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Accelerated Hybrid Mann-type Algorithm for Fixed Point and Variational Inequality Problems

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ABSTRACT

The purpose of this paper is to establish and study an accelerated hybrid Mann-type algorithm for the fixed point of nonexpansive mappings and variational inequality problems of monotone operators with the Lipschitz condition. Based on the Mann algorithm that generates a new iterative vector by a convex combination of the previous two iterative vectors, the advantageous behavior in the construction of a new iterative vector was observed due to the convex combination of three iterative vectors. Furthermore, by combining with the method known as the inertial Tseng's extragradient method, the accelerated hybrid Mann-type algorithm was established. To demonstrate the efficiency and advantages of this new algorithm, we have created some numerical results to compare the advantages of different areas with the previous existing results.

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1. Introduction

Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$. Let C be a nonempty closed convex subset in H. Let $U : H \to H$ be a mapping. A point $x^* \in H$ is called a fixed point of U if $Ux^* = x^*$. The set of fixed points of U is denoted by $Fix(U)$.

A mapping $U : H \to H$ is said to be nonexpansive if $\|Ux - Uy\| \leq \|x - y\|$ for all $x, y \in H$. A mapping $U : H \to H$ with $Fix(U) \neq \emptyset$ is said to be quasi-nonexpansive if $\|Ux - p\| \leq \|x - p\|$ for all $x \in H$ and $p \in Fix(U)$.

Iterative method of fixed points of quasi-nonexpansive mappings has been studied and extended by many authors (see, for example, [9–12, 26, 33]). Notice that every nonexpansive mapping with a nonempty set of fixed points is a quasi-nonexpansive. It is well know that the fixed point problem for the mapping $U : H \to H$ is as follows:

$$\text{Find } x^* \in H \text{ such that } Ux^* = x^*.$$

Most of the problems in nonlinear analysis can be changed the forms to be the problems of finding a fixed point of a nonexpansive mapping and its generalizations. In 1953, Mann [22] created and introduced the explicit iteration procedure for a nonexpansive mapping as follows:

$$x_{n+1} = \mu_n x_n + (1 - \mu_n) U x_n, \quad n \geq 0, \tag{1.1}$$

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where $\{\mu_n\} \subseteq (0,1)$ satisfying $\sum_{n=1}^{\infty} \mu_n(1-\mu_n) = \infty$ if $Fix(U) \neq \emptyset$, then the sequence $\{x_n\}$ generated by (1.1) converges weakly to a fixed point of U.

Let $F : H \to H$ be an operator. The variational inequality problem (VIP) for F on C is to find a point $x^* \in C$ such that

$$\langle Fx^*, x - x^* \rangle \geq 0, \quad \forall x \in C. \tag{1.2}$$

The solution set of VIP (1.2) is denoted by $VI(C,F)$. Variational inequality problems are fundamental in a broad range of mathematical and applied sciences; the theoretical and algorithmic foundations as well as the applications of variational inequality problems have been extensively studied in the literature and continue to attract intensive research, see for instance [2, 13, 18, 19, 23, 36, 37, 39] and the extensive list of references there in.

There are several methods for finding the a common solution of fixed point and variational inequality problem such as the projected gradient method, extragradient method, subgradient extragradient method. Many authors have discovered and introduced several iterative methods for solving VIP (1.2). One of the easiest methods is the following projection method, which can be seen as an extension of the projected gradient method for optimization problems:

$$x_{n+1} = P_C(x_n - \tau F x_n) \tag{1.3}$$

where P_C is denoted by the metric projection from H onto C. Convergence results for (1.3) need F to be Lipschitz continuous with Lipschitz constant L and $\alpha-$strongly monotone and $\tau \in (0, (2\alpha/L^2))$. In [16], He et al. showed that if the strong monotonicity assumption is relaxed to the monotonicity, then the projected gradient method may diverge. Note that, method (1.3) also works for strongly pseudo-monotone VIPs and co-coercive VIPs. To deal with the weakness of the method defined by (1.3). Korpelevich [20] proposed the extragradient method. The method is of the form:

$$x_0 \in C, \quad y_n = P_C(x_n - \tau F x_n), \quad x_{n+1} = P_C(x_n - \tau F y_n) \tag{1.4}$$

where $F : H \to H$ is $L-$Lipschitz continuous and monotone, $\tau \in (0, (1/L))$. Korpelevich showed that if $VI(C,F)$ is nonempty then the sequence $\{x_n\}$ generated by (1.4) converges weakly to an element of $VI(C,F)$. To see the variant forms of the method (1.4), the reader could refer to the recent papers of He et al. [17], Gárciga Otero and Iuzem [14], Solodov and Svaiter [28], Solodov [27]. Recently, Censor et al. [6–8] introduced the subgradient extragradient method as follows:

$$y_n = P_C(x_n - \tau F x_n), \quad x_{n+1} = P_{T_n}(x_n - \tau F y_n) \tag{1.5}$$

where $T_n = \{x \in H \mid x_n - \tau F x_n - y_n, x - y_n \leq 0\}$ and $\tau \in (0, (1/L))$. In method (1.5), they replaced two projections onto C by one projection onto C and one onto a half-space.

In [35], Tseng presented the extragradient method as follows:

$$y_n = P_C(x_n - \tau F x_n), \quad x_{n+1} = y_n - \tau(F y_n - F x_n). \tag{1.6}$$

The method (1.6) and subgradient extragradient method need only to compute one projection onto C in each update. Later, the method (1.6) has gained attention and popularity to solve VIP from many authors (see, e.g. [4, 30, 31, 34, 38] and the references therein).

In 2019, Thong and Hieu [32] introduced some Mann-type algorithms for variational inequality and fixed point problems. They obtained new theorems and good behavior of the numerical results. One of the interesting main theorems is stated as follows:

Theorem 3.1. Let $F : H \to H$ be a monotone and $L-$Lipchitz mapping on H. Assume that the sequence $\{\mu_n\} \subseteq [0, \mu]$, $\mu < \frac{1}{3}$ is non-decreasing and $\{\alpha_n\} \subseteq (\alpha, 0.5], \alpha > 0$ is a sequence of real numbers. Let $\lambda \in (0, (1/L))$ and $U : H \to H$ be a quasi-nonexpansive mapping such that $I - U$ is demiclosed at zero and $Fix(U) \cap VI(C, F) \neq \varnothing$. Let $x_0, x_1 \in H$, the sequence $\{x_n\}$ is defined by

$$\begin{cases} w_n = x_n + \mu_n(x_n - x_{n-1}), \\ y_n = P_C(w_n - \lambda F w_n), \\ z_n = y_n - \lambda(F y_n - F w_n), \\ x_{n+1} = (1 - \alpha_n) w_n + \alpha_n U z_n. \end{cases}$$

Then the sequence $\{x_n\}$ converges weakly to an element of $Fix(U) \cap VI(C, F)$. Notice that the term $\mu_n(x_n - x_{n-1})$ is called an inertial extrapolation term by making use of the previous two iterates x_n and x_{n-1}. The inertial extrapolation term $\mu_n(x_n - x_{n-1})$ is employed in algorithm for the purpose of speeding up the rate of convergence of the algorithm. The vector $(x_n - x_{n-1})$ is acting as an impulsion term and μ_n is acting as a speed regulator (see, e.g. [21, 25]).

On the other hand, for observing the above method especially for the last line updating, we found an anonymous example in the Euclidean space $\mathbb{R}^2$ that provides some advantage geometrical structures of the convex combination of the previous three iterative vectors; w_n, z_n, Uz_n, for updating the new iterative vector x_{n+1}. It can be illustrated via the figures as below:

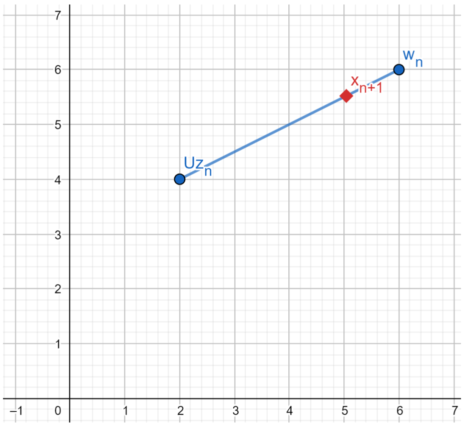

Fig. 1. x_{n+1} lies on a straight line formed by a convex combination of two vectors w_n and Uz_n.

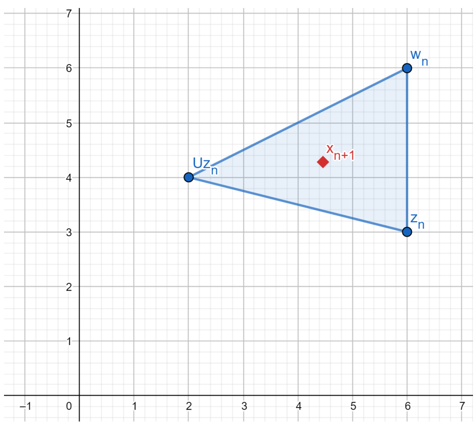

Fig. 2. x_{n+1} lies on a triangle formed by a convex combination of three vectors w_n, z_n and Uz_n.

It is explained by the visual indication of the geometric structure from Figure 1 and Figure 2 that the new vector x_{n+1} that obtained form the convex combination of three iterative vectors is likely to provide better performance than the convex combination of two iterative vectors.

Motivated by the directions mentioned above, in this paper, we aim to introduce and study a new accelerated hybrid Mann-type algorithm by using the convex combination of three iterative vectors for finding a solution of fixed point and variational inequality problems in the framework of Hilbert spaces. Further, we intend to establish some numerical experiments to illustrate the behavior of the new obtained algorithm. For representing the advantage of the main results, we have created some numerical results to compare advantages of different areas with the previous existing results.

2. Preliminaries

Let H be a real Hilbert space and C be a nonempty closed convex subset of H. The weak convergence of $\{x_n\}_{n=1}^{\infty}$ to x is denoted by $x_n \rightharpoonup x$ as $n \to \infty$ while the strong convergence of $\{x_n\}_{n=1}^{\infty}$ to x is written as $x_n \to x$ as $n \to \infty$. For each $x, y, z \in H$ and $\alpha, \beta, \gamma \in \mathbb{R}$ such that

$\alpha+\beta+\gamma=1$, we have

$$\|\alpha x+\beta y+\gamma z\|^2=\alpha\|x\|^2+\beta\|y\|^2+\gamma\|z\|^2-\alpha\beta\|x-y\|^2-\alpha\gamma\|x-z\|^2-\beta\gamma\|y-z\|^2. \tag{2.1}$$

For each point $x\in H$, there exists the unique nearest point in C, denoted by P_Cx such that $\|x-P_Cx\|=\inf_{y\in C}\|x-y\|\leq\|x-y\|,\quad\forall y\in C$. P_C is called the metric projection of H onto C. It is known that P_C is nonexpansive.

Lemma 2.1. [3, 5, 15] *Let C be a nonempty closed convex subset of a real Hilbert space H. Given $x\in H$ and $z\in C$. Then $z=P_Cx\Leftrightarrow\langle x-z,z-y\rangle\geq 0,\quad\forall y\in C$.*

Lemma 2.2. T31,T32,T33 *Let C be a closed and convex subset in a real Hilbert space H, $x\in H$. Then*

(1) $\|P_Cx-P_Cy\|^2\leq\langle P_Cx-P_Cy,x-y\rangle,\quad\forall y\in C$;

(2) $\|P_Cx-y\|^2\leq\|x-y\|^2-\|x-P_Cx\|^2,\quad\forall y\in C$.

Definition 2.3. [3, 5, 15] Assume that $T:H\to H$ is a nonlinear operator with $Fix(T)\neq\emptyset$. Then $I-T$ is said to be demiclosed at zero if for any $\{x_n\}$ in H, the following implication holds:

$$x_n\rightharpoonup x \text{ and } (I-T)x_n\to 0 \quad\Rightarrow\quad x\in Fix(T).$$

Definition 2.4. [3, 5, 15] Let $T:H\to H$ be an operator. Then

- T is called $L-$Lipschitz continuous with $L>0$ if

$$\|Tx-Ty\|\leq\|x-y\|,\quad\forall x,y\in H.$$

- T is called monotone if

$$\langle Tx-Ty,x-y\rangle\geq 0,\quad\forall x,y\in H.$$

Lemma 2.5. [1] *Let $\{\phi_n\},\{\delta_n\}$ and $\{\alpha_n\}$ be sequences in $[0,+\infty)$ such that*

$$\phi_{n+1}\leq\phi_n+\alpha_n(\phi_n-\phi_{n-1})+\delta_n,\quad\forall n\geq 1,\quad\sum_{n=1}^{+\infty}\delta_n<+\infty,$$

and there exists a real number α with $0\leq\alpha_n\leq\alpha<1$ for all $n\in\mathbb{N}$. Then the following hold:

(1) $\sum_{n=1}^{+\infty}[\phi_n-\phi_{n-1}]_+<+\infty$, where $[t]_+:=\max\{t,0\}$;

(2) there exist $\phi^*\in[0,+\infty)$ such that $\lim_{n\to+\infty}\phi_n=\phi^*$.

Definition 2.6. Let H be a real Hilbert space. Then the set

$$\{z\in H\mid\exists\{x_{n_k}\}\subseteq\{x_n\}\text{ such that } x_{n_k}\rightharpoonup z\}$$

is called the set of all **sequential weak cluster point** of $\{x_n\}$.

Lemma 2.7. [24] *Let C be a nonempty set of H and $\{x_n\}$ be a sequence in H such that the following two conditions hold:*

(1) for every $x\in C$, $\lim_{n\to\infty}\|x_n-x\|$ exists;

(2) every sequential weak cluster point of $\{x_n\}$ is in C. Then $\{x_n\}$ converges weakly to a point in C.

Lemma 2.8. [29] *Assume that $F:C\to H$ is a continuous and monotone operator. Then x^* is a solution of (1.2) if and only if x^* is a solution of the following problem:*

$$\textit{find } x\in C \textit{ such that } \langle Fy,y-x\rangle\geq 0,\quad\forall y\in C.$$

3. Main Results

In this section, we introduce the new Mann-type algorithm called the accelerated hybrid Mann-type algorithm for solving some fixed point problems of a quasi-nonexpansive mapping and variational inequality problems of a monotone and $L-$Lipchitz mapping in the frame work of real Hilbert spaces.

Theorem 3.1. *Let $F : H \to H$ be a monotone and $L-$Lipchitz mapping on H. Assume that the sequence $\{\mu_n\} \subseteq [0, \mu]$, $\mu < \frac{1}{5}$ is non-decreasing, $\{\alpha_n\} \subseteq (\alpha, 0.5], \alpha > 0$, $\{\beta_n\} \subseteq [0, 0.5]$ and $\{\gamma_n\} \subseteq [0.5, 1)$ is a sequence of real numbers. Let $\lambda \in (0, (1/L))$ and $U : H \to H$ be a quasi-nonexpansive mapping such that $I - U$ is demiclosed at zero and $Fix(U) \cap VI(C, F) \neq \emptyset$. Let $x_0, x_1 \in H$, the sequence $\{x_n\}$ is defined by*

$$\begin{cases} w_n = x_n + \mu_n(x_n - x_{n-1}), \\ y_n = P_C(w_n - \lambda F w_n), \\ z_n = y_n - \lambda(F y_n - F w_n), \\ x_{n+1} = \gamma_n w_n + \beta_n z_n + \alpha_n U z_n, \end{cases} \tag{3.1}$$

where $\alpha_n + \beta_n + \gamma_n = 1$. Then the sequence $\{x_n\}$ converges weakly to an element of $Fix(U) \cap VI(C, F)$.

Proof. We split the proof into three claims. Let $x^* \in Fix(U) \cap VI(C, F)$.
Claim 1.

$$\|z_n - x^*\|^2 \leq \|w_n - x^*\|^2 - (1 - \lambda^2 L^2)\|y_n - w_n\|^2, \quad \forall n \in \mathbb{N}. \tag{3.2}$$

We have

$$\begin{aligned} \|z_n - x^*\|^2 &= \|y_n - \lambda(F y_n - F w_n) - x^*\|^2 \\ &= \|y_n - x^*\|^2 + \lambda^2\|F y_n - F w_n\|^2 - 2\lambda\langle y_n - x^*, F y_n - F w_n\rangle \\ &= \|w_n - x^*\|^2 + \|w_n - y_n\|^2 + 2\langle y_n - w_n, w_n - x^*\rangle \\ &\quad + \lambda^2\|F y_n - F w_n\|^2 - 2\lambda\langle y_n - x^*, F y_n - F w_n\rangle \\ &= \|w_n - x^*\|^2 + \|w_n - y_n\|^2 - 2\langle y_n - w_n, y_n - w_n\rangle \\ &\quad + 2\langle y_n - w_n, y_n - x^*\rangle + \lambda^2\|F y_n - F w_n\|^2 \\ &\quad - 2\lambda\langle y_n - x^*, F y_n - F w_n\rangle \\ &= \|w_n - x^*\|^2 - \|w_n - y_n\|^2 + 2\langle y_n - w_n, y_n - x^*\rangle \\ &\quad + \lambda^2\|F y_n - F w_n\|^2 - 2\lambda\langle y_n - x^*, F y_n - F w_n\rangle. \end{aligned} \tag{3.3}$$

Since $y_n = P_C(w_n - \lambda F w_n)$, we get

$$\langle y_n - w_n + \lambda F w_n, y_n - x^*\rangle \leq 0,$$

equivalently

$$\langle y_n - w_n, y_n - x^*\rangle \leq -\lambda\langle F w_n, y_n - x^*\rangle. \tag{3.4}$$

Combining (3.3) and (3.4), we obtain

$$\begin{aligned} \|z_n - x^*\|^2 \leq\ & \|w_n - x^*\|^2 - \|w_n - y_n\|^2 - 2\lambda\langle F w_n, y_n - x^*\rangle \\ & + \lambda^2\|F y_n - F w_n\|^2 - 2\lambda\langle y_n - x^*, F y_n - F w_n\rangle \\ =\ & \|w_n - x^*\|^2 - \|w_n - y_n\|^2 + \lambda^2\|F y_n - F w_n\|^2 - 2\lambda\langle y_n - x^*, F y_n\rangle \end{aligned}$$

$$\begin{aligned}
&\leq \|w_n - x^*\|^2 - \|w_n - y_n\|^2 + \lambda^2 L^2 \|y_n - w_n\|^2 \\
&\quad - 2\lambda\langle y_n - x^*, Fy_n - Fx^*\rangle - 2\lambda\langle y_n - x^*, Fx^*\rangle \\
&\leq \|w_n - x^*\|^2 - (1-\lambda^2 L^2)\|y_n - w_n\|^2. \qquad (3.5)
\end{aligned}$$

Claim 2.

$$\lim_{n\to\infty} \|Uz_n - z_n\| = 0. \qquad (3.6)$$

From (3.2), we have

$$\|z_n - x^*\| \leq \|w_n - x^*\|. \qquad (3.7)$$

Consider $\|x_{n+1} - w_n\|^2 = \|\gamma_n w_n + \beta_n z_n + \alpha_n U z_n - w_n\|^2$ and (2.1), we have

$$\begin{aligned}
\|x_{n+1} - w_n\|^2 &= \|\gamma_n w_n + \beta_n z_n + \alpha_n U z_n - w_n\|^2 \\
&= \|\gamma_n (w_n - w_n) + \beta_n (z_n - w_n) + \alpha_n (U z_n - w_n)\|^2 \\
&\leq \beta_n \|z_n - w_n\|^2 + \alpha_n \|U z_n - w_n\|^2.
\end{aligned}$$

Using (2.1) and (3.7) we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \|\gamma_n w_n + \beta_n z_n + \alpha_n U z_n - x^*\|^2 \\
&= \|\gamma_n (w_n - x^*) + \beta_n (z_n - x^*) + \alpha_n (U z_n - x^*)\|^2 \\
&= \gamma_n \|w_n - x^*\|^2 + \beta_n \|z_n - x^*\|^2 + \alpha_n \|U z_n - x^*\|^2 \\
&\quad - \gamma_n \beta_n \|z_n - w_n\|^2 - \gamma_n \alpha_n \|U z_n - w_n\|^2 - \beta_n \alpha_n \|U z_n - z_n\|^2 \\
&\leq \gamma_n \|w_n - x^*\|^2 + \beta_n \|z_n - x^*\|^2 + \alpha_n \|z_n - x^*\|^2 \\
&\quad - \gamma_n \beta_n \|z_n - w_n\|^2 - \gamma_n \alpha_n \|U z_n - w_n\|^2 - \beta_n \alpha_n \|U z_n - z_n\|^2 \\
&\leq \gamma_n \|w_n - x^*\|^2 + \beta_n \|w_n - x^*\|^2 + \alpha_n \|w_n - x^*\|^2 \\
&\quad - \gamma_n \beta_n \|z_n - w_n\|^2 - \gamma_n \alpha_n \|U z_n - w_n\|^2 - \beta_n \alpha_n \|U z_n - z_n\|^2 \\
&= \|w_n - x^*\|^2 - \gamma_n \beta_n \|z_n - w_n\|^2 - \gamma_n \alpha_n \|U z_n - w_n\|^2 \\
&\quad - \beta_n \alpha_n \|U z_n - z_n\|^2 \\
&\leq \|w_n - x^*\|^2 - \gamma_n \beta_n \|z_n - w_n\|^2 - \gamma_n \alpha_n \|U z_n - w_n\|^2 \\
&= \|w_n - x^*\|^2 - \gamma_n \left(\beta_n \|z_n - w_n\|^2 + \alpha_n \|U z_n - w_n\|^2 \right) \\
&\leq \|w_n - x^*\|^2 - \gamma_n \|x_{n+1} - w_n\|^2. \qquad (3.8)
\end{aligned}$$

Moreover

$$\begin{aligned}
\|w_n - x^*\|^2 &= \|(1+\mu_n)(x_n - x^*) - \mu_n (x_{n-1} - x^*)\|^2 \\
&= (1+\mu_n)\|x_n - x^*\|^2 - \mu_n \|x_{n-1} - x^*\|^2 + \mu_n (1+\mu_n)\|x_n - x_{n-1}\|^2. \qquad (3.9)
\end{aligned}$$

We also have

$$\begin{aligned}
\|x_{n+1} - w_n\|^2 &= \|x_{n+1} - x_n - \mu_n (x_n - x_{n-1})\|^2 \\
&= \|x_{n+1} - x_n\|^2 + \mu_n^2 \|x_n - x_{n-1}\|^2 - 2\mu_n \langle x_{n+1} - x_n, x_n - x_{n-1}\rangle \\
&\geq \|x_{n+1} - x_n\|^2 + \mu_n^2 \|x_n - x_{n-1}\|^2 - 2\mu_n \|x_{n+1} - x_n\| \|x_n - x_{n-1}\| \\
&\geq (1-\mu_n)\|x_{n+1} - x_n\|^2 + (\mu_n^2 - \mu_n)\|x_n - x_{n-1}\|^2. \qquad (3.10)
\end{aligned}$$

Combining (3.8), (3.9) and (3.10) we obtain

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\le (1+\mu_n)\|x_n-x^*\|^2-\mu_n\|x_{n-1}-x^*\|^2+(1+\mu_n)\mu_n\|x_n-x_{n-1}\|^2\\ &\quad -\gamma_n(1-\mu_n)\|x_{n+1}-x_n\|^2-\gamma_n(\mu_n^2-\mu_n)\|x_n-x_{n-1}\|^2\\ &= (1+\mu_n)\|x_n-x^*\|^2-\mu_n\|x_{n-1}-x^*\|^2-\gamma_n(1-\mu_n)\|x_{n+1}-x_n\|^2\\ &\quad (\mu_n+\mu_n^2-\gamma_n\mu_n^2+\gamma_n\mu_n)\|x_n-x_{n-1}\|^2\\ &\le (1+\mu_n)\|x_n-x^*\|^2-\mu_n\|x_{n-1}-x^*\|^2-\gamma_n(1-\mu_n)\|x_{n+1}-x_n\|^2\\ &\quad ((1-\gamma_n)\mu_n^2+(1+\gamma_n)\mu_n)\|x_n-x_{n-1}\|^2\\ &\le (1+\mu_n)\|x_n-x^*\|^2-\mu_n\|x_{n-1}-x^*\|^2\\ &\quad -\gamma_n(1-\mu_n)\|x_{n+1}-x_n\|^2+2\mu_n\|x_n-x_{n-1}\|^2\\ &\le (1+\mu_{n+1})\|x_n-x^*\|^2-\mu_n\|x_{n-1}-x^*\|^2\\ &\quad -\gamma_n(1-\mu_n)\|x_{n+1}-x_n\|^2+2\mu_n\|x_n-x_{n-1}\|^2.\end{aligned} \tag{3.11}$$

This follows that

$$\begin{aligned}&\|x_{n+1}-x^*\|^2-\mu_{n+1}\|x_n-x^*\|^2+2\mu_{n+1}\|x_{n+1}-x_n\|^2\\ &\le \|x_n-x^*\|^2-\mu_n\|x_{n-1}-x^*\|^2\\ &\quad +2\mu_n\|x_n-x_{n-1}\|^2+2\mu_{n+1}\|x_{n+1}-x_n\|^2-\gamma_n(1-\mu_n)\|x_{n+1}-x_n\|^2.\end{aligned}$$

Put $\Lambda_n := \|x_n-x^*\|^2-\mu_n\|x_{n-1}-x^*\|^2+2\mu_n\|x_n-x_{n-1}\|^2$. We get

$$\Lambda_{n+1}-\Lambda_n \le -(\gamma_n(1-\mu_n)-2\mu_{n+1})\|x_{n+1}-x_n\|^2.$$

It follows from $\mu_n \le \mu < \frac{1}{5}$ that $\gamma_n(1-\mu_n)-2\mu_{n+1} \ge 0.5-2.5\mu > 0$. Therefore, we obtain

$$\Lambda_{n+1}-\Lambda_n \le -\delta\|x_{n+1}-x_n\|^2 \le 0 \tag{3.12}$$

where $\delta = 0.5-2.5\mu$. This implies that the sequence $\{\Lambda_n\}$ is nonincreasing. And we have

$$\begin{aligned}\Lambda_n &= \|x_n-x^*\|^2-\mu_n\|x_{n-1}-x^*\|^2+2\mu_n\|x_n-x_{n-1}\|^2\\ &\ge \|x_n-x^*\|^2-\mu_n\|x_{n-1}-x^*\|^2.\end{aligned}$$

This implies that

$$\begin{aligned}\|x_n-x^*\|^2 &\le \mu_n\|x_{n-1}-x^*\|^2+\Lambda_n\\ &\le \mu\|x_{n-1}-x^*\|^2+\Lambda_1\\ &\le \ldots \le \mu^n\|x_0-x^*\|^2+\Lambda_1(\mu^{n-1}+\ldots+1)\\ &\le \mu^n\|x_0-x^*\|^2+\frac{\Lambda_1}{1-\mu}.\end{aligned} \tag{3.13}$$

We have

$$\begin{aligned}\Lambda_{n+1} = &\ \|x_{n+1}-x^*\|^2-\mu_{n+1}\|x_n-x^*\|^2+2\mu_{n+1}\|x_{n+1}-x^*\|^2\\ \ge &\ -\mu_{n+1}\|x_n-x^*\|^2.\end{aligned} \tag{3.14}$$

From (3.13) and (3.14) we obtain

$$-\Lambda_{n+1} \le \mu_{n+1}\|x_n-x^*\|^2 \le \mu\|x_n-x^*\|^2 \le \mu^{n+1}\|x_0-x^*\|^2+\frac{\mu\Lambda_1}{1-\mu}.$$

It follows from (3.12) that

$$\delta \sum_{n=1}^{k} \|x_{n+1} - x_n\|^2 \leq \Lambda_1 - \Lambda_{k+1} \leq \mu^{k+1}\|x_0 - x^*\|^2 + \frac{\Lambda_1}{1-\mu}$$
$$\leq \|x_0 - x^*\|^2 + \frac{\Lambda_1}{1-\mu}.$$

This implies

$$\sum_{n=1}^{\infty} \|x_{n+1} - x_n\|^2 < +\infty. \tag{3.15}$$

We obtain

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.16}$$

We have

$$\begin{aligned} \|x_{n+1} - w_n\| &= \|x_{n+1} - x_n - \mu_n(x_n - x_{n-1})\| \leq \|x_{n+1} - x_n\| + \mu_n \|x_n - x_{n-1}\| \\ &\leq \|x_{n+1} - x_n\| + \mu \|x_n - x_{n-1}\|. \end{aligned} \tag{3.17}$$

From (3.16) and (3.17) we obtain

$$\lim_{n\to\infty} \|x_{n+1} - w_n\| = 0.$$

From (3.11) we get

$$\|x_{n+1} - x^*\|^2 \leq (1+\mu_n)\|x_n - x^*\|^2 - \mu_n\|x_{n-1} - x^*\|^2 + 2\mu\|x_n - x_{n-1}\|^2. \tag{3.18}$$

By (3.15), (3.18) and Lemma 2.7 we have

$$\lim_{n\to\infty} \|x_n - x^*\| = l. \tag{3.19}$$

And by (3.9) we obtain

$$\lim_{n\to\infty} \|w_n - x^*\| = l. \tag{3.20}$$

We also have

$$0 \leq \|x_n - w_n\| \leq \mu \|x_n - x_{n-1}\| \to 0. \tag{3.21}$$

From

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\leq \gamma_n\|w_n - x^*\|^2 + \beta_n\|z_n - x*\|^2 + \alpha_n\|Uz_n - x*\|^2 \\ &\leq \gamma_n\|w_n - x^*\|^2 + \beta_n\|z_n - x^*\|^2 + \alpha_n\|z_n - x^*\|^2 \\ &= (1-(\alpha_n+\beta_n))\|w_n - x^*\|^2 + (\alpha_n+\beta_n)\|z_n - x^*\|^2. \end{aligned}$$

This implies that

$$\begin{aligned} \|z_n - x^*\|^2 &\geq \frac{\|x_{n+1} - x^*\|^2 - \|w_n - x^*\|^2}{(\alpha_n+\beta_n)} + \|w_n - x^*\|^2 \\ &> \frac{\|x_{n+1} - x^*\|^2 - \|w_n - x^*\|^2}{\alpha} + \|w_n - x^*\|^2. \end{aligned} \tag{3.22}$$

It implies from (3.19), (3.20) and (3.22) that

$$\lim_{n\to\infty} \|z_n - x^*\|^2 \geq \lim_{n\to\infty} \|w_n - x^*\|^2 = l. \tag{3.23}$$

By (3.7) we get

$$\lim_{n\to\infty} \|z_n - x^*\|^2 \le \lim_{n\to\infty} \|w_n - x^*\|^2 = l. \tag{3.24}$$

Combining (3.23) and (3.24) we obtain

$$\lim_{n\to\infty} \|z_n - x^*\|^2 = l.$$

From (3.5) we have

$$(1-\lambda^2L^2)\|y_n - w_n\|^2 \le \|w_n - x^*\|^2 - \|z_n - x^*\|^2.$$

This implies that

$$\lim_{n\to\infty} \|y_n - w_n\| = 0. \tag{3.25}$$

It also holds

$$\|z_n - y_n\| = \lambda \|Fy_n - Fw_n\| \le \lambda L \|y_n - w_n\| \to 0. \tag{3.26}$$

Combining (3.25) and (3.26) we obtain

$$\lim_{n\to\infty} \|z_n - w_n\| = 0. \tag{3.27}$$

From

$$Uz_n - w_n = \frac{1}{\alpha_n}(x_{n+1} - w_n - \beta_n(z_n - w_n))$$

we have

$$\|Uz_n - w_n\| = \left\|\frac{1}{\alpha_n}(x_{n+1} - w_n - \beta_n(z_n - w_n))\right\| \le \frac{1}{\alpha_n}\|x_{n+1} - w_n\| + \frac{\beta_n}{\alpha_n}\|z_n - w_n\|. \tag{3.28}$$

From $\alpha_n \ge \alpha$, it follows from (3.16), (3.27) and (3.28) that

$$\lim_{n\to\infty} \|Uz_n - w_n\| = 0. \tag{3.29}$$

Combining (3.27) and (3.29) we obtain

$$\|Uz_n - z_n\| \le \|Uz_n - w_n\| + \|z_n - w_n\| \to 0.$$

Claim 3. The sequence $\{x_n\}$ converges weakly to an element of $Fix(U) \cap VI(C,F)$. Indeed, since $\{x_n\}$ is a bounded sequence, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $z \in H$ such that $x_{n_k} \rightharpoonup z$. By (3.21) we get $w_{n_k} \rightharpoonup z$ and by (3.27) $z_{n_k} \rightharpoonup z$. It follows from (3.6) and demiclosedness of $I - U$ that $z \in Fix(U)$.

From $y_{n_k} = P_C(w_{n_k} - \lambda F w_{n_k})$ and F is monotone, we have for every $x \in C$ that

$$\begin{aligned}
0 &\le \langle y_{n_k} - w_{n_k} + \lambda F w_{n_k}, x - y_{n_k}\rangle \\
&= \langle y_{n_k} - w_{n_k}, x - y_{n_k}\rangle + \lambda \langle F w_{n_k}, x - y_{n_k}\rangle \\
&= \langle y_{n_k} - w_{n_k}, x - y_{n_k}\rangle + \lambda \langle F w_{n_k}, w_{n_k} - y_{n_k}\rangle + \lambda \langle F w_{n_k}, x - w_{n_k}\rangle \\
&\le \langle y_{n_k} - w_{n_k}, x - y_{n_k}\rangle + \lambda \langle F w_{n_k}, w_{n_k} - y_{n_k}\rangle + \lambda \langle Fx, x - w_{n_k}\rangle.
\end{aligned}$$

Passing to the limit, we get

$$\langle Fx, x - z\rangle \ge 0 \quad \forall x \in C.$$

By Lemma 2.8 we have $z \in VI(C,F)$. Therefore, we have shown that for every $x^* \in Fix(U) \cap VI(C,F)$, $\lim_{n\to\infty} \|x_n - x^*\|$ exists and each sequential weak cluster point of sequence $\{x_n\}$ is in $Fix(U) \cap VI(C,F)$. By Lemma 2.7 the sequence $\{x_n\}$ converges weakly to $z \in Fix(U) \cap VI(C,F)$. ∎

Corollary 3.2 (Thong and Hieu [32, Theorem 3.1]). *Let $F : H \to H$ be a monotone and $L-$Lipchitz mapping on H. Assume that the sequence $\{\mu_n\} \subseteq [0, \mu]$, $\mu < \frac{1}{5}$ is non-decreasing, $\{\alpha_n\} \subseteq (\alpha, 0.5]$, $\alpha > 0$ is a sequence of real numbers. Let $\lambda \in (0, (1/L))$ and $U : H \to H$ be a quasi-nonexpansive mapping such that $I - U$ is demiclosed at zero and $Fix(U) \cap VI(C, F) \neq \varnothing$. Let $x_0, x_1 \in H$ the sequence $\{x_n\}$ is defined by*

$$\begin{cases} w_n = x_n + \mu_n(x_n - x_{n-1}) \\ y_n = P_C(w_n - \lambda F w_n) \\ z_n = y_n - \lambda(F y_n - F w_n) \\ x_{n+1} = (1 - \alpha_n) w_n + \alpha_n U z_n. \end{cases} \tag{3.30}$$

Then the sequence $\{x_n\}$ converges weakly to an element of $Fix(U) \cap VI(C, F)$.

Proof. If we set $\beta_n = 0$ for all $n \in \mathbb{N}$, then $\gamma_n = 1 - \alpha_n$. Therefore, Theorem 3.1 can be reduced to Corollary 3.2 as required. ∎

4. Numerical Experiments

In this section, we compare the advantages of the new algorithm with the previous exiting algorithm introduced by Thong and Hieu [32, Theorem 3.1].

Example 4.1. [32] Let $H = \mathbb{R}$, $C = [-2, 5]$ and $F : \mathbb{R} \to \mathbb{R}$ be given by

$$Fx := x - 3 + \sin(x - 3)$$

and $U : \mathbb{R} \to \mathbb{R}$ be given by

$$Ux = \frac{x + 3}{\frac{x^2}{9} + 1} \quad \forall x \in \mathbb{R}.$$

The solution of the problem is $x^* = 3$. The stopping criterion is defined by Error $= \|x_{n+1} - x_n\| < 10^{-4}$. Choose $x_0 = 5$ and $x_1 = 4$. Figure 3 and figure 4 show a comparison of the numerical behavior of an accelerated hybrid Mann-type algorithm (3.1) with an advantage over Mann-type algorithm (3.30).

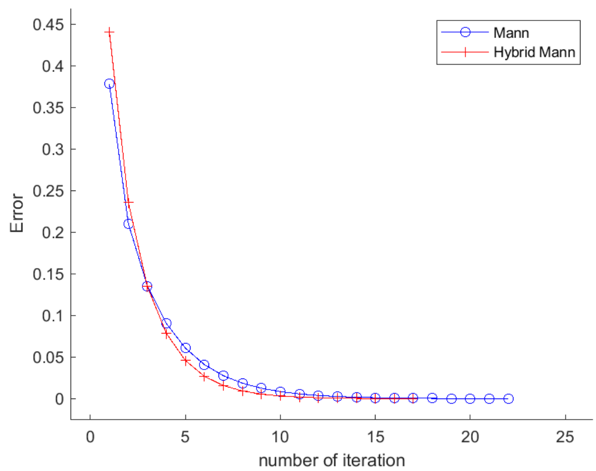

Fig. 3. Convergence behavior of $\{x_n\}$ of Example 4.1.

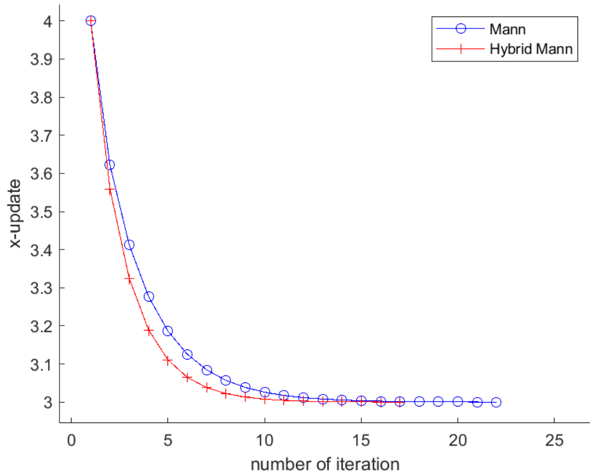

Fig. 4. $x-$update converges to solution x^* of Example 4.1.

Example 4.2. [32] Consider a nonlinear operator $F : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$F(x, y) = (x + y + \sin x, -x + y + \sin y)$$

and the feasible set C is a box defined by $C = [-2, 5] \times [-2, 5]$. Let E be a 2×2 matrix defined by

$$E = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

Mapping $U : \mathbb{R}^2 \to \mathbb{R}^2$ by $Uz = \|E\|^{-1}Ez$, where $z = (x, y)^T$. The solution of the problem is $x^* = (0, 0)^T$. The stopping criterion is defined by Error $= \|x_{n+1} - x_n\| < 10^{-4}$. Choose $x_0 = (7, 7)^T$ and $x_1 = (4, 3)^T$. By using this example, Figure 5 - Figure 9 show the advantage of (3.1) via numerical results.

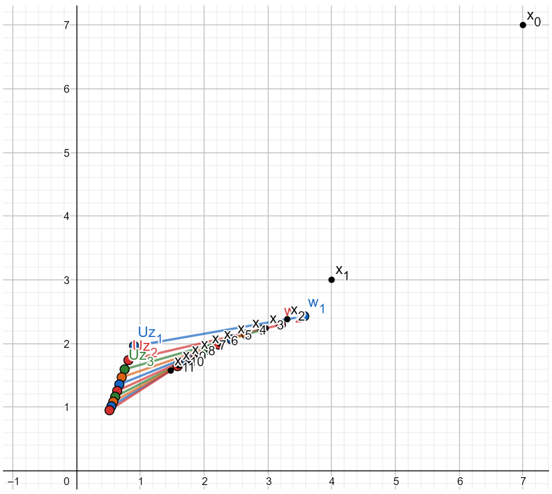

Fig. 5. The behavior of each x-update which lies on a straight line formed by a convex combination of two iterative vectors w_n and Uz_n.

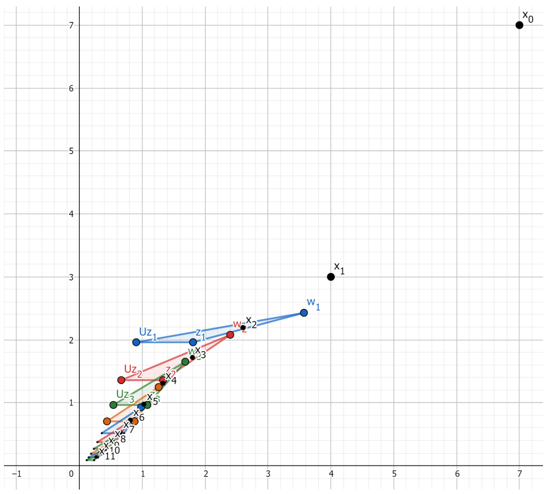

Fig. 6. The behavior of each x-update which lies on a triangle formed by a convex combination of three iterative vectors w_n, z_n and Uz_n.

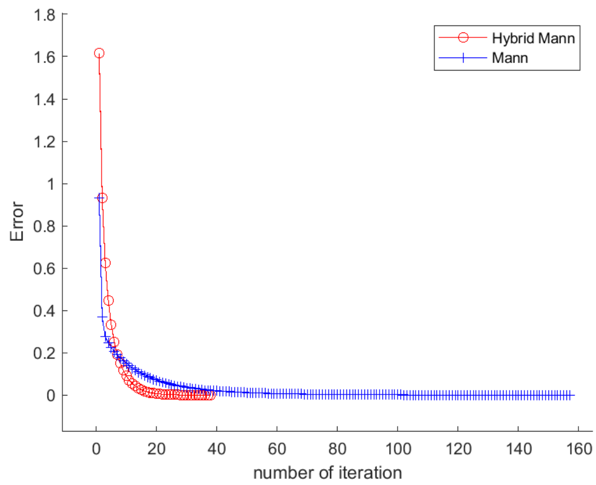

Fig. 7. Convergence behavior of $\{x_n\}$ of Example 4.2.

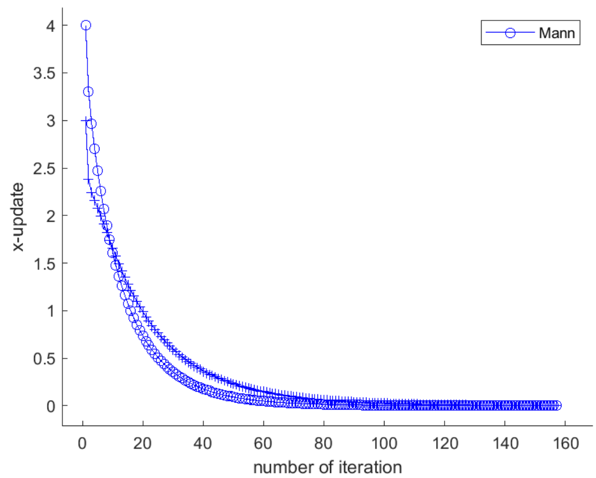

Fig. 8. Mann-type: x-update converges to solution $x^* = (0,0)^T$ of Example 4.2.

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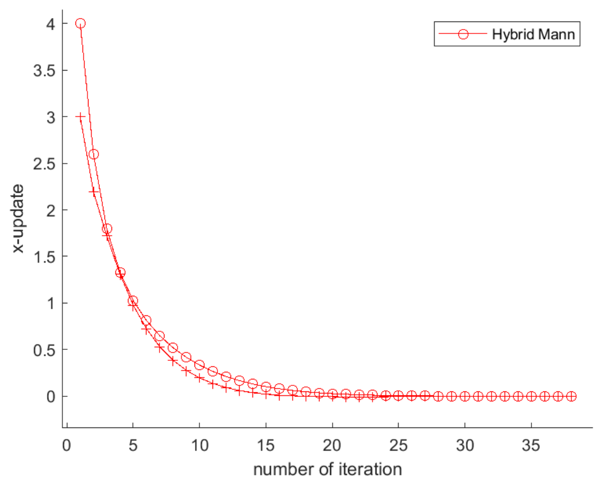

Fig. 9. Hybrid Mann-type: x-update converges to solution $x^* = (0,0)^T$ of Example 4.2.

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5. Conclusions

We introduced and studied the new Mann-type algorithm which is called the accelerated hybrid Mann-type algorithm and established the main theorem as follows:

Theorem 3.1. Let $F : H \to H$ be a monotone and $L-$Lipchitz mapping on H. Assume that the sequence $\{\mu_n\} \subseteq [0, \mu]$, $\mu < \frac{1}{5}$ is non-decreasing, $\{\alpha_n\} \subseteq (\alpha, 0.5], \alpha > 0$, $\{\beta_n\} \subseteq [0, 0.5]$ and $\{\gamma_n\} \subseteq [0.5, 1)$ is a sequence of real numbers. Let $\lambda \in (0, (1/L))$ and $U : H \to H$ be a quasi-nonexpansive mapping such that $I - U$ is demiclosed at zero and $Fix(U) \cap VI(C, F) \neq \varnothing$. Let $x_0,\ x_1 \in H$, the sequence $\{x_n\}$ is defined by

$$\begin{cases} w_n = x_n + \mu_n(x_n - x_{n-1}), \\ y_n = P_C(w_n - \lambda F w_n), \\ z_n = y_n - \lambda(F y_n - F w_n), \\ x_{n+1} = \gamma_n w_n + \beta_n z_n + \alpha_n U z_n, \end{cases}$$

where $\alpha_n + \beta_n + \gamma_n = 1$. Then the sequence $\{x_n\}$ converges weakly to an element of $Fix(U) \cap VI(C, F)$.

The above theorem not only extends the theoretical concepts of the previous research work, but also provides numerical results that have an advantage over the previous work proposed by Thong and Hieu [32, Theorem 3.1]. It can be clearly seen in section 4 of this paper.

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