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On the Result of $\mathcal{P}$-contraction Operators

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ABSTRACT

In this work, we propose and introduce a random $\mathcal{P}$-contraction operator and prove an existence theorem of random fixed points for this operator which is a Banach Contraction Principle random version proof by famous Polish mathematician Stefan Banach in the year 1982 which 100 years ago (1892 – 1945). Moreover, we obtain an existence result for a solution of non-linear stochastic integral equations in a separable partially ordered Banach space.

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1. Introduction

A random fixed point is a stochastic generalization of a classical fixed point. Random fixed point theorems for random contraction mappings on separable complete metric spaces were first proved by Spacek [1] and Hans [2, 3]. In 1966, Mukherjee [4] gave a random fixed point theorem version of Schaduer's fixed point theorem on an atomic probability measure space. In 1976, Bharucha-Reid [5] gave sufficient conditions for a stochastic version of the well-known Schauder's fixed point theorem. In 1979, Itoh [6] extended Spacek's and Hans's theorems to multivalued contraction mappings. Random fixed point theorems with an application to random differential equations in Banach spaces are obtained by [6]. In 1984, Sehgal and Waters [7] obtained several theorems of random fixed points including a stochastic generalization of Rothe's fixed point theorems [8]. In 1993, Beg and Shahzad [9] proved random coincidence points and common random fixed points for a pair of compatible random multivalued oper-

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ators in Polish spaces. Saha [10], Saha and Debnath [11] proved some random fixed point theorems over a separable Banach space and a separable Hilbert space with a probability measure. On the other hand, Padgett [12] studied the existence and uniqueness of a random solution of a non-linear stochastic integral equation of the Hammerstein type. In 2012, Saha and Dey [13] proved random fixed point theorems for $(\theta, L)-$weak contractions in a separable Banach space. Recently, Saha and Ganguly [14] proved a random fixed point theorem in a separable Banach space equipped with a complete probability measure for a certain class of contractive mappings.

In 2002, Kumam's master thesis [15] studied fixed point property in modular spaces, he extended and investigated the necessary and sufficient conditions to guarantee existence and uniqueness of the fixed point of such mappings in that spaces (see more [16]). Later in 2003 the first group of Thai mathematicians leader by Professor Sompong Dhompongsa attend to joint fixed point's conference in Valencia, Spain. Then we can study fixed point theory for further mathematical research for further development in Thailand.

A random fixed point was first studied in Thailand by Dhomponsa, Plubtieng and Kumam before 2005 (see in [17, 18]). In 2005, Kumam and Plubtieng [19] studied some random fixed point theorems for set-valued nonexpansive non-self operator.

In yeas 2004-2005 Professor Poom Kumam first studied by his Ph.D. thesis under supervisor Professor Somyot Plubtueng and proved some fixed point theorems for set-valued random nonexpansive operators in the framework of Banach spaces with characteristic of noncompact convexity associated to the separation measure of noncompactness condition. (see more in Kumam and Plubtieng [20]) Later in 2006, Kumam and Plubtieng [21–23] proved a random fixed point theorem for non-expansive non-self random operators and multivalued non-expansive non-self random operators in Banach spaces. In 2007, Kumam and Plubtieng [24] established a random coincidence point and random common fixed point for non-linear multivalued random operators. In the same year, Kumam and Plubtieng [25] studied the characteristic of noncompact convexity and random fixed point theorem for set-valued operators. In 2009, Kumam and Plubtieng [26] proved random common fixed point theorems for a pair of multi-valued and single-valued nonexpansive random operators in a separable Banach space. In the same year, Kumam and Plubtieng [27] proved random fixed point theorems for asymptotically regular random operators. In 2009, Kumam and Plubtieng [28] studied viscosity approximation methods of random fixed point solutions and random variational inequalities in Hilbert spaces. Recently, Kumam et al proved some remarkable results on random fixed point theorems in a series of papers (see in [29–47]).

Recently, in 2021, Dhompongsa and P. Kumam [49] studied and new proof of the Caristi's fixed point theorem and the Brouwer fixed point theorem which extended related with equilibrium via best proximity pairs in abstract economies and optimization (see [54, 55]). More [51–53] studied more about another proofs Brouwer fixed point and classical fixed point theorems.

Banach's contraction principle [48] is one of the most important result of non-linear analysis. It has been the source of metric fixed point theory and its significance rests in its various applicability in different branches of mathematics. In the general setting of a complete metric space, this theorem runs as follows (see in [56, 57]).

Fig. 1. From left to right: Sompong Dhompongsa (CMU), Satit Saejung (KKU), Attapol Kaewkhao (CMU), Somyos Pulubting (NU) and Poom Kumam (KMUTT) in The 7th International Conference on Fixed Point Theorem and its Applications (ICFPTA), July 2003, Valencia Spain. Available from: https://sites.google.com/site/fixedpointthailand

Theorem 1.1. *(Banach's contraction principle) If (X, d) be a complete metric space and $T : X \to X$ be a self-mapping such that,*

$$d(Tx, Ty) \leq \alpha d(x, y), \tag{1.1}$$

for each $x, y \in X$ for some $\alpha \in [0, 1)$, then T has a unique fixed point.

In 2000, Ciric [58] dealt with a class of mappings (not necessarily continuous) which are defined on a metric space and proved the following fixed point theorem which is a double generalization of Gregus [59].

Theorem 1.2. *Let C be a closed convex subset of a complete convex metric space X and $T : C \to C$ be a mapping satisfying*

$$d(Tx, Ty) \leq ad(x, y) + b\max\{d(x, Tx), d(y, Ty)\} + c[d(x, Ty) + d(y, Tx)]. \tag{1.2}$$

where $0 < a < 1, a + b = 1, c \leq \frac{4-a}{8-b}$ for all $x, y \in C$. Then T has a unique fixed point.

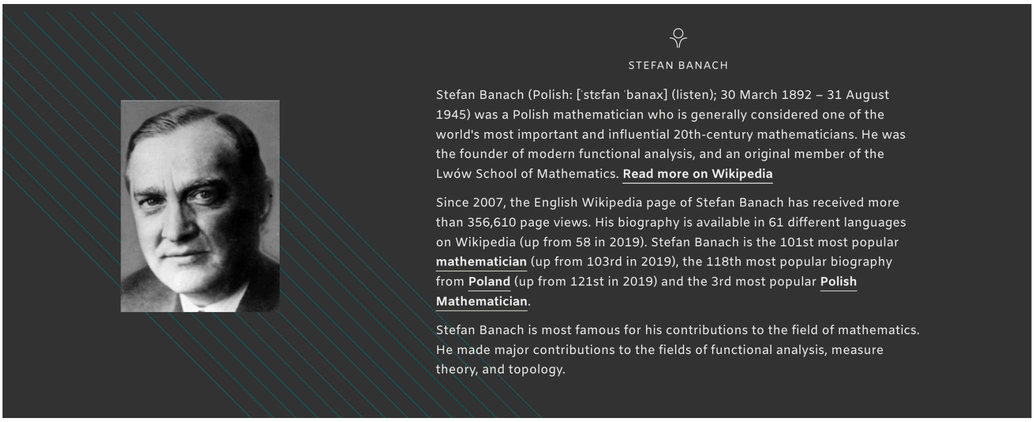

Fig. 2. Stefan Banach (1892–1945). Available from: https://pantheon.world/profile/person/Stefan_Banach

Recently, Chaipunya, Sintunavarat and Kumam [60] first introduced a new type of a contractive condition defined on a partially ordered space, namely a $\mathcal{P}$-contraction, which generalizes the weak contraction, as follows.

Fig. 3. From left to right: Wutiphol Sintunavarat (TU), Parin Chaipunya (KMUTT) and Poom Kumam (KMUTT) first introduced a $\mathcal{P}$-contraction (2012)

A relation $\sqsubseteq$ is called a partial ordering on a set X if it is reflexive, anti-symmetric and transitive. For $x, y \in X$, we may write $x \sqsupseteq y$ for $y \sqsubseteq x$ to emphasize some particular cases.

For $x, y \in X$ are called comparable if $x \sqsubseteq y$ or $y \sqsubseteq x$. If a set X has a partial ordering $\sqsubseteq$ defined on it,we say that it is a partially ordered set and denote it by $(X, \sqsubseteq)$. If any two elements in X are comparable, then $(X, \sqsubseteq)$ is called a totally ordered set. Moreover, it is called a sequentially ordered set if each element of a convergent sequence in X is comparable with its limit.

Definition 1.3. Let $(X, \sqsubseteq, d)$ be a partially ordered metric space. A function $\varrho : X \times X \to \mathbb{R}$ is called a P-function w.r.t. $\sqsubseteq$ in X if it satisfies the following conditions:

(a) $\varrho(x, y) \geq 0$ for every comparable $x, y \in X$;

(b) for any sequences $\{x_n\}_{n=1}^{+\infty}, \{y_n\}_{n=1}^{+\infty}$ in X such that x_n and y_n are comparable at each $n \in N$, if $\lim_{n\to+\infty} x_n = x$ and $\lim_{n\to+\infty} y_n = y$, then $\lim_{n\to+\infty} \varrho(x_n, y_n) = \varrho(x, y)$;

(c) for any sequences $\{x_n\}_{n=1}^{+\infty}, \{y_n\}_{n=1}^{+\infty}$ in X such that x_n and y_n are comparable at each $n \in N$, if $\lim_{n\to+\infty} \varrho(x_n, y_n) = 0$, then $\lim_{n\to+\infty} d(x_n, y_n) = 0$.

If, in addition, the following condition is also satisfied:

(A) for any sequences $\{x_n\}_{n=1}^{+\infty}, \{y_n\}_{n=1}^{+\infty}$ in X such that x_n and y_n are comparable at each $n \in N$, if $\lim_{n\to+\infty} d(x_n, y_n)$ exists, then $\lim_{n\to+\infty} \varrho(x_n, y_n)$ also exists,

then ϱ is said to be a $\mathcal{P}$-function of type (A) w.r.t. $\sqsubseteq$ in X.

Definition 1.4. Let $(X, \sqsubseteq, d)$ be a partially ordered metric space, a mapping $f : X \to X$ is called a $\mathcal{P}$-contraction w.r.t. $\sqsubseteq$ if there exists a $\mathcal{P}$-function $\varrho : X \times X \to \mathbb{R}$ w.r.t. $\sqsubseteq$ in X such that

$$d(fx, fy) \leq d(x, y) - \varrho(x, y) \tag{1.3}$$

for any comparable $x, y \in X$. If in addition ϱ is a $\mathcal{P}$-function of type (A) w.r.t. $\sqsubseteq$ in X, then f is said to be a $\mathcal{P}$-contraction of type (A) w.r.t. $\sqsubseteq$.

Also recently, in the sense of random fixed points, Saha and Ganguly [14] proved a theorem of random fixed point in a separable Banach space equipped with a complete probability measure for a certain class of contractive mappings as follows.

Theorem 1.5. *Let X be a separable Banach space and (Ω, β, μ) be a complete probability measure space. Let $T : \Omega \times X \to X$ be a continuous random operator such that for $\omega \in \Omega$, T satisfies*

$$\begin{aligned}\|T(\omega, x_1) - T(\omega, x_2)\| \leq a(\omega) \max\{\|x_1 - x_2\|, c(\omega)[\|x_1 - T(\omega, x_1)\| + \|x_2 - T(\omega, x_1)\|]\} \\ + b(\omega) \max\{\|x_1 - T(\omega, x_1)\|, \|x_2 - T(\omega, x_2)\|\}\end{aligned}$$

for all $x_1, x_2 \in X$ where $a(\omega), b(\omega), c(\omega)$ are real-valued random variables such that $0 < a(\omega) < 1$, $a(\omega) + b(\omega) = 1$, $c(\omega) \leq \frac{4-a(\omega)}{8-a(\omega)}$ almost surely. Then there exists a unique random fixed point of T in X.

The purpose of this paper is to prove a random fixed point theorem for a random $\mathcal{P}$-contraction operator. The paper is organized as follows. Sections 1 and 2 contains Introduction and Preliminaries, respectively. The main results are presented in section 3. The last section contains some application to a random non-linear integral equations.

2. Preliminaries

Let (X, β_X) be a separable Banach space, where β_X is a σ-algebra of Borel subsets of X, (Ω, β, μ) be a complete probability measure space. More details we refer to the paper of Joshi et.al. [61].

Definition 2.1.

(1) $x : \Omega \to X$ is called an X-*valued random variable* if $x^{-1}(B) \in \beta$ for any $B \in \beta_X$

(2) $x : \Omega \to X$ is called a *finitely valued random variable* if it is constant on any finite number of disjoint sets $A_i \in \beta$ and is equal to 0 over $\Omega \setminus (\bigcup_{i=1}^{n} A_i)$. The mapping x is said to be a *simple random variable* if it's finitely valued and $\mu\{\omega : \|x(\omega)\| > 0\} < \infty$.

(3) $x : \Omega \to X$ is called a *strong random variable* if there is a sequence of simple random variables $\{x_n(\omega)\}$ converges to $x(\omega)$ almost surely, that is, there is a set $A_0 \in \beta$ with $\mu(A_0) = 0$ so that $\lim_{n\to\infty} x_n(\omega) = x(\omega)$ for any $\omega \in \Omega \setminus A_0$.

(4) $x : \Omega \to X$ is called a *weak random variable* if the function $x^*(x(\cdot))$ is a real valued random variable for any $x^* \in X^*$, where X^* denotes the first normed dual space of X.

In a separable Banach space X, the notions of strong and weak random variables coincide ([61]).

Theorem 2.2. [61] *Assume $x, y : \Omega \to X$ be strong random variables and α, β be constants. Then the following assumption hold:*

(1) $\alpha x + \beta y$ is a strong random variable.

(2) If $f : \Omega \to \mathbb{R}$ is a real valued random variable, then fx is a strong random variable.

(3) If x_n is a sequence of strong random variables converges strongly to x almost surely, then x is a strong random variable.

Definition 2.3. Let Y be another Banach space.

(1) $F : \Omega \times X \to Y$ is called a *random mapping* if $F(\cdot, x)$ is a Y-valued random variable $\forall x \in X$.

(2) $F : \Omega \times X \to Y$ is called a *continuous random mapping* if $\mu(\{\omega \in \Omega : F(\omega, x)$ is a continuous function of $x\}) = 1$.

(3) $F : \Omega \times X \to Y$ is called a *demicontinuous* at $x \in X$ if $\|x_n - x\| \to 0$ implies $F(\cdot, x_n) \rightharpoonup F(\cdot, x)$ almost surely.

Theorem 2.4. [61] *Let $F : \Omega \times X \to Y$ be a demicontinuous random mapping where Y is a separable Banach space. Then, for any X-valued random variable x, the function $F(\cdot, x(\cdot))$ is a Y-valued random variable.*

Following Joshi et.al. [61], we recall some necessary Definitions and results:

Definition 2.5.

(1) $F(\omega, x(\omega)) = x(\omega)$ is said to be a *random fixed point equation*, where F is a random mapping.

(2) For each $x : \Omega \to X$ which satisfies the random fixed point equation almost surely is called a *wide sense solution* of the fixed point equation.

(3) For each X-valued random variable x which satisfies $\mu\{\omega : F(\omega, x(\omega)) = x(\omega)\} = 1$ is called a random fixed point of $F : \Omega \to X$.

3. The Main Results

In this section, we propose the definition of $\mathcal{P}$-contraction in the sense of random fixed point and prove the existence of a random fixed point for this contraction in a separable partially ordered Banach space.

Motivated and inspired by Definition 1.4 and Theorem 1.5 we propose the definition as follows.

Definition 3.1. Let $T : \Omega \times X \to X$ be a continuous random operator such that for $\omega \in \Omega$ almost surely, T is said to be a random $\mathcal{P}$-contraction if we have

$$\|T(\omega, x_1) - T(\omega, x_2)\| \leq \|x_1 - x_2\| - \varrho(\omega, x_1, x_2) \tag{3.1}$$

for all $x_1, x_2 \in X$ and $\varrho(\omega, \cdot, \cdot) : X \times X \to \mathbb{R}$ satisfies the condition in Definition 1.3.

Next, we prove the existence of a random fixed point of T in X.

Theorem 3.2. *Let X be a separable partially ordered Banach space and (Ω, β, μ) be a complete probability measure space. Let $T : \Omega \times X \to X$ be a continuous random operator such that for $\omega \in \Omega$ almost surely, T satisfies a random $\mathcal{P}$-contraction in Definition 3.1. Then there exists a random fixed point of T.*

Proof. Let $A = \{\omega \in \Omega : T(\omega, x)$ is a continuous function of $x\}$ and for $x_1, x_2 \in X$, $B_{x_1,x_2} = \{\omega \in \Omega : \|T(\omega, x_1) - T(\omega, x_2)\| \leq \|x_1 - x_2\| - \varrho(\omega, x_1, x_2)\}$.

Let S be a countable dense subset of X. Now, we prove that

$$\bigcap_{x_1,x_2 \in X}(B_{x_1,x_2} \cap A) = \bigcap_{s_1,s_2 \in S}(B_{s_1,s_2} \cap A).$$

Let $\omega \in \bigcap_{s_1,s_2 \in S}(B_{s_1,s_2} \cap A)$ for any $s_1, s_2 \in S$, we obtain

$$\begin{aligned}\|T(\omega, x_1) - T(\omega, x_2)\| &\leq \|T(\omega, x_1) - T(\omega, s_1)\| + \|T(\omega, s_2) - T(\omega, x_2)\| \\ &\quad + \|T(\omega, s_1) - T(\omega, s_2)\| \\ &\leq \|T(\omega, x_1) - T(\omega, s_1)\| + \|T(\omega, s_2) - T(\omega, x_2)\|) \\ &\quad + \|s_1 - s_2\| - \varrho(\omega, s_1, s_2) \\ &\leq \|T(\omega, x_1) - T(\omega, s_1)\| + \|T(\omega, s_2) - T(\omega, x_2)\|) \\ &\quad + \|s_1 - x_1\| + \|x_1 - x_2\| + \|x_2 - s_2\| - \varrho(\omega, s_1, s_2).\end{aligned} \tag{3.2}$$

Let $\varepsilon > 0$, we choose $\delta > 0$ so that

$$\|T(\omega, x) - T(\omega, y)\| < \varepsilon \qquad \text{whenever} \qquad \|x - y\| < \delta.$$

Choose sequences $\{s_{1n}\}$, $\{s_{2n}\}$ in S, we get

$$\|x_1 - s_{1n}\| < \delta = \frac{1}{n}$$

and

$$\|x_2 - s_{2n}\| < \delta = \frac{1}{n}$$

for each n. Plug in $\{s_{1n}\}$, $\{s_{2n}\}$ into (3.2), we obtain

$$\|T(\omega, x_1) - T(\omega, x_2)\| \leq 4\varepsilon + \|x_1 - x_2\| - \varrho(\omega, s_{1n}, s_{2n}).$$

Without loss of generality, we assume that $\{s_{1n}\}$ and $\{s_{2n}\}$ are comparable for each n. Thus, by the condition (b) in Definition 3.1, we get

$$\lim_{n\to\infty} \varrho(\omega, s_{1n}, s_{2n}) = \varrho(\omega, x_1, x_2).$$

Therefore,

$$\|T(\omega, x_1) - T(\omega, x_2)\| = 4\varepsilon + \|x_1 - x_2\| - \varrho(\omega, x_1, x_2).$$

As $\varepsilon > 0$ is arbitrary, if follows that

$$\|T(\omega, x_1) - T(\omega, x_2)\| \quad \leq \quad \|x_1 - x_2\| - \varrho(\omega, x_1, x_2). \tag{3.3}$$

Thus we have $\omega \in \bigcap_{x_1,x_2\in X}(B_{x_1,x_2} \cap A)$, which implies that

$$\textstyle\bigcap_{s_1,s_2\in S}(B_{s_1,s_2} \cap A) \subset \bigcap_{x_1,x_2\in X}(B_{x_1,x_2} \cap A).$$

Also, we have

$$\textstyle\bigcap_{x_1,x_2\in X}(B_{x_1,x_2} \cap A) \subset \bigcap_{s_1,s_2\in S}(B_{s_1,s_2} \cap A).$$

Therefore, we get

$$\textstyle\bigcap_{s_1,s_2\in S}(B_{s_1,s_2} \cap A) = \bigcap_{x_1,x_2\in X}(B_{x_1,x_2} \cap A).$$

Let $N' = \bigcap_{s_1,s_2\in S}(B_{s_1,s_2} \cap A)$. Then $\mu(N') = 1$. Next, we prove that $\forall\omega \in N'$, $T(\omega, x)$ is a deterministic continuous operators satisfying the mapping referred in [60].

Let $x : \Omega \to X$ be a random variable defined for some $x^* \in X$ by

$$x(\omega) = \begin{cases} x_\omega, & \omega \in N' \\ x^*, & \omega \notin N'. \end{cases}$$

Next, we show that $x(\omega)$ is the random variable. We construct a sequence of random variable $x_n(\omega)$ as follows. Let $x_0(\omega)$ be an arbitrary random variable and $x_1(\omega) = T(\omega, x_0(\omega))$. So $x_1(\omega)$ is a random variable. Next, we get $x_{n+1}(\omega) = T(\omega, x_n(\omega))$, by repeated generating, it gives that $\{x_n(\omega)\}_{n=1,2,\dots}$ is a random variables sequence converge to $x(\omega)$. Thus, $x(\omega)$ is a random variable.

Finally, we prove that $x(\omega)$ is a unique. Let $y : \Omega \to X$ be another random fixed point. We want to prove that $x(\omega) = y(\omega)$ almost surely. Let $M = \{\omega \in N' : x(\omega) = y(\omega)\}$. To prove $\mu(M) = 0$. Suppose $\mu(M) > 0$, thus $\mu(M \cap N') > 0$ implies $M \cap N' \neq \emptyset$, for all $\omega \in M \cap N'$. Let $\omega \in M \cap N'$, thus $x(\omega) \neq y(\omega)$. But $x(\omega)$ and $y(\omega)$ are fixed point of $T(\omega, \cdot) : X \to X$, thus $x(\omega) = y(\omega)$. So $\mu(M) = 0$ which is contradiction. Thus, $x(\omega)$ is a unique. Therefore, $x(\omega)$ is a unique random fixed point of T. This completes the proof. ∎

If we do not consider the $\mathcal{P}$ function, we obtain the corollary as follows.

Corollary 3.3. *Assume that* (Ω, β, μ) *be a complete probability measure space and* T *be a operator satisfying*

$$\|T(\omega, x_1) - T(\omega, x_2)\| < \|x_1 - x_2\|$$

for all $x_1, x_2 \in X$*, where* X *be a separable Banach space. Then a random fixed point of* T *exists in* X*.*

Proof. Suppose

$$A = \{\omega \in \Omega : T(\omega, x) \quad \text{is a continuous of} \quad x\}$$

and

$$B_{x_1,x_2} = \{\omega \in \Omega : \|T(\omega, x_1) - T(\omega, x_2)\| < \|x_1 - x_2\|\}.$$

Suppose S be a set of countable dense, $S \subset X$. Now, we prove that

$$\bigcap_{x_1,x_2 \in X} (B_{x_1,x_2} \cap A) = \bigcap_{s_1,s_2 \in S} (B_{s_1,s_2} \cap A).$$

Then for all $s_1, s_2 \in S$, we get

$$\|T(\omega, s_1) - T(\omega, s_2)\| < \|s_1 - s_2\|. \tag{3.4}$$

Next, following the proof in Theorem 3.2. ∎

4. Application to a Random Non-linear Integral Equation

Many years ago Professor Poom Kumam team studied many stochastics version of fixed point and also some related topics of random fixed point for some applications part you can see in the references therein see [38–47].

Now, we use Theorem 3.2 to show a solution of a non-linear stochastic integral equation exist in a Banach space. Assume that S is a locally compact metric space and (Ω, β, μ) is the probability measure space with β being σ-algebra and μ the probability measure. We can write this equation of the Hammerstein type ([12]) as follows:

$$x(t_1; \omega) = h(t_1; \omega) + \int_S k(t_1; t_2; \omega) f(t_2; x(t_2; \omega)) d\mu(t_2), \tag{4.1}$$

where

(a) d is a metric imposed on product cartesian of S;

(b) μ_0 is a complete σ-finite measure imposed on the collection of Borel subsets of S;

(c) $\omega \in \Omega$ where Ω is the supporting set of (Ω, β, μ);
(d) $x(t_1;\omega)$ is the unknown vector valued random variable for any $t_1 \in S$;
(e) $h(t_1;\omega)$ is the stochastic free term imposed for $t_1 \in S$;
(f) $k(t_1, t_2;\omega)$ is the stochastic kernel imposed for $t_1, t_2 \in S$;
(h) $f(t_1, x)$ is a vector valued function for $t_1 \in S$ and x.

Note that (4.1) is called a Bochner integral (see in [62]).

Next, we suppose that the union of a countable family $\{C_n\}$ of compact sets by $C_{n+1} \subset C_n$ is imposed as S so that, for each another compact set in S, there is C_i which contains it (see [63]).

We impose a space of all continuous functions from S into $L_2(\Omega, \beta, \mu)$ by $C = C(S, L_2(\Omega, \beta, \mu))$ by the topology of uniform convergence on compact sets of S, that is, $x(t_1;\omega)$ is a vector valued random variable for any fixed $t_1 \in S$ so that

$$\|x(t_1;\omega)\|^2_{L_2(\Omega,\beta,\mu)} = \int_\Omega |x(t_1;\omega)|^2 d\mu(\omega) < \infty.$$

Observe that $C(S, L_2(\Omega, \beta, \mu))$ is a locally convex space([62]) who topology is given by

$$\|x(t_1;\omega)\|_n = \sup_{t_1 \in C_n} \|x(t_1;\omega)\|_{L_2(\Omega,\beta,\mu)} \tag{4.2}$$

which is the countable family of semi-norms, for any $n \geq 1$. Moreover, because $L_2(\Omega, \beta, \mu)$ is complete, then $C(S, L_2(\Omega, \beta, \mu))$ is complete relative to (4.2).

Later, we impose a Banach space of all bounded continuous functions from S into $L_2(\Omega, \beta, \mu)$ by $BC = BC(S, L_2(\Omega, \beta, \mu))$ by the norm

$$\|x(t_1;\omega)\|_{BC} = \sup_{t_1 \in S} \|x(t_1;\omega)\|_{L_2(\Omega,\beta,\mu)}.$$

$BC \subset C$ is a space of all second order vector valued stochastic processes imposed on S which are bounded and continuous in mean square.

Now, we consider the functions $h(t_1;\omega)$ and $f(t_1, x(t_1;\omega))$ to be in $C(S, L_2(\Omega, \beta, \mu))$ space by respect to the stochastic kernel and suppose that, for any pair (t_1, t_2), $k(t_1, t_2;\omega) \in L_\infty(\Omega, \beta, \mu)$ and the norm denoted by

$$|||k(t_1, t_2;\omega)||| = \|k(t_1, t_2;\omega)\|_{L_\infty(\Omega,\beta,\mu)} = \mu - ess \sup_{\omega \in \Omega} |k(t_1, t_2;\omega)|.$$

Also, we assume that $k(t_1, t_2;\omega) \in L_\infty(\Omega, \beta, \mu)$ is so that

$$|||k(t_1, t_2;\omega)||| = \|x(t_2;\omega)\|_{L_2(\Omega,\beta,\mu)}$$

is μ-integrable by respect to t_2 for any $t_1 \in S$ and $x(t_2;\omega) \in C(S, L_2(\Omega, \beta, \mu))$ and there is a real valued function G μ-a.e. on S so that $G(S)\|x(t_2;\omega)\|_{L_2(\Omega,\beta,\mu))}$ is μ-integrable and, for any (t_1, t_2) in $S \times S$,

$$|||k(t_1, u;\omega) - k(t_2, u;\omega)||| \cdot \|x(u;\omega)\|_{L_2(\Omega,\beta,\mu)} \leq G(u)\|x(u;\omega)\|_{L_2(\Omega,\beta,\mu)} \quad \mu - a.e..$$

Later, suppose that, for almost everywhere $t_2 \in S$, $k(t_1, t_2;\omega)$ is continuous in t_1 from S into $L_\infty(\Omega, \beta, \mu)$.

Now, we impose the random integral operator T on $C(S, L_2(\Omega, \beta, \mu))$ by

$$(Tx)(t_1;\omega) = \int_S k(t_1, t_2;\omega)x(t_2;\omega)d\mu(t_2), \tag{4.3}$$

which is called a Bochner integral. By the assumptions on $k(t_1, t_2; \omega)$, it follows that, for each $t_1 \in S$, $(Tx)(t_1;\omega) \in L_2(\Omega, \beta, \mu)$ and $(Tx)(t_1;\omega)$ is continuous in mean square by Lebesgue's dominated convergence theorem, that is, $(Tx)(t_1;\omega) \in C(S, L_2(\Omega, \beta, \mu))$.

Lemma 4.1. [12] *The linear operator T defined by (4.3) is continuous from $C(S, L_2(\Omega, \beta, \mu))$ into itself.*

Definition 4.2. [64], [65] Let B and D be Banach spaces. The pair (B, D) is called *admissible* by respect to a linear operator T if $T(B) \subset D$.

Lemma 4.3. [12] *If T is a continuous linear operator from $C(S, L_2(\Omega, \beta, \mu))$ into itself and $B, D \subset C(S, L_2(\Omega, \beta, \mu))$ are Banach spaces stronger than $C(S, L_2(\Omega, \beta, \mu))$ so that (B, D) is admissible by respect to T, then T is continuous from B into D.*

By a *random solution* of (4.1), we mean a function

$$x(t_1;\omega) \in C(S, L_2(\Omega, \beta, \mu))$$

which satisfies (4.1) $\mu - a.e.$.

By using Theorem 3.2, we are now in state prove the theorem as follows.

Theorem 4.4. *If (4.1) is subject to the assumptions as follows:*

(1) B and D are Banach spaces stronger than $C(S, L_2(\Omega, \beta, \mu))$ so that (B, D) is admissible by respect to the integral operator imposed by (4.3);

(2) $x(t_1;\omega) \mapsto f(t_1, x(t_1;\omega))$ is an operator from $Q(\rho) = \{x(t_1;\omega) : x(t_1;\omega) \in D, \|x(t_1;\omega)\|_D \leq \rho\}$ into B satisfying

$$\|f(t_1, x_1(t_1,\omega)) - f(t_1, x_2(t_1,\omega))\|_B \leq \|x_1(t_1,\omega) - x_2(t_1,\omega)\| - \varrho(x_1(t_1,\omega), x_2(t_1,\omega)) \tag{4.4}$$

for any $x_1(t_1,\omega), x_2(t_1,\omega) \in Q(\rho)$;

(3) $h(t_1;\omega) \in D$,

then a unique stochastic solution of (4.1) exist in $Q(\rho)$ provided

$$\|h(t_1,\omega)\|_D + \varsigma(\omega)\|f(t_1, 0)\|_B \leq \rho(1 - \varsigma(\omega)),$$

where the norm of $T(\omega)$ is denoted by $\varsigma(\omega)$.

Proof. Let a mapping $\mathcal{U}(\omega) : Q(\rho) \to D$ defined by

$$(\mathcal{U}x)(t_1,\omega) = h(t_1,\omega) + \int_S k(t_1, t_2, \omega) f(s, x(t_2,\omega)) d_{\mu_0}(s).$$

Then we get

$$\begin{aligned} \|(\mathcal{U}x)(t_1,\omega)\|_D &\leq \|h(t_1,\omega)\|_D + \varsigma(\omega)\|f(t_1, x(t_1,\omega))\|_B \\ &= \|h(t_1,\omega)\|_D + \varsigma(\omega)\|f(t_1,0) + f(t_1, x(t_1,\omega)) - f(t_1,0)\|_B \\ &\leq \|h(t_1,\omega)\|_D + \varsigma(\omega)\|f(t_1,0)\|_B + \varsigma(\omega)\|f(t_1, x(t_1,\omega)) - f(t_1,0)\|_B. \end{aligned}$$

Thus, it follows by (4.4) that

$$\|f(t_1, x(t_1,\omega)) - f(t_1,0)\|_B < \|x(t_1,\omega)\|_D - \varrho(\omega, x(t_1,\omega), 0)$$

which implies that

$$\|f(t_1, x(t_1,\omega)) - f(t_1,0)\|_B < \|x(t_1,\omega)\|_D.$$

Therefore, we obtained

$$\|f(t_1, x(t_1,\omega)) - f(t_1,0)\|_B < \rho. \tag{4.5}$$

Thus, by (4.5), we have

$$\begin{aligned} \|(\mathcal{U}x)(t_1,\omega)\|_D &\leq \|h(t_1,\omega)\|_D + \varsigma(\omega)\|f(t_1,0)\|_B + \varsigma(\omega)\|f(t_1, x(t_1,\omega)) - f(t_1,0)\|_B \\ &< \|h(t_1,\omega)\|_D + \varsigma(\omega)\|f(t_1,0)\|_B + \varsigma(\omega)\rho \\ &< \rho \end{aligned} \tag{4.6}$$

and so, by (4.6), $(\mathcal{U}x)(t_1,\omega) \in Q(\rho)$. Thus, for any $x_1(t_1,\omega)$, $x_2(t_1,\omega) \in Q(\rho)$ and, by condition (2), we get

$$\begin{aligned} \|(\mathcal{U}x_1)(t_1,\omega) - (\mathcal{U}x_2)(t_1,\omega)\|_D &= \left\| \int_S k(t_1, t_2, \omega)[f(t_2, x_1(t_2,\omega)) - f(t_2, x_2(t_2,\omega))] d\mu_0(s) \right\|_D \\ &\leq \varsigma(\omega)\|f(t_2, x_1(t_2,\omega)) - f(t_2, x_2(t_2,\omega))\|_B \\ &\leq \|x_1(t_1,\omega) - x_2(t_1,\omega)\|_D. \end{aligned}$$

Consequently, $\mathcal{U}(\omega)$ is a random contraction mapping over $Q(\rho)$. Therefore, by Theorem 3.2, there is a unique $x^*(t_1,\omega) \in Q(\rho)$, which is a random fixed point of $\mathcal{U}$, i.e., x^* is a stochastic solution of equation (4.1). This completes the proof. ∎

Example 4.5. Consider the non-linear stochastic integral equation as follows:

$$x(t_1;\omega) = \int_0^\infty \frac{e^{-t_1-t_2}}{8(1+|x(t_2;\omega)|)} dt_2. \tag{4.7}$$

Next, we compare between equations (4.1) and (4.7), we get that $h(t_1;\omega) = 0$, $k(t_1;t_2;\omega) = \frac{1}{2}e^{-t_1-t_2}$ and $f(t_2; x(t_2;\omega)) = \frac{1}{4(1+|x(t_2;\omega)|)}$. Then, the equation (4.4) is hold.

Also, comparing with integral equation (4.3), we get that $\varsigma(\omega) = \frac{1}{2}$ which $\varsigma(\omega)$ is the norm of $T(\omega)$. Thus, all assumption of Theorem 4.4 are satisfied and therefore, random operator T has a random fixed point.

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