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Generalized Projection Algorithm for Convex Feasibility Problems on Hadamard Manifolds

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ABSTRACT

We present a two-step cyclic algorithm for solving convex feasibility problems on Hadamard manifolds in this study. On Hadamard manifolds, the convergent result and linear convergent results are proven. In addition, to support the main results, a numerical example on the Poincaré plane is provided.

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1. Introduction

Let H be a Hilbert space and $C_1, \dots, C_m$ are closed convex subsets of H with nonempty intersection $\bigcap_{i=1}^{m} C_i$. Finding a point at the intersection of convex sets is a challenge in mathematics and physical sciences. The problem is known as a convex feasibility problem, and it is defined as follows:

$$\text{Find a point } x \in \bigcap_{i=1}^{m} C_i.$$

A point x solving this problem is said to be a *feasibility point*. The *projection algorithm*, in which each iterative step is to project onto an individual set corresponding to a control sequence (see, for example, [7] for the definition of the control sequence), is one of the most widely studied methods for determining such feasibility points. For more information, see [4, 3, 13, 12, 28, 22, 2] and the references therein. Convex inequalities [17, 18], convex minimization problems [32, 26, 31], medical imaging [8] and computerized tomography [24, 1] are some of the applications of the projection method.

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Many nonlinear problems, such as fixed point theory, convex analysis, variational inequalities, equilibrium problems, and optimization problems, have been extended from linear spaces to the setting of manifolds in the last decade because the problems cannot be posted in linear space and necessarily require a Riemannian manifold structure, see for examples [30, 19, 23, 15, 5, 11, 27, 20, 10] and the reference therein.

Returning to the convex feasibility problems, Bauschke et al. [4] proposed the following general procedure with an initial point $x_0 \in H$ in Hilbert spaces:

$$x_{n+1} := (1-\alpha_n)x_n + \alpha_n \sum_{i=1}^{m} \mu_i^{(n)} T_i^{(n)}(x_n), \quad \forall n \in \mathbb{N}, \tag{1.1}$$

where each $\alpha_n \in [0,2]$ is a relaxation parameter, $\{\mu_i^{(n)} : 1,\dots,m\} \subseteq [0,1]$ is weight satisfying $\sum_{i=1}^{m} \mu_i^{(n)} = 1$, and each $T_i^{(n)} : H \to H$ is a firmly nonexpansive satisfying $\operatorname{Fix} T_i^{(n)} \supseteq C_i$. Some works [7, 16, 21] discuss the case where the weights $\left\{\mu_i^{(n)}\right\}$ satisfy the condition that

$$\mu_i^{(n)} = \delta_{i_n,i} := \begin{cases} 1, & \text{if } i = i_n, \\ 0, & \text{otherwise,} \end{cases}$$

where $\{i_n\}_{n=0}^{\infty}$ is a so-called *control sequence*, and each $T_i^{(n)}$ is the projection onto a hyperplane separating C_{i_n} from x_n. Censor [9] proposed the cyclic subgradient projection algorithm for the case where each convex is given as a sublevel set of a convex function, i.e., the algorithm uses weights $\{\mu_i^{(n)}\}$ satisfying the following condition

$$\mu_i^{(n)} := \begin{cases} 1, & \text{if } i = n \pmod{m} + 1, \\ 0, & \text{otherwise,} \end{cases} \tag{1.2}$$

and each $T_i^{(n)}$ is the projection operator of the corresponding subgradient. In the Hadamard manifolds, Wang et al. [30] extended the cyclic method (1.1) using the weights $\{\mu_i^{(n)}\}$ satisfying the condition (1.2). The following is the Riemannian version of the cyclic algorithm:

Algorithm 1.1. Let M be an Hadamard manifold and $x_0 \in M$ be an initial point. Define x_{n+1} by

$$x_{n+1} := \exp_{x_n} \alpha_n \exp_{x_n}^{-1} T_{i_n}^{(n)}(x_n), \quad \forall n \in \mathbb{N}, \tag{1.3}$$

where $T_{i_n}^{(n)} : M \to M$ be a family of firmly nonexpansive mapping, $\{\alpha_n\} \subseteq (0,2)$ is a relaxation parameter sequence, and set $i_n = n \pmod{m} + 1$. The authors [30] also established that Algorithm 1.1 is convergent and linearly convergent in Hadamard manifolds.

The goal of this paper is to extend the cyclic algorithm (1.3) to two-step projection cyclic algorithms on Hadamard manifolds, as motivated and inspired by the previous efforts. Furthermore, we show that Algorithm 3.1 is convergent under certain conditions. We show that this approach is linearly convergent when the algorithm is linearly focusing and the family of convex sets is linearly regular. On the Poincar'e plane, a numerical example is shown.

The rest of this paper is organized in the following: Section 2, we give some basic concept and fundamental results of Riemannian geometry. For solving convex feasibility problems, the two-step cyclic algorithm is described in Section 3. Furthermore, we show that any sequence generated by the proposed method converges to feasibility points. The linear convergence of the two-step projection cyclic projection algorithm is given in Section 4. Finally, Section 5 gives a numerical example of our method for approximating convex feasibility problem solutions on the Poincaré plane.

2. Preliminaries

In this section, we recall some fundamental definitions, properties, useful results, and notations of Riemannian geometry. For more information, readers can consult several textbooks [25, 14, 29].

Let M be a connected finite-dimensional manifold. For $p \in M$, we denote T_pM the *tangent space* of M at p which is a vector space of the same dimension as M, and by $TM = \bigcup_{p\in M} T_pM$ the *tangent bundle* of M. We always suppose that M can be endowed with a Riemannian metric $\langle \cdot, \cdot \rangle_p$, with corresponding norm denoted by $\| \cdot \|_p$, to become a *Riemannian manifold*. The angle $\angle_p(u, v)$ between $u, v \in T_pM$ $(u, v \neq \mathbf{0})$ is set by $\cos \angle_p(u, v) = \dfrac{\langle u, v\rangle_p}{\|u\|_p\|v\|_p}$. If there is no confusion, we denote $\langle \cdot, \cdot \rangle := \langle \cdot, \cdot \rangle_p$, $\| \cdot \| := \| \cdot \|_p$ and $\angle(u, v) := \angle_p(u, v)$. Let $\gamma : [a, b] \to M$ be a piecewise smooth curve joining $\gamma(a) = p$ to $\gamma(b) = q$, we define the length of the curve γ by using the metric as

$$L(\gamma) = \int_a^b \|\gamma^{'}(t)\| dt,$$

minimizing the length function over the set of all such curves, we obtain a Riemannian distance $d(p, q)$ which induces the original topology on M.

Let ∇ be a Levi-Civita connection associated to $(M, \langle \cdot, \cdot \rangle)$. Given γ a smooth curve, a smooth vector field X along γ is said to be *parallel* if $\nabla_{\gamma'} X = \mathbf{0}$, where $\mathbf{0}$ is the zero section of TM. If $\gamma^{'}$ itself is parallel, we say that γ is a *geodesic*, and in this case $\|\gamma^{'}\|$ is a constant. When $\|\gamma^{'}\| = 1$, then γ is said to be *normalized*. A geodesic joining p to q in M is said to be a *minimal geodesic* if its length equals to $d(p, q)$.

A Riemannian manifold is complete if for any $p \in M$ all geodesic emanating from p are defined for all $t \in \mathbb{R}$. From the Hopf-Rinow theorem we know that if M is complete then any pair of points in M can be joined by a minimal geodesic. Moreover, (M, d) is a complete metric space and every bounded closed subset is compact.

Let M be a complete Riemannian manifold and $p \in M$. The exponential map $\exp_p : T_pM \to M$ is defined as $\exp_p v = \gamma_v(1, x)$, where $\gamma(\cdot) = \gamma_v(\cdot, x)$ is the geodesic starting at p with velocity v (i.e., $\gamma_v(0, p) = p$ and $\gamma_v^{'}(0, p) = v$). Then, for any value of t, we have $\exp_p tv = \gamma_v(t, p)$ and $\exp_p \mathbf{0} = \gamma_v(0, p) = p$. Note that the exponential $\exp_p$ is differentiable on T_pM for all $p \in M$. It well known that the derivative $D\exp_p(\mathbf{0})$ of $\exp_p(\mathbf{0})$ is equal to the identity vector of T_pM. Therefore, by the inverse mapping theorem, there exists an inverse exponential map $\exp^{-1} : M \to T_pM$. Moreover, for any $p, q \in M$, we have $d(p, q) = \| \exp_p^{-1} q\|$.

A complete simply connected Riemannian manifold of non-positive sectional curvature is said to be an *Hadamard manifold.* Throughout the remainder of the paper, we always assume that M is a finite-dimensional Hadamard manifold. The following proposition is well-known and will be useful.

Proposition 2.1. [25] *Let $p \in M$. The $\exp_p : T_pM \to M$ is a diffeomorphism, and for any* *two points $p, q \in M$ there exists a unique normalized geodesic joining p to q, which is can be expressed by the formula*

$$\gamma(t) = \exp_p t \exp_p^{-1} q, \quad \forall t \in [0,1].$$

This proposition yields that M is diffeomorphic to the Euclidean space $\mathbb{R}^n$. Then, M has same topology and differential structure as $\mathbb{R}^n$. Moreover, Hadamard manifolds and Euclidean spaces have some similar geometrical properties. One of the most important proprieties is illustrated in the following propositions.

A geodesic triangle $\triangle(p_1, p_2, p_3)$ of a Riemannian manifold M is a set consisting of three points p_1, p_2 and p_3, and three minimal geodesics γ_i joining p_i to p_{i+1} where $i = 1, 2, 3 \pmod 3$.

Proposition 2.2. [25] *Let $\triangle(p_1, p_2, p_3)$ be a geodesic triangle in Hadamard manifolds M. For each $i = 1, 2, 3 \pmod 3$, given $\gamma_i : [0, l_i] \to M$ the geodesic joining p_i to p_{i+1} and set $l_i := L(\gamma_i)$, $\alpha_i : \angle(\gamma_i^{'}(0), -\gamma_{i-1}^{'}(l_{i-1}))$. Then*

$$\alpha_1 + \alpha_2 + \alpha_3 \leq \pi; \tag{2.1}$$

$$l_i^2 + l_{i+1}^2 - 2 l_i l_{i+1} \cos \alpha_{i+1} \leq l_{i-1}^2. \tag{2.2}$$

In the terms of the distance and the exponential map, the inequality (2.2) can be rewritten as

$$d^2(p_i, p_{i+1}) + d^2(p_{i+1}, p_{i+2}) - 2\langle \exp_{p_{i+1}}^{-1} p_i, \exp_{p_{i+1}}^{-1} p_{i+2} \rangle \leq d^2(p_{i-1}, p_i), \tag{2.3}$$

where $\langle \exp_{p_{i+1}}^{-1} p_i, \exp_{p_{i+1}}^{-1} p_{i+2} \rangle = d(p_i, p_{i+1}) d(p_{i+1}, p_{i+2}) \cos \alpha_{i+1}$.

The following relation between geodesic triangles in Riemannian manifolds and triangles in $\mathbb{R}^2$ can be referred to [6].

Lemma 2.3. [6] *Let $\triangle(p_1, p_2, p_3)$ be a geodesic triangle in M. Then there exists a triangle $\triangle(\overline{p_1}, \overline{p_2}, \overline{p_3})$ for $\triangle(p_1, p_2, p_3)$ such that $d(p_i, p_{i+1}) = \|\overline{p_i} - \overline{p_{i+1}}\|$, indices taken modulo 3; it is unique up to an isometry of $\mathbb{R}^2$.*

The triangle $\triangle(\overline{p_1}, \overline{p_2}, \overline{p_3})$ in Lemma 2.3 is said to be a *comparison triangle* for $\triangle(p_1, p_2, p_3)$. The geodesic side from x to y will be denoted $[x, y]$. A point $\overline{x} \in [\overline{p_1}, \overline{p_2}]$ is said to be a *comparison point* for $x \in [p_1, p_2]$ if $\|\overline{x} - \overline{p_1}\| = d(x, p_1)$. The interior angle of $\triangle(\overline{p_1}, \overline{p_2}, \overline{p_3})$ at $\overline{p_1}$ is said to be the *comparison angle* between $\overline{p_2}$ and $\overline{p_3}$ at $\overline{p_1}$ and is denoted $\angle_{\overline{p_1}}(\overline{p_2}, \overline{p_3})$. With all notation as in the statement of Proposition 2.2, according to the law of cosine, (2.2) is valid if and only if

$$\langle \overline{p_2} - \overline{p_1}, \overline{p_3} - \overline{p_1} \rangle_{\mathbb{R}^2} \leq \langle \exp_{p_1}^{-1} p_2, \exp_{p_1}^{-1} p_3 \rangle \tag{2.4}$$

or,

$$\alpha_1 \leq \angle_{\overline{p_1}}(\overline{p_2}, \overline{p_3})$$

or, equivalent, $\triangle(p_1, p_2, p_3)$ satisfies the CAT(0) inequality and that is, given a comparison triangle $\overline{\triangle} \subset \mathbb{R}^2$ for $\triangle(p_1, p_2, p_3)$ for all $x, y \in \triangle$,

$$d(x, y) \leq \|\overline{x} - \overline{y}\|, \tag{2.5}$$

where $\overline{x}, \overline{y} \in \overline{\triangle}$ are the respective comparison points of x, y.

Definition 2.4. A subset Q is said to be *geodesic convex* if for any two points p and q in Q, the geodesic joining p to q is contained in Q, that is, if $\gamma : [a, b] \to M$ is a geodesic such that $p = \gamma(a)$ and $q = \gamma(b)$, then $\gamma((1-t)a + tb) \in Q$ for all $t \in [0, 1]$.

Definition 2.5. A real function f defined on M is said to be *geodesic convex* if for any geodesic γ of M, the composition function $f \circ \gamma : [a, b] \to \mathbb{R}$ is convex, that is,

$$(f \circ \gamma)(ta + (1-t)b) \leq t(f \circ \gamma)(a) + (1-t)(f \circ \gamma)(b)$$

where $a, b \in \mathbb{R}$, and $t \in [0, 1]$.

Proposition 2.6. [25] *Let $d : M \times M \to \mathbb{R}$ be the distance function. Then d is a convex function with respect to the Riemannian metric, that is, for any pair of geodesics $\gamma_1 : [0, 1] \to M$ and $\gamma_2 : [0, 1] \to M$ the following inequality holds for all $t \in [0, 1]$*

$$d(\gamma_1(t), \gamma_2(t)) \leq (1-t)d(\gamma_1(0)\gamma_2(0)) + td(\gamma_1(t), \gamma_2(t)).$$

In particular, for each $p \in M$, the function $d(\cdot, p) : M \to \mathbb{R}$ is a geodesic convex function.

A nonempty, closed geodesic convex set in M shall be denoted by Q from here on. Let $T : Q \to M$ be a mapping. We say that T is *nonexpansive* if for any two points $x, y \in Q$ such that

$$d(T(x), T(y)) \leq d(x, y).$$

Let $F(T)$ denote the set of all fixed points of T, i.e.,

$$F(T) := \{x \in Q : T(x) = x\}.$$

The definition of firmly nonexpansive mappings on Hadamard manifolds was introduce by Li et.al. [23].

Definition 2.7. [23] Let $T : Q \to M$ be a mapping. Then T is said to be *firmly nonexpansive* if for any $x, y \in Q$, the function $\sigma : [0, 1] \to [0, +\infty]$ defined by

$$\sigma(t) : d(\exp_x t \exp_x^{-1} T(x), \exp_x t \exp_y^{-1} T(y)), \quad \forall t \in [0, 1],$$

is nonincreasing.

By definition, it easy to see that any firmly nonexpansive mapping T is nonexpansive.

Next, we follow the definition of a *distance function* $d(\cdot, Q) : M \to \mathbb{R}$ and a *projection operator* $P_Q(\cdot) : M \to Q$, which are defined for every $x \in M$ by

$$d(x, Q) := \inf_{y \in Q} d(x, y)$$

and

$$P_Q(x) := \{z : d(x, z) \leq d(x, y), \quad \forall y \in Q\},$$

respectively. The projection operator P_Q is firmly nonexpansive as described in the following proposition.

Proposition 2.8. [23] *Let $Q \subseteq M$ be a nonempty, closed and convex set. Then the following assertions holds:*

(i) P_Q *is single valued and firmly nonexpansive;*

(ii) *For every* $x \in M$, $z = P_Q(x)$ *if and only if*

$$\langle \exp_z^{-1} x, \exp_z^{-1} y \rangle \leq 0, \quad \forall y \in Q.$$

We end this section with the following crucial results.

A sequence $\{x_n\}$ is said to br *Fejér monotone* w.r.t. Q if for all $x \in Q$ and $n \in \mathbb{N}$,

$$d(x_{n+1}, x) \leq d(x_n, x).$$

The following lemma provides some properties of Fejér monotonicity sequences that are useful for establishing convergence and linear convergence results.

Lemma 2.9. [30] *Let* $\{x_n\} \subseteq M$ *be a Fejér monotone sequence w.r.t* Q*. Then the following conditions hold:*

(i) $\{x_n\}$ *is bounded, and,* $\lim_{n\to+\infty} x_n = x$ *if* x *is a cluster point of* $\{x_n\}$ *and* $x \in Q$.

(ii) *Let* $\alpha > 0$ *be such that*

$$\alpha d^2(x_n, Q) \leq d^2(x_n, Q) - d^2(x_{n+1}, Q), \quad \forall n \in \mathbb{N}. \tag{2.6}$$

Then $\{x_n\}$ *converges linearly to a point* x *in* Q*:*

$$d(x_n, x) \leq 2(\sqrt{1-\alpha})^n d(x_0, Q), \quad \forall n \in \mathbb{N}. \tag{2.7}$$

3. Two-step Cyclic Algorithm and Its Convergence

In this section, we introduce an iterative method for solving convex feasibility problems in the setting of Hadamard manifolds.

We first recall the concept of convex feasibility problem in Hadamard manifolds. Let $I := \{1, 2, \dots m\}$ and $\{C_i : i \in I\}$ be a family of nonempty closed geodesic convex subset of M. Then the problem is to find

$$x^* \in C := \bigcap_{i=1}^{m} C_i, \tag{3.1}$$

where C is assumed to be a nonempty set. Set $i_n := n \pmod{m} + 1$ and let $\{T_{i_n}^{(n)}\}$ be a family of firmly nonexpansive mappings from M to itself satisfying

$$F\left(T_{i_n}^{(n)}\right) \supseteq C_{i_n}, \quad \forall n \in \mathbb{N}.$$

The multi-step cyclic algorithm is defined as follows:

Algorithm 3.1. Let $x_0 \in M$ be an initial point and define a sequence $\{x_n\}$ by

$$\begin{cases} x_{n+1} = \exp_{x_n} \alpha_n \exp_{x_n}^{-1} T_{i_n}^{(n)}(y_n) \\ y_n = \exp_{x_n} \beta_n \exp_{x_n}^{-1} T_{i_n}^{(n)}(x_n), \quad \forall n \in \mathbb{N}, \end{cases} \tag{3.2}$$

where $\{\alpha_n\}, \{\beta_n\} \subseteq (0, 1)$ are *relaxation parameter sequences*.

The following lemma is required to prove our main convergent theorem.

Lemma 3.2. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.1 and $x \in C := \bigcap_{i=1}^{m} C_i$. Then the following assertions hold for all $n \in \mathbb{N}$*

(i)

$$d^2(x_{n+1}, x_n) \leq \frac{\alpha_n}{1-\alpha_n}(d^2(x_n, x) - d^2(x_{n+1}, x)). \tag{3.3}$$

(ii) The sequence $\{x_n\}$ is Fejér monotone w.r.t C. If furthermore

$$\liminf_{n\to+\infty} \alpha_n(1-\alpha_n) > 0, \tag{3.4}$$

then

$$\lim_{n\to+\infty} d(x_{n+1}, x_n) = 0, \tag{3.5}$$

and

$$\liminf_{n\to+\infty} \alpha_n\beta_n(1-\beta_n) > 0, \tag{3.6}$$

then

$$\lim_{n\to+\infty} d(x_n, y_n) = 0. \tag{3.7}$$

Proof. (i) Fix $n \in \mathbb{N}$, let $x \in C$ and $\gamma_n : [0,1] \to M$ be geodesic joining x_n to $T_{i_n}^{(n)}(x_n)$. Thus, (3.2) can be written as $y_n = \gamma_n(\beta_n)$. By using geodesic convexity of Riemannian distance, we get

$$\begin{aligned} d(y_n, x) &= d(\gamma_n(\beta_n), x) \\ &\leq (1-\beta_n)d(x_n, x) + \beta_n d\left(T_{i_n}^{(n)}(x_n), T_{i_n}^{(n)}(x)\right) \\ &\leq (1-\beta_n)d(x_n, x) + \beta_n d(x_n, x) \\ &= d(x_n, x). \end{aligned} \tag{3.8}$$

Let $\triangle\left(x, x_n, T_{i_n}^{(n)}(y_n)\right) \subseteq M$ be a geodesic triangle with vertices x, x_n and $T_{i_n}^{(n)}(y_n)$, and $\triangle\left(\overline{x}, \overline{x_n}, \overline{T_{i_n}^{(n)}(x_n)}\right) \subseteq \mathbb{R}^2$ be the corresponding comparison triangle. Then, we have

$$\begin{aligned} &d(x, x_n) = \|\overline{x} - \overline{x_n}\|, d\left(x_n, T_{i_n}^{(n)}(y_n)\right) = \left\|\overline{x_n} - \overline{T_{i_n}^{(n)}(y_n)}\right\|, \text{and} \\ &d\left(T_{i_n}^{(n)}(y_n), x\right) = \left\|\overline{T_{i_n}^{(n)}(y_n)} - \overline{x}\right\|. \end{aligned} \tag{3.9}$$

Recall from (3.2) that $x_{n+1} = \exp_{x_n} \alpha_n \exp_{x_n}^{-1} T_{i_n}^{(n)}(y_n)$, then

$$\overline{x_{n+1}} = (1-\alpha_n)\overline{x_n} + \alpha_n \overline{T_{i_n}^{(n)}(y_n)}.$$

In view of (2.5), we have

$$d(x_{n+1}, x) \leq \|\overline{x_{n+1}} - \overline{x}\|. \tag{3.10}$$

From expression (3.9), yields

$$\begin{aligned} d^2(x_{n+1},x) &\leq \|\overline{x_{n+1}}-\overline{x}\|^2 \\ &= \left\|(1-\alpha_n)\overline{x_n}+\alpha_n\overline{T_{i_n}^{(n)}(y_n)}-\overline{x}\right\|^2 \\ &= \left\|(1-\alpha_n)(\overline{x_n}-\overline{x})+\alpha_n\left(\overline{T_{i_n}^{(n)}(y_n)}-\overline{x}\right)\right\|^2 \\ &= (1-\alpha_n)\|\overline{x_n}-\overline{x}\|^2+\alpha_n\left\|\overline{T_{i_n}^{(n)}(y_n)}-\overline{x}\right\|^2-\alpha_n(1-\alpha_n)\left\|\overline{x_n}-\overline{T_{i_n}^{(n)}(y_n)}\right\|^2 \\ &= (1-\alpha_n)d^2(x_n,x)+\alpha_n d^2\left(T_{i_n}^{(n)}(y_n),x\right)-\alpha_n(1-\alpha_n)d^2\left(x_n,T_{i_n}^{(n)}(y_n)\right) \\ &\leq (1-\alpha_n)d^2(x_n,x)+\alpha_n d^2\left(y_n,x\right)-\alpha_n(1-\alpha_n)d^2\left(x_n,T_{i_n}^{(n)}(y_n)\right). \end{aligned} \tag{3.11}$$

Since $x_{n+1}=\exp_{x_n}\alpha_n\exp_{x_n}^{-1}T_{i_n}^{(n)}(y_n)$, then

$$d(x_{n+1},x_n)=\alpha_n d\left(x_n,T_{i_n}^{(n)}(y_n)\right). \tag{3.12}$$

Substituting (3.8) and (3.12) into (3.11), we have

$$\begin{aligned} d^2(x_{n+1},x) &\leq (1-\alpha_n)d^2(x_n,x)+\alpha_n d^2\left(x_n,x\right)-\frac{(1-\alpha_n)}{\alpha_n}d^2\left(x_{n+1},x_n\right) \\ &= d^2(x_n,x)-\frac{(1-\alpha_n)}{\alpha_n}d^2(x_{n+1},x_n), \end{aligned} \tag{3.13}$$

and we further have

$$d^2(x_{n+1},x_n)\leq\frac{\alpha_n}{1-\alpha_n}(d^2(x_n,x)-d^2(x_{n+1},x)). \tag{3.14}$$

As a result, condition (i) holds.

(ii) From (3.14), we have

$$d^2(x_{n+1},x_n) \leq \frac{\alpha_n}{1-\alpha_n}d^2(x_n,x)-\frac{\alpha_n}{1-\alpha_n}d^2(x_{n+1},x),$$

which implies that

$$\begin{aligned} \frac{\alpha_n}{1-\alpha_n}d^2(x_{n+1},x) &\leq \frac{\alpha_n}{1-\alpha_n}d^2(x_n,x)-d^2(x_{n+1},x_n) \\ &\leq \frac{\alpha_n}{1-\alpha_n}d^2(x_n,x). \end{aligned}$$

Thus, $d(x_{n+1},x)\leq d(x_n,x)$ for all $n\in\mathbb{N}$, which means that $\{x_n\}$ is Fejér monotone w.r.t. C. Next, we show that $\lim_{n\to+\infty}d(x_{n+1},x_n)=0$. Suppose that (3.4) holds. Then there exists $N\in\mathbb{N}$ and $\epsilon>0$ such that $\alpha_n(1-\alpha_n)\geq\epsilon$ for each $n\geq N$. Furthermore, we can verify that

$$\frac{\alpha_n}{1-\alpha_n}\leq\frac{1}{\epsilon},\quad \forall n\geq N.$$

From (3.14), we have

$$d^2(x_{n+1}, x_n) \leq \frac{1}{\epsilon}(d^2(x_n, x) - d^2(x_{n+1}, x)), \quad \forall n \geq N.$$

Since $\{x_n\}$ is a Fejér monotone w.r.t. C, it follows that $\lim_{n\to+\infty} d(x_n, x)$ exists. By letting $n \to \infty$ to the last inequality, we can have $\lim_{n\to+\infty} d(x_{n+1}, x_n) = 0$.

Assume that (3.6) holds. Now, we show that $\lim_{n\to+\infty} d(x_n, y_n) = 0$. Fix $n \in \mathbb{N}$, let $\triangle\left(x, x_n, T_{i_n}^{(n)}(x_n)\right) \subseteq M$ be a geodesic triangle with vertices x, x_n and $T_{i_n}^{(n)}(x_n)$, and let $\triangle\left(\overline{x}, \overline{x_n}, \overline{T_{i_n}^{(n)}(x_n)}\right) \subseteq \mathbb{R}^2$ be the corresponding comparison triangle. Then, we obtain

$$\begin{aligned} &d(x, x_n) = \|\overline{x} - \overline{x_n}\|, \quad d\left(x_n, T_{i_n}^{(n)}(x_n)\right) = \left\|\overline{x_n} - \overline{T_{i_n}^{(n)}(x_n)}\right\|, \text{ and} \\ &d\left(T_{i_n}^{(n)}(x_n), x\right) = \left\|\overline{T_{i_n}^{(n)}(x_n)} - \overline{x}\right\|. \end{aligned} \tag{3.15}$$

Recall from (3.2) that $y_n = \exp_{x_n} \beta_n \exp_{x_n}^{-1} T_{i_n}^{(n)}(x_n)$ and set

$$\overline{y_n} = (1 - \beta_n)\overline{x_n} + \beta_n \overline{T_{i_n}^{(n)}(x_n)}.$$

In view of (2.5) and (3.15), yields

$$\begin{aligned} d^2(y_n, x) &\leq \|\overline{y_n} - \overline{x}\|^2 \\ &= \left\|(1-\beta_n)\overline{x_n} + \beta_n \overline{T_{i_n}^{(n)}(x_n)} - \overline{x}\right\|^2 \\ &= \left\|(1-\beta_n)(\overline{x_n} - \overline{x}) + \beta_n\left(\overline{T_{i_n}^{(n)}(x_n)} - \overline{x}\right)\right\|^2 \\ &= (1-\beta_n)\|\overline{x_n} - \overline{x}\|^2 + \beta_n\left\|\overline{T_{i_n}^{(n)}(x_n)} - \overline{x}\right\|^2 - \beta_n(1-\beta_n)\left\|\overline{x_n} - \overline{T_{i_n}^{(n)}(x_n)}\right\|^2 \\ &= (1-\beta_n)d^2(x_n, x) + \beta_n d^2\left(T_{i_n}^{(n)}(x_n), x\right) - \beta_n(1-\beta_n)d^2\left(x_n, T_{i_n}^{(n)}(x_n)\right) \\ &\leq (1-\beta_n)d^2(x_n, x) + \beta_n d^2(x_n, x) - \beta_n(1-\beta_n)d^2\left(x_n, T_{i_n}^{(n)}(x_n)\right) \\ &= d^2(x_n, x) - \beta_n(1-\beta_n)d^2\left(x_n, T_{i_n}^{(n)}(x_n)\right). \end{aligned} \tag{3.16}$$

Since $y_n = \exp_{x_n} \beta_n \exp_{x_n}^{-1} T_{i_n}^{(n)}(x_n)$, we deduce that

$$d(x_n, y_n) = \beta_n d\left(x_n, T_{i_n}^{(n)}(x_n)\right). \tag{3.17}$$

Substitution (3.17) into (3.16), we get

$$d^2(y_n, x) \leq d^2(x_n, x) - \frac{(1-\beta_n)}{\beta_n} d^2(x_n, y_n). \tag{3.18}$$

By combining (3.11) and (3.18), we have

$$\begin{aligned} d^2(x_{n+1}, x) &\leq (1-\alpha_n)d^2(x_n, x) + \alpha_n d^2(y_n, x) \\ &\leq (1-\alpha_n)d^2(x_n, x) + \alpha_n\left(d^2(x_n, x) - \frac{(1-\beta_n)}{\beta_n}d^2(x_n, y_n)\right) \\ &= d^2(x_n, x) - \frac{\alpha_n(1-\beta_n)}{\beta_n}d^2(x_n, y_n), \end{aligned}$$

and we further have

$$d^2(x_n, y_n) \le \frac{\beta_n}{\alpha_n(1-\beta_n)}(d^2(x_n, x) - d^2(x_{n+1}, x)). \tag{3.19}$$

Since (3.6) holds, then there exists $N \in \mathbb{N}$ and $\eta > 0$ such that $\alpha_n\beta_n(1-\beta_n) \ge \eta$ for all $n \ge N$. It is easy to check that

$$\frac{\beta_n}{\alpha_n(1-\beta_n)} \le \frac{1}{\eta}, \quad \forall n \ge N.$$

From (3.19), implies that

$$d^2(x_n, y_n) \le \frac{1}{\eta}(d^2(x_n, x) - d^2(x_{n+1}, x)), \quad \forall n \ge N.$$

Recall that $\{x_n\}$ is Fejér monotone w.r.t. C so that $\lim_{n\to+\infty} d(x_n, x)$ exists. Hence, from the above inequality, we have $\lim_{n\to+\infty} d(x_n, y_n) = 0$. The proof is therefore completed. ∎

Following the definitions of focusing algorithm and linearly focusing algorithm, we will prove that the sequence $\{x_n\}$ generated by Algorithm 3.1 converges to a point in C.

Definition 3.3. [4] An algorithm is said to be

(i) *focusing* if for all $j \in I$ and every subsequence $\{x_{n_k}\}$ of $\{x_n\}$,

$$\left.\begin{array}{l} x_{n_k} \to x \\ d(x_{n_k}, T_j^{(n_k)}(x_{n_k})) \to 0 \\ i_{n_k} = j \quad \text{for all } k \in \mathbb{N}. \end{array}\right\} \Longrightarrow x \in C_j; \tag{3.20}$$

(ii) *linearly focusing* if there is $\lambda > 0$ such that

$$\lambda d(x_n, C_{i_n}) \le d(x_n, C_{i_n}^{(n)}) \quad \text{for all } n \in \mathbb{N}, \tag{3.21}$$

where $\{x_n\}$ is a sequence generated by Algorithm 3.1 and $C_{i_n}^{(n)}$ is a closed geodesic convex nonempty set containing C_{i_n}.

Every a linearly focusing algorithm is a focusing algorithm.

Remark 3.4. [30] In the case when the sequence $\left\{T_{i_n}^{(n)}\right\}$ of firmly nonexpansive mappings satisfies that $F(T_{i_n}) = C_{i_n}$ and $T_{i_n}^{(n)} = T_{i_n}$ for all $n \in \mathbb{N}$, the algorithm is linearly focusing; in particular the algorithm is linearly focusing when $T_{i_n}^{(n)} = P_{C_{i_n}}$ for all $n \in \mathbb{N}$.

We can now present the main result as follows.

Theorem 3.5. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.1. Assume that Algorithm 3.1 is focusing and $\{\alpha_n\}$, $\{\beta_n\}$ satisfy (3.4), (3.6), respectively. Then the sequence $\{x_n\}$ converges to a point in C.*

Proof. As a consequence of Lemma 3.2, $\{x_n\}$ is Fejér monotone w.r.t C and is bounded by Lemma 2.9. Thus there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\lim_{k\to+\infty} x_{n_k} = x^* \in M$. We shall verify $x^* \in C$ in this proof.

Noting the definition of $\{i_{n_k}\}$, without loss of generality we suppose that $i_{n_k} = m$ for all k. Let $j \in I$ and consider the subsequence $\{x_{n_k+j}\}_{k=0}^{\infty}$. From (3.5), we have

$$\lim_{k\to\infty} x_{n_k+j} = x^* \quad \text{and} \quad i_{n_k+j} = j. \tag{3.22}$$

Because $\{\beta_n\} \subseteq (0,1)$ then $\{\beta_n\}$ is bounded below by some positive numbers. In view of (3.17) and the fact that $\lim_{n\to+\infty} d(x_n, y_n) = 0$, we get

$$\begin{aligned} \lim_{k\to+\infty} d\left(x_{n_k+j}, T_j^{(n_k+j)}(x_{n_k+j})\right) &= \lim_{k\to+\infty} \frac{1}{\beta_{n_k}} d\left(x_{n_k+j}, y_{n_k+j}\right) \\ &= 0. \end{aligned}$$

This, together with (3.22), yields that $x^* \in C_j$ since Algorithm 3.1 is focusing. Therefore, $x^* \in C$ as $j \in I$ arbitrary. Moreover, by Lemma 2.9, $\lim_{n\to+\infty} x_n = x^*$ as required. The proof is therefore completed.

∎

According ot Remark 3.4, we can have the following corollary.

Corollary 3.6. *Suppose that $T_{i_n}^{(n)} :\equiv P_{C i_n}$ and $\{\alpha_n\}$, $\{\beta_n\}$ satisfy condition (3.4) and (3.6), respectively. Then any sequence $\{x_n\}$ generated by Algorithm 3.1 converges to a point in C.*

Proof. From Proposition 2.8, P_{C_i} is firmly nonexpansive for any $i \in I$. Furthermore, Algorithm 3.1 is focusing by Remark 3.4. Follows from the proof of Theorem 3.5, and is thus omitted. ∎

4. Linear convergence of Two-step Cyclic Projection Algorithm

In this section, we discuss the linear convergence of the Algorithm 3.1 where each $T_{i_n}^{(n)}$ is the projection onto some closed convex nonempty set $C_{i_n}^{(n)}$ containing C_{i_n}, i.e.,

$$T_{i_n}^{(n)} := P_{C_{i_n}^{(n)}} \text{ and } C_{i_n}^{(n)} \supseteq C_{i_n}, \quad \forall n \in \mathbb{N}. \tag{4.1}$$

Next, let us present the concept of linear convergence.

Definition 4.1. Let $\{x_n\} \subset M$ such that $\{x_n\}$ converges to a point $x \in M$. Then, the convergence is said to be *linear convergence* if and only if there exist a constant $\theta < 1$ and a positive $N \in \mathbb{N}$ such that

$$d(x_n, \overline{x}) \leq \theta d(x_{n-1}, x), \quad \forall n > N.$$

For establishing the linear convergence results, we need the definitions of linear regularity and bounded linear regularity.

Definition 4.2. [30] A family $\{C_i : i \in I\}$ is called

(i) *linearly regular* if there exists $\tau > 0$ such that

$$d(x, C) \leq \tau \max_{i \in I}\{d(x, C_i)\}, \tag{4.2}$$

for all $x \in M$.

(ii) *bounded linearly regular* if, for any bounded subset $S \subseteq M$, there exist $\tau_S > 0$ such that (4.2) holds for any $x \in S$ with $\tau = \tau_S$.

Next, we present and prove the linear convergence theorem for the two-step cyclic projection algorithm.

Theorem 4.3. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.1. Suppose that Algorithm 3.1 is linearly focusing, the family $\{C_i : i \in I\}$ is bounded linearly regular, and the conditions (3.4) and (3.6) hold. Then $\{x_n\}$ converges linearly to a point $x \in C$.*

Proof. Lemma 3.2 and Theorem 3.5 are applicable according to the assumptions. Then, $\{x_n\}$ is Fejér monotone w.r.t. C and converges to a point $x \in C$. Next, we show that subsequence $\{x_{km}\}_{k=0}^{+\infty}$ converges linearly, i.e. there is some $\gamma > 0$ and $\theta \in (0, 1)$ such that

$$d(x_{km}, x) \leq \gamma\theta^k, \quad \forall k \in \mathbb{N}. \tag{4.3}$$

To verify (4.3), without loss of generality, since (3.4) holds, then there exists $\epsilon > 0$ such that

$$\alpha_n(1 - \alpha_n) \geq \epsilon > 0, \quad \forall n \in \mathbb{N}. \tag{4.4}$$

This implies that

$$\frac{\alpha_n}{1 - \alpha_n} \leq \frac{1}{\epsilon}, \quad \forall n \in \mathbb{N}. \tag{4.5}$$

Let $i \in I$ and $k \in \mathbb{N}$. It is easy to see that

$$d(x_{km}, x_{km+i}) \leq \sum_{j=0}^{m-1} d(x_{km+j}, x_{km+j+1}).$$

From the last inequality, we get

$$\begin{aligned} d^2(x_{km}, x_{km+i}) &\leq \left(\sum_{j=0}^{m-1} 1 \cdot d(x_{km+j}, x_{km+j+1})\right)^2 \\ &\leq \left(\sum_{j=0}^{m-1} 1^2\right)\left(\sum_{j=0}^{m-1} d^2(x_{km+j}, x_{km+j+1})\right) \\ &= m\sum_{j=0}^{m-1} d^2(x_{km+j}, x_{km+j+1}). \end{aligned}$$

Substitution (3.3) into the last inequality, we obtain

$$d^2(x_{km}, x_{km+i}) \leq m\sum_{j=0}^{m-1} \frac{\alpha_{km+j}}{1 - \alpha_{km+j}}(d^2(x_{km+j}, z) - d^2(x_{km+j+1}, z))$$

for any $z \in C$, and it follows from (4.5),

$$d^2(x_{km}, x_{km+i}) \leq \frac{m}{\epsilon}(d^2(x_{km}, C) - d^2(x_{(k+1)m}, C)). \tag{4.6}$$

Similarly, from (3.6) holds, then there exist $\eta > 0$ such that

$$\alpha_n \beta_n (1 - \beta_n) \geq \eta > 0, \quad \forall n \in \mathbb{N}. \tag{4.7}$$

This implies that

$$\frac{\beta_n}{\alpha_n(1 - \beta_n)} \leq \frac{1}{\eta}, \quad \forall n \in \mathbb{N}. \tag{4.8}$$

We also have

$$\frac{1}{\alpha_n \beta_n (1 - \beta_n)} \leq \frac{1}{\eta} \Longrightarrow \frac{1}{\beta_n} \leq \frac{\alpha_n(1 - \beta_n)}{\eta} \leq \frac{1}{\eta}, \quad \forall n \in \mathbb{N}. \tag{4.9}$$

For $i \in I$ and $k \in \mathbb{N}$. It easy to see that

$$d(x_{km+i}, y_{km+i}) \leq \sum_{j=1}^{m} d(x_{km+j}, y_{km+j}).$$

Following from the above inequality, we obtain

$$\begin{aligned} d^2(x_{km+i}, y_{km+i}) &\leq \left(\sum_{j=1}^{m} 1 \cdot d(x_{km+j}, y_{km+j}) \right)^2 \\ &\leq \left(\sum_{j=1}^{m} 1^2 \right) \left(\sum_{j=0}^{m} d^2(x_{km+j}, y_{km+j}) \right) \\ &= m \sum_{j=1}^{m} d^2(x_{km+j}, y_{km+j}). \end{aligned}$$

In view of (3.19), we conclude that

$$d^2(x_{km+i}, y_{km+i}) \leq m \sum_{j=1}^{m} \frac{\beta_{km+j}}{\alpha_{km+j}(1 - \beta_{km+j})} (d^2(x_{km+j}, z) - d^2(x_{km+j+1}, z))$$

for any $z \in C$. Summing up the last inequality and applying (4.8), we get

$$d^2(x_{km+i}, y_{km+i}) \leq \frac{m}{\eta}(d^2(x_{km}, C) - d^2(x_{(k+1)m}, C)). \tag{4.10}$$

From algorithm is linearly focusing, there exists $\lambda > 0$ (independent of i and k) such that

$$\lambda d(x_{km+i}, C_i) \leq d\left(x_{km+i}, C_i^{(km+i)}\right). \tag{4.11}$$

In view of (3.2) and (4.1), we get

$$\begin{aligned} d(x_{km+i}, C_i^{(km+i)}) &= d\left(x_{km+i}, T_i^{(km+i)}(x_{km+i})\right) \\ &= \frac{1}{\beta_{km+i}} d(x_{km+i}, y_{km+i}). \end{aligned} \tag{4.12}$$

Substitution (4.9) and (4.10) into the last inequality,

$$\begin{aligned} d^2(x_{km+i}, C_i) &\leq \frac{1}{\lambda^2\beta^2_{km+i}}\left[\frac{m}{\eta}(d^2(x_{km}, C) - d^2(x_{(k+1)m}, C))\right] \\ &\leq \frac{m}{\lambda^2\eta^3}\left(d^2(x_{km}, C) - d^2(x_{(k+1)m}, C)\right). \end{aligned} \tag{4.13}$$

Consider,

$$\begin{aligned} d^2(x_{km}, C_i) &\leq (d(x_{km}, x_{km+i}) + d(x_{km+i}, C_i))^2 \\ &\leq 2d^2(x_{km}, x_{km+i}) + 2d^2(x_{km+i}, C_i). \end{aligned} \tag{4.14}$$

By combing (4.6), (4.13) and (4.14), we obtain

$$d^2(x_{km}, C_i) \leq \left(\frac{2m}{\epsilon} + \frac{2m}{\lambda^2\eta^3}\right)(d^2(x_{km}, C) - d^2(x_{(k+1)m}, C)). \tag{4.15}$$

From the fact that the family $\{C_i : i \in I\}$ is bounded linearly regular and $\{x_n\}$ is bounded, then there exists $\tau > 0$ such that

$$d(x_n, C) \leq \tau \max_{i\in I}\{d(x_n, C_i)\}, \quad \forall\ n \in \mathbb{N}.$$

Thereby,

$$\begin{aligned} d^2(x_{km}, C) &\leq \tau^2 \max_{i\in I}\{d^2(x_{km}, C_i)\} \\ &\leq \tau^2\left(\frac{2m}{\epsilon} + \frac{2m}{\lambda^2\eta^3}\right)(d^2(x_{km}, C) - d^2(x_{(k+1)m}, C)). \end{aligned}$$

The subsequence $\{x_{km}\}_{k=0}^{+\infty}$ is linearly converges to $x \in C$ by using (ii) of Lemma 2.9. This implies that (4.3) holds. Fix $n \in \mathbb{N}$, and set

$$n = km + r \quad \text{where } r \in \{0, 1, \ldots, m-1\}.$$

Then we conclude

$$d(x_n, x) \leq d(x_{km}, x) \leq \gamma(\theta^{\frac{1}{m}})^{km} = \frac{\gamma(\theta^{\frac{1}{m}})^{km+r}}{\theta^{\frac{r}{m}}} \leq \frac{\gamma}{\theta}(\theta^{\frac{1}{m}})^n,$$

and complete the proof. ∎

We obtain the following corollary from Remark 3.4 and Theorem 4.3 in the spacial case when $C_{i_n}^{(n)} = C_{i_n}$ in (4.1) for all $n \in \mathbb{N}$.

Corollary 4.4. *Let $\{x_n\}$ be a sequence generated by the two-step cyclic projection algorithm. Suppose that conditions (3.4), (3.6) hold, the family $\{C_i : i \in I\}$ is boundedly linearly regular. Suppose further that, for all $n \in \mathbb{N}$, $C_{i_n}^{(n)} = C_{in}$ in (4.1). Then $\{x_n\}$ converges linearly to a point in C.*

5. Numerical Example

In this section, we provide a numerical examples in Hadamard manifolds to illustrate the convergence behavior of Algorithms 3.1. All the programs are written in `Matlab R2016b` and computed on `PC Intel(R) Core(TM) i7 @1.80 GHz` and a `8 GB 1600 MHz DDR3` Memory.

Let $M = \mathbb{H} := \{(t_1, t_2) \in \mathbb{R}^2 \mid t_2 > 0\}$ be the *Poincaré plane* endowed with the Riemannian metric defined by

$$g_{11} = g_{22} := \frac{1}{t_2^2}, g_{12} := 0 \text{ for any } (t_1, t_2) \in \mathbb{H}. \tag{5.1}$$

The sectional curvature of $\mathbb{H}$ is equal to -1 and the geodesics of the Poincaré plane are the semilines $\gamma_a : t_1 = a, t_2 > 0$ and the semicircles $\gamma_{b,r} : (t_1 - b)^2 + t_2^2 = r^2, t_2 > 0$; or admit the following natural parameterizations

$$\begin{aligned} &\gamma_a : t_1 = a, \ t_2 = e^s, && s \in (-\infty, +\infty); \\ &\gamma_{b,r} : t_1 = -r \tanh s, \ t_2 = \frac{r}{\cosh s}, && s \in (-\infty, +\infty); \end{aligned} \tag{5.2}$$

see e.g., [29]. Furthermore, consider two points $y = (t_1^y, t_2^y)$ and $z = (t_1^z, t_2^z)$ in $\mathbb{H}$. Then the Riemannian distance between y, z is given by

$$d_{\mathbb{H}}(y, z) = \begin{cases} \left| \ln \dfrac{t_2^z}{t_2^y} \right|, & \text{if } t_1^y = t_1^z, \\ \left| \ln \dfrac{t_1^y - b + r}{t_1^z - b + r} \cdot \dfrac{t_2^z}{t_2^y} \right|, & \text{if } t_1^y \neq t_1^z, \end{cases}$$

where

$$b = \frac{(t_1^y)^2 + (t_2^y)^2 - ((t_1^z)^2 + (t_2^z)^2)}{2(t_1^y - t_1^z)} \quad \text{and} \quad r = \sqrt{(t_1^y - b)^2 + (t_2^y)^2}.$$

To get the expression of $\exp_y^{-1} z$, we consider a smooth geodesic curve γ joining y to z defined by

$$\gamma(s) := (\gamma_1(s), \gamma_2(s)), \quad s \in [0, 1]$$

where for each $s \in [0, 1]$, $\gamma_1(s)$ and $\gamma_2(s)$ are respectively defined by

$$\gamma_1(s) := \begin{cases} t_1^y, & \text{if } t_1^y = t_1^z, \\ b - r \tanh\left((1 - s) \cdot \operatorname{arctanh} \dfrac{b - t_1^y}{r} + s \cdot \operatorname{arctanh} \dfrac{b - t_1^z}{r}\right), & \text{if } t_1^y \neq t_1^z, \end{cases}$$

and

$$\gamma_2(s) := \begin{cases} e^{(1-s)\cdot \ln t_2^y + s \cdot \ln t_2^z}, & \text{if } t_1^y = t_1^z, \\ \dfrac{r}{\cosh\left((1 - s) \cdot \operatorname{arctanh} \dfrac{b - t_1^y}{r} + s \cdot \operatorname{arctanh} \dfrac{b - t_1^z}{r}\right)}, & \text{if } t_1^y \neq t_1^z. \end{cases}$$

By the Riemannian metric endowed on $\mathbb{H}$ (c.f. (5.1)), one checks that

$$\gamma'(0) = \left(\frac{d\gamma_1(s)}{ds}, \frac{d\gamma_2(s)}{ds} \right) \bigg|_{s=0};$$

see [14], page 7. Therefore, by elementary calculus, we get that

$$\exp_y^{-1} z = \gamma'(0) = \begin{cases} \left(0, t_2^y \ln \frac{t_2^z}{t_2^y}\right), & \text{if } t_1^y = t_1^z, \\ \frac{t_2^y \left(\operatorname{arctanh} \frac{b - t_1^y}{r} - \operatorname{arctanh} \frac{b - t_1^z}{r}\right)}{r} (t_2^y, b - t_1^y), & \text{if } t_1^y \neq t_1^z. \end{cases} \tag{5.3}$$

Example 5.1. Follows from [30], the Example 5.1: Let $M = \mathbb{H}^2$ and C_1, C_2 be closed convex subsets of M defined as

$$C_1 := \{(t_1, t_2) \in M : t_2 \geq 1\}$$

and

$$C_2 := \{(t_1, t_2 \in M : t_1^2 + t_2^2 \leq 1)\}.$$

From [29], page 301, C_1 and C_2 are convex because that C_1 is level set convex function $f : M \to \mathbb{R}$ defined by

$$f(y) = \frac{1}{t_2}, \ \forall y \in (t_1, t_2) \in M$$

and $\gamma := \{(t_1, t_2) \in M : t_1 + t_2 = 1\}$ is a geodesic of M, receptively. Moreover $C = C_1 \cap C_2 = \{(0, 1)\}$,

$$P_{C_1}(x) = (t_1, 1), \ \forall x = (t_1, t_2) \notin C_1$$

and

$$P_{C_2}(x) = \left(\frac{2t_1}{t_1^2 + t_2^2 + 1}, \sqrt{1 - \left(\frac{2t_1}{t_1^2 + t_2^2 + 1}\right)}\right) \text{ for any } x = (t_1, t_2) \notin C_2.$$

We implement the projection algorithm to find $x^* = (0, 1) \in C_1 \cap C_2$ which is defined as: Choose $x_0 \in M$ and defined x_{n+1} by

$$\begin{cases} x_{n+1} = \exp_{x_n} \alpha_n \exp_{x_n}^{-1} T_{i_n}^{(n)}(y_n) \\ y_n = \exp_{x_n} \beta_n \exp_{x_n}^{-1} T_{i_n}^{(n)}(x_n), \quad \forall n \in \mathbb{N}, \end{cases}$$

where $i_n := (n \mod 2) + 1$ and $\{\alpha_n\}, \{\beta_n\} \in (0, 1)$ are constants. The numerical results are listed in Table 1 and Table 2 with an initial point $x_0 = (1, 1)$ and $x_0 = (3, 2)$, respectively; and each one shows the results for three different constant relaxation parameters, $\alpha_n = 0.3, 0.6, 0.9$ and $\beta_n = 0.3, 0.6, 0.9$, respectively. Moreover, the numerical results displayed on Figure 1 and Figure 2 which depicts the " Distance to Solution" versus "Iteration Number". The numerical results show the convergence tendency of the algorithm as predicted by Theorem 3.5; furthermore, we observe that the bigger the relaxation parameters α_n and β_n, the faster the algorithm converges.

Table 1. The comparison of Projection Algorithm for the initial point is $x_0 = (1, 1)$ with relaxation parameters α_n and β_n.

Iteration No.	Initial point $x_0 = (1, 1)$		
	$\alpha_n = 0.3, \beta_n = 0.3$	$\alpha_n = 0.6, \beta_n = 0.6$	$\alpha_n = 0.9, \beta_n = 0.9$
1	(0.962023, 0.980092)	(0.857514, 0.914993)	(0.715276, 0.796372)
2	(0.962023, 0.981867)	(0.857514, 0.944729)	(0.715276, 0.957661)
3	(0.928696, 0.962795)	(0.773302, 0.878510)	(0.609786, 0.835884)
4	(0.928696, 0.966086)	(0.773302, 0.920445)	(0.609786, 0.966513)
5	(0.899310, 0.947932)	(0.716982, 0.867794)	(0.542662, 0.870441)
6	(0.899310, 0.952505)	(0.716982, 0.913244)	(0.542662, 0.973981)
7	(0.873262, 0.935291)	(0.675205, 0.869113)	(0.494071, 0.893398)
8	(0.873262, 0.940939)	(0.675205, 0.914132)	(0.494071, 0.978810)
9	(0.850044, 0.924649)	(0.641820, 0.875387)	(0.456705, 0.909423)
10	(0.850044, 0.931191)	(0.64182, 0.918349)	(0.456705, 0.982122)
⋮	⋮	⋮	⋮
50	(0.630933, 0.904178)	(0.382564, 0.970729)	(0.232119, 0.995622)
⋮	⋮	⋮	⋮
100	(0.520102, 0.933159)	(0.284848, 0.983952)	(0.167478, 0.997740)

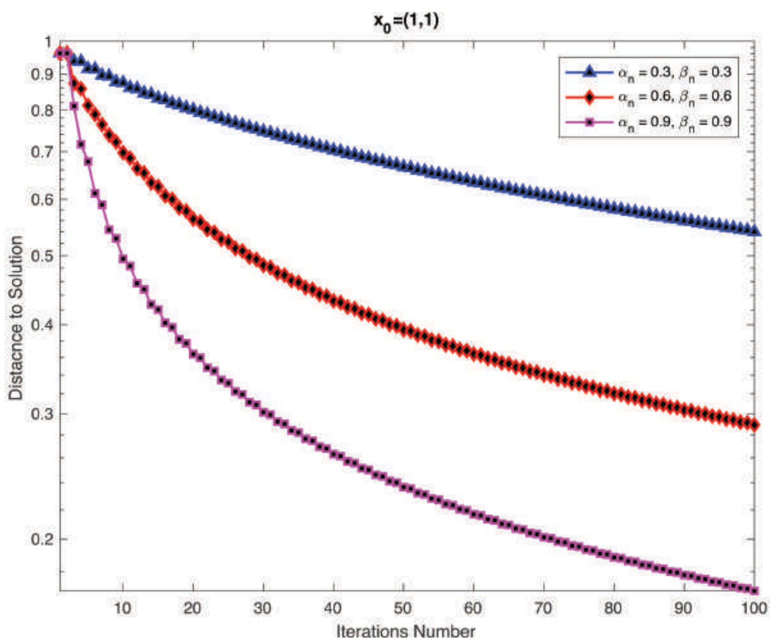

Fig. 1. Distance to solution $x^* = (0, 1)$ of each iteration number where the initial point is $x_0 = (1, 1)$.

Table 2. The comparison of Projection Algorithm for the initial point is $x_0 = (3, 2)$ with relaxation parameters α_n and β_n.

Iteration No.	Initial point $x_0 = (3, 2)$		
	$\alpha_n = 0.3, \beta_n = 0.3$	$\alpha_n = 0.6, \beta_n = 0.6$	$\alpha_n = 0.9, \beta_n = 0.9$
1	(2.694926, 1.972871)	(1.842587, 1.799538)	(0.784290, 1.157324)
2	(2.694926, 1.855837)	(1.842587, 1.456475)	(0.784290, 1.028150)
3	(2.413940, 1.912864)	(1.227856, 1.354538)	(0.617482, 0.850046)
4	(2.413940, 1.804398)	(1.227856, 1.214357)	(0.617482, 0.969603)
5	(2.162268, 1.832029)	(0.947803, 1.084283)	(0.546490, 0.869407)
6	(2.162268, 1.734876)	(0.947803, 1.053153)	(0.947803, 1.053153)
7	(1.941625, 1.740411)	(0.811160, 0.957026)	(0.496873, 0.892178)
8	(1.941625, 1.655744)	(0.811160, 0.972280)	(0.496873, 0.978556)
9	(1.751247, 1.645610)	(0.733998, 0.903834)	(0.458905, 0.908521)
10	(1.751247, 1.573466)	(0.733998, 0.937339)	(0.458905, 0.981937)
⋮	⋮	⋮	⋮
50	(0.708503, 0.918746)	(0.395949, 0.968588)	(0.232401, 0.995611)
⋮	⋮	⋮	⋮
100	(0.552972, 0.924655)	(0.290245, 0.983330)	(0.167584, 0.997737)

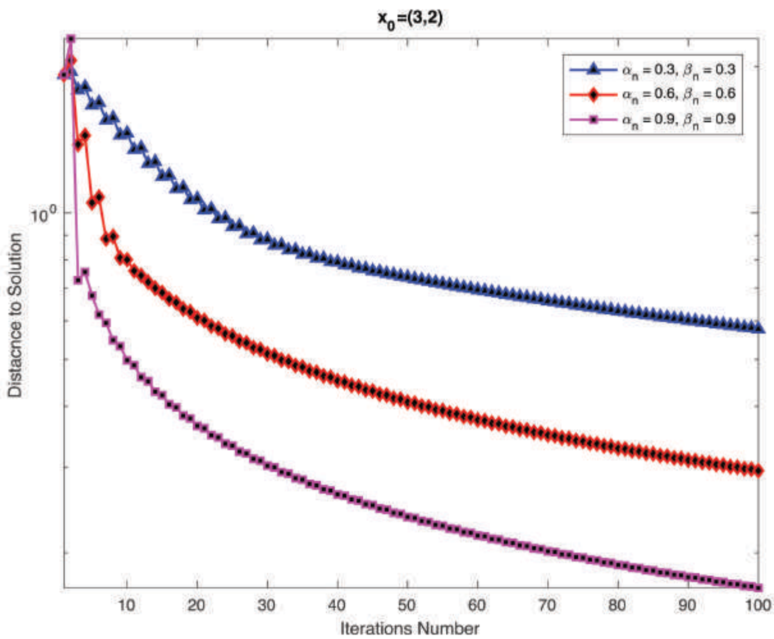

Fig. 2. Distance to solution $x^* = (0, 1)$ of each iteration number where the initial point is $x_0 = (2, 3)$.

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