] J. Jachymski, Converses to fixed point theorems of Zermelo and Caristi, Nonlinear Analysis, 52 (2003), 1455–1463.

[29] J.L. Li, Several extensions of the Abian-Brown fixed point theorem and their applications to extended and generalized Nash equilibria on chain-complete posets, J. Math. Anal. Appl., 409 (2014), 1084–1092.

[30] J.L. Li, Fixed point theorems on partially ordered topological vector spaces and their applications to equilibrium problems with incomplete preferences, Fixed Point Theory Appl., 2014 (2014), 192.

[31] S. Abian and A.B. Brown, A theorem on partially ordered sets, with applications to fixed point theorems, Canad. J. Math., 13 (1961), 78–82.

[32] P. Howard and J.E. Rubin, Consequences of the Axiom of Choice, Providence, RI: American Mathematical Society, RI, (1998).

[33] J. Jachymski, Order-theoretic aspects of metric fixed point theory, In: Handbook of Metric Fixed Point Theory, Springer, (2001), 613–641.

[34] A.C. Davis, A characterization of complete lattice, Pacific J. Math., 5 (1955), 311–319.

[35] T. Fujimoto, An extension of Tarski's fixed point theorem and its applications to isotone complementarity problems, Math. Program., 28 (1984), 116–118.

[36] N. Shahzad, O. Valero, M.A. Alghamdi and M.A. Alghamdi, On fixed point theory in topological posets, extended quasi-metric spaces and an application to asymptotic complexity of algorithms, Fixed Point Theory Appl., 2015 (1) (2015), 1–13.

[37] J. Dugundji and A. Granas, Fixed Point Theory I, Polish Scientific Publishers, Warszawa, 1982.

[38] J. Jachymski, Fixed point theorems in metric and uniform spaces via the Knaster-Tarski Principle, Nonlinear Anal., 32 (1998), 225–233.

[39] J. Jachymski, L. Gajek and P. Pokarowski, The Tarski-Kantorovitch principle and the theory of iterated function systems, Bull. Austral. Math. Soc., 61 (2000), 247–261.