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Idempotent and Regular Elements on Some Semigroups of the Generalized Cohypersubstitutions of type $\tau = (2)$

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ABSTRACT

A generalized cohypersubstitution σ of type $\tau = (n_i)_{i\in I}$ is a mapping which maps every n_i-ary cooperation symbol f_i to the coterm $\sigma(f)$ of type τ. We denoted the set of all generalized cohypersubstitutions of type τ by $Cohyp_G(\tau)$. In this study, we focus on the semigroups $(Cohyp_G(2), +_{CG})$ and $(Cohyp_G(2), \oplus_{CG})$ where $+_{CG}$ and $\oplus_{CG}$ are binary operations the set $Cohyp_G(2)$. We characterize the set of all idempotent and regular elements of these semigroups.

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1. Introduction

The concept of cohypersubstitution of type τ was first introduced by K. Denecke and K. Saengsura [3] in 2009. They used as the main tool in the study of cohyperidentities. They defined coalgebras, coidentities, cohyperidentities and applied all the concepts to construct the monoid of cohypersubstitutions of type τ. After that, in 2013, S. Jermjitpornchai and N. Seangsura [5] generalized the concepts of K. Denecke and K. Saengsura [3] by studying on the generalized cohypersubstitutions of type $\tau = (n_i)_{i\in I}$, introduced coterms, generalized superpositions, some algebraic-structural properties and constructed the monoid of generalized cohypersubstitutions. Later that, in the same year, N. Seangsura and S. Jermjitpornchai [8] fixed type $\tau = (2)$ and characterized all idempotent and regular elements of the generalized cohypersubstitutions of type $\tau = (2)$. After the study, the structural properties and special elements of the monoid of generalized cohypersubstitutions of type $\tau = (2), \tau = (3)$ and $\tau = (n)$ have been stydied by many other authors, see in [1], [5] and [8]. Moreover, in 2021, N. Chansuriya and S. Phuapong gave some structural properties and the relationship among submonoids of the monoid of generalized cohypersubstitutions of type τ by using the concepts in [6] and [7]. They also defined two new binary operations $+_{CG}$ and $\oplus_{CG}$ on the set of all generalized cohypersubstitutions of type τ, $Cohyp_G(\tau)$, and showed that

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$(Cohyp_G(\tau), +_{CG})$,$(Cohyp_G(\tau), \oplus_{CG})$ were semigroups.

In this study, we fix type $\tau = (2)$ and focus on the semigroups $(Cohyp_G(2), +_{CG})$ and $(Cohyp_G(2), \oplus_{CG})$. We characterize the set of all idempotent and regular elements of this semigroups.

2. Preliminaries

In this section, we provide the basic concept of the monoid of set of all generalized cohypersubstitutions which is very useful to this research.

Let A be a non-empty set and $n \in \mathbb{N}^+ = \mathbb{N} \cup \{0\}$. Define the union of n disjoint copies of A by $A^{\sqcup n} := \underline{n} \times A$ where $\underline{n} = \{1, 2, \ldots, n\}$, so it is called the n-th copower of A. An element (i, a) in this copower corresponds to the element a in the i-th copy of A where $i \in \underline{n}$. A mapping $f^A : A \to A^{\sqcup n}$ is a co-operation on A; the natural number n is called the arity of the co-operation f^A. Every n-ary co-operation f^A on the set A can be uniquely expressed as the pair of mappings (f_1^A, f_2^A) where $f_1^A : A \to \underline{n}$ gives the labelling used by f^A in mapping elements to copies of A, and $f_2^A : A \to A$ shows what element of A any element is mapped to, so $f^A(a) = (f_1^A(a), f_2^A(a))$. We denote the set of all n-ary co-operations defined on A by $cO_A^{(n)} = \{f^A : A \to A^{\sqcup n}\}$.

Let $\tau = (n_i)_{i \in I}$ and let $(f_i)_{i \in I}$ be an indexed set of co-operation symbols which f_i has arity n_i for each $i \in I$. Let $\bigcup\{e_j^n \mid n \geq 1, n \in \mathbb{N}^+, 0 \leq j \leq n-1\}$ be a set of symbols which disjoint from $\{f_i \mid i \in I\}$ such that e_j^n has arity n for each $0 \leq j \leq n-1$. An *coterms* of type τ are defined as follows:

(i) For every $i \in I$, the co-operation symbol f_i is an n-ary coterm of type τ.

(ii) For every $n \geq 1$ and $0 \leq j \leq n-1$ the symbol e_j^n is an n-ary coterm of type τ.

(iii) If $t_1, \ldots, t_{n_i}$ are n-ary coterms of type τ, then $f_i[t_1, \ldots, t_{n_i}]$ is an n-ary coterm of type τ for every $i \in I$, and if $t_0, \ldots, t_{n-1}$ are m-ary coterms of type τ, then $e_j^n[t_0, \ldots, t_{n-1}]$ is an n-ary coterm of type τ for every $0 \leq j \leq n-1$.

Let $CT_\tau^{(n)}$ be the set of all n-ary coterms of type τ, and $CT_\tau := \bigcup\limits_{n \geq 1} CT_\tau^{(n)}$ the set of all coterms of type τ.

For example, let us consider the type $\tau = (2)$ with one binary co-operation symbol f and the set of all injection symbols $E := \{e_j^n \mid n, j \in \mathbb{N}^+ := \mathbb{N} \cup \{0\}\}$. Then some example of coterm of type $\tau = (2)$ are:

$$e_0^2, e_1^2, f[e_0^2, e_1^2], f[e_1^2, e_2^2], f[f[e_1^2, e_0^2], e_2^2], f[e_0^2, f[e_1^2, e_1^2]], f[f[e_0^2, e_3^2, f[e_1^2, e_4^2]].$$

Definition 2.1. [5] Let $m \in \mathbb{N}^+ = \mathbb{N} \cup \{0\}$. A generalized superposition of coterms $S^m : CT_\tau^{m+1} \to CT_\tau$ is defined inductively by the following steps:

(i) If $t = e_i^n$ and $0 \leq i \leq m-1$, then $S^m(e_i^n, t_0, \ldots, t_{m-1}) = t_i$, where $t_0, \ldots, t_{m-1} \in CT_\tau$.

(ii) If $t = e_i^n$ and $0 < m \leq i \leq n-1$, then $S^m(e_i^n, t_0, \ldots, t_{m-1}) = e_i^n$, where $t_0, \ldots, t_{m-1} \in CT_\tau$.

(iii) If $t = f_i[s_1, \dots, s_{n_i}]$, then $S^m(t, t_1, \dots, t_m) = f_i(S^m(s_1, t_1, \dots, t_m), \dots, S^m(s_{n_i}, t_1, \dots, t_m))$, where $S^m(s_1, t_1, \dots, t_m), \dots, S^m(s_{n_i}, t_1, \dots, t_m) \in CT_\tau$.

The above definition can be written as the following forms:

(i) If $t = e_i^n$ and $0 \le i \le m-1$, then $e_i^n[t_0, \dots, t_{m-1}] = t_i$, where $t_0, \dots, t_{m-1} \in CT_\tau$.

(ii) If $t = e_i^n$ and $0 < m \le i \le n-1$, then $e_i^n[t_0, \dots, t_{m-1}] = e_i^n$, where $t_0, \dots, t_{m-1} \in CT_\tau$.

(iii) If $t = f_i[s_1, \dots, s_{n_i}]$, then $(f_i[s_1, \dots, s_{n_i}])[t_1, \dots, t_m] = f_i(s_1[t_1, \dots, t_m], \dots, s_{n_i}[t_1, \dots, t_m])$, where $s_1[t_1, \dots, t_m], \dots, s_{n_i}[t_1, \dots, t_m] \in CT_\tau$.

Definition 2.2. [5] A generalized cohypersubstitution of type τ is a mapping $\sigma : \{f_i \mid i \in I\} \to CT_\tau$. The extension of σ is a mapping $\widehat{\sigma} : CT_\tau \to CT_\tau$ which is inductively defined by the following steps :

(i) $\widehat{\sigma}(e_j^n) := e_j^n$ for every $n \ge 1$ and $0 \le j \le n-1$,

(ii) $\widehat{\sigma}(f_i) := \sigma(f_i)$ for every $i \in I$,

(iii) $\widehat{\sigma}(f_i[t_1, \dots, t_{n_i}]) := \sigma(f_i)[\widehat{\sigma}(t_1), \dots, \widehat{\sigma}(t_{n_i})]$ for $t_1, \dots t_{n_i} \in CT_\tau^{(n)}$.

Let $Cohyp_G(\tau)$ be the set of all generalized cohypersubstitutions of type τ.

Proposition 2.3. [5] *If $t, t_1, \dots, t_n \in CT_\tau$ and $\sigma \in Cohyp_G(\tau)$, then*

$$\widehat{\sigma}(t[t_1, \dots, t_n]) = \widehat{\sigma}(t)[\widehat{\sigma}(t_1), \dots, \widehat{\sigma}(t_n)].$$

On the set $Cohyp_G(\tau)$ of all generalized cohypersubstitutions of type τ, we may define an operation $\circ_{CG} : Cohyp_G(\tau) \times Cohyp_G(\tau) \to Cohyp_G(\tau)$ by $\sigma_1 \circ_{CG} \sigma_2 := \widehat{\sigma_1} \circ \sigma_2$ for all $\sigma_1, \sigma_2 \in Cohyp_G(\tau)$ where $\circ$ is the usual composition of mappings. Let σ_{id} be the generalized cohypersubstitution such that $\sigma_{id}(f_i) := f_i[e_0^n, e_1^n, \dots, e_{n_i-1}^n]$ for all $i \in I$. Then σ_{id} is an identity element in $Cohyp_G(\tau)$. Thus $\underline{Cohyp_G(\tau)} := (Cohyp_G(\tau), \circ_{CG}, \sigma_{id})$ is a monoid and called the *monoid of generalized cohypersubstitutions of type* τ. A algebraic-structural properties of the monoid $\underline{Cohyp_G(\tau)}$ can be found in [5].

In [2], a new binary operation "$+_{CG}$" on the set $Cohyp_G(\tau)$ was defined by

$$(\sigma_1 +_{CG} \sigma_2)(f_i) := \sigma_2(f_i)[\underbrace{\sigma_1(f_i), \dots, \sigma_1(f_i)}_{n_i-terms}] \in CT_{(\tau)},$$

for all $\sigma_1, \sigma_2 \in Cohyp_G(\tau)$. Then $(Cohyp_G(\tau), +_{CG})$ is a semigroup. Furthermore, they also defined another new binary operation "$\oplus_{CG}$" on the set $Cohyp_G(\tau)$ by

$$(\sigma_1 \oplus_{CG} \sigma_2)(f_i) := \sigma_1(f_i)[\underbrace{\sigma_2(f_i), \dots, \sigma_2(f_i)}_{n_i-terms}] \in CT_{(\tau)},$$

for all $\sigma_1, \sigma_2 \in Cohyp_G(\tau)$. So, $(Cohyp_G(\tau), \oplus_{CG})$ forms a semigroup.

Example 2.4. Let $\tau = (2)$ and $t = f[f[e_0^2, e_2^2], e_1^2],\ s = f[e_3^2, f[e_0^2, e_1^2]] \in CT_{(2)}$. Then

$$\begin{aligned}(\sigma_{f[f[e_0^2,e_2^2],e_1^2]} +_{CG} \sigma_{f[e_3^2,f[e_0^2,e_1^2]]})(f) &= \sigma_{f[e_3^2,f[e_0^2,e_1^2]]}(f)[\sigma_{f[f[e_0^2,e_2^2],e_1^2]}(f), \sigma_{f[f[e_0^2,e_2^2],e_1^2]}(f)] \\ &= f[e_3^2, f[e_0^2, e_1^2]][f[f[e_0^2, e_2^2], e_1^2], f[f[e_0^2, e_2^2], e_1^2]] \\ &= f[e_3^2, f[f[f[e_0^2, e_2^2], e_1^2], f[f[e_0^2, e_2^2], e_1^2]]], \text{ and}\end{aligned}$$

$$\begin{aligned}(\sigma_{f[f[e_0^2,e_2^2],e_1^2]} \oplus_{CG} \sigma_{f[e_3^2,f[e_0^2,e_1^2]]})(f) &= \sigma_{f[f[e_0^2,e_2^2],e_1^2]}(f)[\sigma_{f[e_3^2,f[e_0^2,e_1^2]]}(f), \sigma_{f[e_3^2,f[e_0^2,e_1^2]]}(f)] \\ &= f[f[e_0^2, e_2^2], e_1^2][f[e_3^2, f[e_0^2, e_1^2]], f[e_3^2, f[e_0^2, e_1^2]]] \\ &= f[f[f[e_3^2, f[e_0^2, e_1^2]], e_2^2], f[e_3^2, f[e_0^2, e_1^2]]].\end{aligned}$$

Throughout this paper, we denote:
σ_t := the generalized cohypersubstitution σ of type τ which maps f to the coterm t,
e_j^n := the injection symbol for all $0 \le j \le n-1, n \in \mathbb{N}$,
E := the set of all injection symbols, i.e., $E := \{e_j^n \mid n, j \in \mathbb{N}^+ := \mathbb{N} \cup \{0\}\}$,
$E(t)$:= the set of all injection symbols occuring in the coterm t.

3. Main Results

In this section, we focus on the set $Cohyp_G(2)$ of all generalized cohypersubstitutions of type $\tau = (2)$ with a binary operation "$+_{CG}$" on the set $Cohyp_G(2)$ defined by $(\sigma_1 +_{CG} \sigma_2)(f) := (\sigma_2(f))[\sigma_1(f), \sigma_1(f)]$ for all $\sigma_1, \sigma_2 \in Cohyp_G(2)$. Then we have $(Cohyp_G(2), +_{CG})$ is a semigroup. We describe idempotent and regular elements in $Cohyp_G(2)$. Firstly, we recall the definition of an idempotent element in the semigroup $(Cohyp_G(2), +_{CG})$.

Definition 3.1. Let $(Cohyp_G(2), +_{CG})$ be a semigroup. An element $\sigma_t \in Cohyp_G(2)$ is called idempotent if $\sigma_t +_{CG} \sigma_t = \sigma_t$. Denoted by $\mathcal{E}^{+_{CG}}(Cohyp_G(2))$ the set of all idempotent elements of $Cohyp_G(2)$.

Theorem 3.2. *Let $t, s \in CT_{(2)}$. Then the following statements hold.*

(i) If $E(t) \cap \{e_0^2, e_1^2\} = \{e_0^2\}$, then $t[s, s] = t$ if and only if $s = e_0^2$.

(ii) If $E(t) \cap \{e_0^2, e_1^2\} = \{e_1^2\}$, then $t[s, s] = t$ if and only if $s = e_1^2$.

(iii) If $E(t) \cap \{e_0^2, e_1^2\} = \emptyset$, then $t[s, s] = t$.

Proof. (i) Let $t, s \in CT_{(2)}$ where $E(t) \cap \{e_0^2, e_1^2\} = \{e_0^2\}$.

Let $t = f[t_1, t_2]$ where $t_1, t_2 \in CT_{(2)}$ and assume that $t[s, s] = t$. Suppose that $s \neq e_0^2$. Then

$$t[s, s] = (f[t_1, t_2])[s, s] = f[t_1[s, s], t_2[s, s]].$$

Since $E(t) \cap \{e_0^2, e_1^2\} = \{e_0^2\}$ and $s \neq e_0^2$, this force that $t_1[s, s] \neq t_1$ and $t_2[s, s] \neq t_2$. Thus $t[s, s] = (f[t_1, t_2])[s, s] = f[t_1[s, s], t_2[s, s]] \neq f[t_1, t_2] = t$, which is a contradiction. Hence, $s = e_0^2$.

Conversely, assume that $s = e_0^2$. We give a proof by indection on the complexity of the coterm t. If $t = e_0^2$, then $e_0^2[s, s] = e_0^2$. If $t = e_j^2$ for $j \geq 2$, then $e_j^2[s, s] = e_j^2$. If $t = f[t_1, t_2]$ and suppose that $t_1[s, s] = t_1$ and $t_2[s, s] = t_2$, then $t[s, s] = (f[t_1, t_2])[s, s] = f[t_1[s, s], t_2[s, s]] = f[t_1, t_2] = t$.

Similarly, we can proof (ii) and (iii). ∎

Theorem 3.3. *The generalized cohypersustitution σ_t of type $\tau = (2)$ is idempotent if and only if $t[\sigma_t(f), \sigma_t(f)] = t$.*

Proof. Let $t \in CT_{(2)}$. Assume that σ_t is an idempotent. Then
$t[\sigma_t(f), \sigma_t(f)] = (\sigma_t(f) +_{CG} \sigma_t)(f) = \sigma_t(f) = t$.

Conversely, assume that $t[\sigma_t(f), \sigma_t(f)] = t$. Then $(\sigma_t +_{CG} \sigma_t)(f) = (\sigma_t(f)[\sigma_t(f), \sigma_t(f)] = t[\sigma_t(f), \sigma_t(f)] = t = \sigma_t(f)$. Thus σ_t is an idempotent. ∎

Next, we study on the set of all projection generalized cohypersubstitutions of type $\tau = (2)$ which define as following.

Definition 3.4. Let $\tau = (2)$. A generalized cohypersubstitution σ of type $\tau = (2)$ is called a projection generalized cohypersubstitution if the coterm $\sigma(f_i)$ is the injection symbol for each $i \in I$. Let $\sigma_t \in P_{CG}^{inj}(2)$ be the set of all projection generalized cohypersubstitutions of type $\tau = (2)$, i.e., $\sigma_t \in P_{CG}^{inj}(2) := \{\sigma_{e^2,j} \mid e_j^2 \in E\}$.

By applying the Theorem 3.2 and Theorem 3.3, we have the following corollary.

Corollary 3.5. *Every $\sigma_t \in P_{CG}^{inj}(2)$ is idempotent.*

Corollary 3.6. *If $\sigma_t \in Cohyp_G(2)$ and $E(t) \cap \{e_0^2, e_1^2\} = \emptyset$, then σ_t is idempotent.*

Lemma 3.7. *Let $t_1, t_2 \in CT_{(2)}$ and $\sigma_t \in Cohyp_G(2)$. Then the following statements hold.*

(i) If $t = f[e_0^2, t_2]$ where $E(t) \cap \{e_0^2, e_1^2\} = \{e_0^2\}$, then σ_t is not idempotent.

(ii) If $t = f[t_1, e_1^2]$ where $E(t) \cap \{e_0^2, e_1^2\} = \{e_1^2\}$, then σ_t is not idempotent.

(iii) If $t = f[t_1, t_2]$ where $\{e_0^2, e_1^2\} \subseteq E(t)$, then σ_t is not idempotent.

Proof. (i) Let $\sigma_t \in Cohyp_G(2)$ where $t = f[e_0^2, t_2]$, $E(t) \cap \{e_0^2, e_1^2\} = \{e_0^2\}$ and $t_2 \in CT_{(2)}$.
Consider

$$\begin{aligned}(\sigma_{f[e_0^2,t_2]} +_{CG} \sigma_{f[e_0^2,t_2]})(f) &= \sigma_{f[e_0^2,t_2]}(f)[\sigma_{f[e_0^2,t_2]}(f), \sigma_{f[e_0^2,t_2]}(f)] \\ &= f[e_0^2, t_2][f[e_0^2, t_2], f[e_0^2, t_2]] \\ &= f[f[e_0^2, t_2], t_2[f[e_0^2, t_2], f[e_0^2, t_2]]] \\ &\neq f[e_0^2, t_2].\end{aligned}$$

Hence, σ_t is not idempotent.

Similarly, we can proof (ii).

(iii) Let $\sigma_t \in Cohyp_G(2)$ where $t = f[t_1, t_2]$, $t_1, t_2 \in CT_{(2)}$ and $\{e_0^2, e_1^2\} \subseteq E(t)$.
Consider

$$\begin{aligned}(\sigma_{f[t_1,t_2]} +_{CG} \sigma_{f[t_1,t_2]})(f) &= \sigma_{f[t_1,t_2]}(f)[\sigma_{f[t_1,t_2]}(f), \sigma_{f[t_1,t_2]}(f)] \\ &= f[t_1, t_2][f[t_1, t_2], f[t_1, t_2]] \\ &= f[t_1[f[t_1, t_2], f[t_1, t_2]], t_2[f[t_1, t_2], f[t_1, t_2]]].\end{aligned}$$

Since $\{e_0^2, e_1^2\} \subseteq E(t)$, then we have $t_1[f[t_1, t_2], f[t_1, t_2]] \neq t_1$ and $t_2[f[t_1, t_2], f[t_1, t_2]] \neq t_2$. So, $(\sigma_{f[t_1,t_2]} +_{CG} \sigma_{f[t_1,t_2]})(f) \neq \sigma_{f[t_1,t_2]}$. Hence, σ_t is not idempotent. ∎

Lemma 3.8. *Let $t_1, t_2 \in CT_{(2)}$ and $\sigma_t \in Cohyp_G(2)$. Then the following statements hold.*

(i) If $t = f[t_1, e_0^2]$ where $E(t) \cap \{e_0^2, e_1^2\} = \{e_0^2\}$, then σ_t is not idempotent.

(ii) If $t = f[e_1^2, t_2]$ where $E(t) \cap \{e_0^2, e_1^2\} = \{e_1^2\}$, then σ_t is not idempotent.

Proof. The proof of this lemma is similarl to Lemma 3.7. ∎

We now set $E^* := \{\sigma_t \mid E(t) \cap \{e_0^2, e_1^2\} = \emptyset\}$. So, we have the following theorem.

Theorem 3.9. *$\mathcal{E}^{+_{CG}}(Cohyp_G(2)) := P_{CG}^{inj}(2) \cup E^*$ is the set of all idempotent elements of $(Cohyp_G(2), +_{CG})$.*

Proof. The proof is directly optained from Corollary 3.5, Corollary 3.6, Lemma 3.7 and Lemma 3.8. ∎

By applying the method in [1], we have the following lemma.

Lemma 3.10. *$\mathcal{E}^{+_{CG}}(Cohyp_G(2))$ is a maximal idempotent subsemigroup of $(Cohyp_G(2), +_{CG})$.*

Proof. It is easy to see that $\mathcal{E}^{+_{CG}}(Cohyp_G(2)) \subset Cohyp_G(2)$ and it is closed under the operation $+_{CG}$. So, $\mathcal{E}^{+_{CG}}(Cohyp_G(2))$ is an idempotent subsemigroup of $(Cohyp_G(2), +_{CG})$. We next to show that it is a maximal idempotent subsemigroup.

Let $\mathcal{M}$ be a proper idempotent subsemigroup of $(Cohyp_G(2), +_{CG})$ such that $\mathcal{E}^{+_{CG}}(Cohyp_G(2)) \subseteq \mathcal{M} \subset Cohyp_G(2)$. Let $\sigma_t \in \mathcal{M}$. Then σ_t is an idempotent element. Suppose that $\sigma_t \neq \mathcal{E}^{+_{CG}}(Cohyp_G(2))$. Then, by Lemma 3.7 and Lemma 3.8, σ_t is not idempotent, which is a contradiction. So, $\sigma_t \in \mathcal{E}^{+_{CG}}(Cohyp_G(2))$. Hence, $\mathcal{M} = \mathcal{E}^{+_{CG}}(Cohyp_G(2))$.

Therefore, $\mathcal{E}^{+_{CG}}(Cohyp_G(2))$ is a maximal idempotent subsemigroup of $(Cohyp_G(2), +_{CG})$. ∎

Now, we will describe regular elements in the semigroup $(Cohyp_G(2), +_{CG})$.

Definition 3.11. Let $(Cohyp_G(2), +_{CG})$ be a semigroup. An element $\sigma_t \in Cohyp_G(2)$ is call regular if there exists $\sigma_s \in Cohyp_G(2)$ such that $\sigma_t +_{CG} \sigma_s +_{CG} \sigma_t = \sigma_t$. Denoted by $\mathcal{R}^{+_{CG}}(Cohyp_G(2))$ the set of all regular elements of $Cohyp_G(2)$.

Theorem 3.12. *For any type $\tau = (2)$, $\mathcal{E}^{+_{CG}}(Cohyp_G(2)) = \mathcal{R}^{+_{CG}}(Cohyp_G(2))$.*

Proof. Since every idempotent elements is regular element, so we have $\mathcal{E}^{+_{CG}}(Cohyp_G(2)) \subseteq \mathcal{R}^{+_{CG}}(Cohyp_G(2))$. We will show that $\mathcal{R}^{+_{CG}}(Cohyp_G(2)) = \mathcal{E}^{+_{CG}}(Cohyp_G(2))$. Let $\sigma_t \in \mathcal{R}^{+_{CG}}(Cohyp_G(2))$. Then there exists $\sigma_s \in Cohyp_G(2)$ such that $\sigma_t +_{CG} \sigma_s +_{CG} = \sigma_t$.

So,

$$\begin{aligned}(\sigma_t(f))[(\sigma_t +_{CG} \sigma_s(f))(f), (\sigma_t +_{CG} \sigma_s(f))(f)] &= \sigma_t(f)\\ t[s[t,t], s[t,t]] &= t.\end{aligned}$$

This force that $e_0^2, e_1^2 \notin E(t)$. Thus $\sigma_t \in E^*$.

Assume that $t \neq e_i^2; i \in \mathbb{N}^*$, then $s[t,t] \neq e_i^2; i \in \mathbb{N}^*$. So, by Theorem 3.2 (i),(ii), we obtain that $t[s[t,t], s[t,t]] \neq t$. We get a cotradiction. Thus $t = e_i^2; i \in \mathbb{N}^*$ which implies that $\sigma_t \in P_{CG}^{inj}(2)$. Hence $\sigma_t \in E^* \cup P_{CG}^{inj}(2) := \mathcal{E}^{+_{CG}}(Cohyp_G(2))$.

Therefore, $\mathcal{E}^{+_{CG}}(Cohyp_G(2)) = \mathcal{R}^{+_{CG}}(Cohyp_G(2))$. ∎

In the last of this section, we study on the set $Cohyp_G(2)$ of all generalized cohypersubstitutions of type $\tau = (2)$ together with a binary operation "$\oplus_{CG}$" on the set $Cohyp_G(2)$ defined by $(\sigma_1 \oplus_{CG} \sigma_2)(f) := (\sigma_1(f))[\sigma_2(f), \sigma_2(f)]$ for all $\sigma_1, \sigma_2 \in Cohyp_G(2)$. Then we have that $(Cohyp_G(2), \oplus_{CG})$ is a semigroup. We describe idempotent and regular elements in $Cohyp_G(2)$ by using the following definitions.

Definition 3.13. Let $(Cohyp_G(2), \oplus_{CG})$ be a semigroup. An element $\sigma_t \in Cohyp_G(2)$ is call idempotent if $\sigma_t \oplus_{CG} \sigma_t = \sigma_t$. Denoted by $\mathcal{E}^{\oplus_{CG}}(Cohyp_G(2))$ the set of all idempotent elements of $Cohyp_G(2)$.

Definition 3.14. Let $(Cohyp_G(2), \oplus_{CG})$ be a semigroup. An element $\sigma_t \in Cohyp_G(2)$ is call regular if there exists $\sigma_s \in Cohyp_G(2)$ such that $\sigma_t \oplus_{CG} \sigma_s \oplus_{CG} \sigma_t = \sigma_t$. Denoted by $\mathcal{R}^{\oplus_{CG}}(Cohyp_G(2))$ the set of all regular elements of $Cohyp_G(2)$.

We can see that every idempotent element in $(Cohyp_G(2), +_{CG})$ is idempotent element in $(Cohyp_G(2), \oplus_{CG})$ and also regular element. So, we have the following results.

Proposition 3.15. *For any type $\tau = (2)$, $\mathcal{E}^{\oplus_{CG}}(Cohyp_G(2)) = \mathcal{R}^{\oplus_{CG}}(Cohyp_G(2))$.*

Proof. The proof is similar to Theorem 3.12. ∎

So, we have the following corollary.

Corollary 3.16. *For any type $\tau = (2)$, $\mathcal{R}^{+_{CG}}(Cohyp_G(2)) = \mathcal{E}^{+_{CG}}(Cohyp_G(2)) = \mathcal{E}^{\oplus_{CG}}(Cohyp_G(2)) = \mathcal{R}^{\oplus_{CG}}(Cohyp_G(2))$.*

4. Conclusion

This study focuses on the semigroups $(Cohyp_G(2), +_{CG})$ and $(Cohyp_G(2), \oplus_{CG})$ of generalized cohypersubstitutions of type $\tau = (2)$. We characterize the idempotent and regular elements on these semigroups. The main results of the study shown that any regular elements are idempotent elements. Moreover, we can see that the set of all idempotent and regular elements of the semigroup $(Cohyp_G(2), +_{CG})$ equal to the set of all idempotent and regular elements of the semigroup $(Cohyp_G(2), \oplus_{CG})$.

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