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# A Spectral Conjugate Gradient-like Method for Convex Constrained Nonlinear Monotone Equations and Signal Recovery

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## ABSTRACT

Many real-world problems can be formulated as systems of nonlinear equations. Thus, finding their solutions is of paramount importance. Traditional approaches such as Newton and quasi-Newton methods for solving these systems require computing Jacobian matrix or an approximation to it at every iteration, which is very expensive especially when the dimension of the systems is large. In this work, we propose a derivative-free algorithm for solving these systems. The proposed algorithm is a combination of the popular conjugate gradient method for unconstrained optimization problems and the projection method. We prove the global convergence of the proposed algorithm under Lipschitz continuity and monotonicity assumptions on the underlying mapping. We perform numerical experiments on some test problems, and the proposed algorithm proves to be more efficient in comparison with some existing works. Finally, we give an application of the proposed algorithm in signal recovery.

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## 1. Introduction

In recent years, spectral and conjugate gradient projection based methods have received much attention in solving convex constrained systems of nonlinear monotone equations given as:

$$J(x) = 0, \quad \text{subject to} \quad x \in \Lambda \subseteq \mathbb{R}^n, \tag{1.1}$$



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where $\Lambda$ is nonempty, closed and convex, and $J : \mathbb{R}^n \to \mathbb{R}^n$ is continuous and monotone. Monotone means

$$\langle J(x) - J(y),\ x - y\rangle \geq 0 \quad \text{for all} \quad x, y \in \mathbb{R}^n.$$

Problem (1.1) arises in a number of applications from engineering and other branches of sciences. For example, in economic equilibrium problems [11], power flow equations [28] and chemical equilibrium systems [22]. Moreover, algorithms for solving systems of nonlinear monotone equations are used in signal and image recovery, (see [14, 29, 30, 3, 1, 4, 17, 2, 6]).

Classical methods for solving (1.1) include Newton and quasi-Newton methods which have fast convergence from good initial guess. However, the main problem associated with these methods include solving linear system using a Jacobian matrix or its approximation at every iteration. As a result, these methods are not suitable to handle large scale problems. On the other hand, spectral and conjugate gradient methods proposed for unconstrained optimization problems do not require any Jacobian matrix or its approximation and are simple to implement. These important properties make them suitable for solving large scale optimization problems. Motivated by these nice properties, researchers extended these methods to solve system of nonlinear equations [9, 18, 19]. Conjugate gradient method produces sequence of iterations using the formula:

$$x_{k+1} = x_k + \alpha_k d_k, \tag{1.2}$$

where $\alpha_k$ is a step size, and $d_k$ is a search direction. To solve (1.1), the definition of the search direction is given as:

$$d_k = \begin{cases} -J(x_k), & \text{if } k = 0, \\ -J(x_k) + \beta_k d_{k-1}, & \text{if } k \geq 1, \end{cases} \tag{1.3}$$

where the parameter $\beta_k$ is a scalar known as the conjugate gradient parameter.

Following the success of the projection technique proposed by Solodov and Svaiter [26], many conjugate gradient projection based methods have been proposed to handle large scale systems of nonlinear equations. For example, Cheng [8] combined the projection technique in [26] and the Polak–Ribière–Polyak (PRP) method [24, 25] to propose a conjugate gradient projection based method for solving nonlinear monotone equations. They proved the global convergence of the method under monotonicity and Lipschitz continuity assumption of the mapping considered. The numerical results presented proved that their method is promising in solving large scale problems. In the work of Xiao and Zhou [30], based on the projection technique in [26], an extension of the popular descent conjugate gradient method (CG_Descent) [15] is proposed for solving convex constrained monotone nonlinear equations. Liu and Li [21] presented another extension of the CG_Descent method to solve convex constrained nonlinear monotone equations. They showed that their proposed method is globally convergent and has some advantages numerically when compared with the method proposed in [30]. Liu and Feng [20] propose a spectral conjugate gradient method for solving convex constrained monotone nonlinear equations. Their work is also a combination of the projection technique [26] and the popular Dai-Yuan conjugate gradient parameter [10]. The numerical results proved that their method is more efficient than the ones proposed in [31] and [21].

Inspired by the above contributions, and the success of the projection technique in [26], we propose a spectral conjugate gradient projection based method for solving (1.1). Among the advantages of the proposed method is that it inherits the low storage requirement property of the spectral conjugate gradient method, and thus, it is suitable to solve large scale nonlinear monotone equations. It is derivative-free, and the global convergence is established without any differentiability assumption. Moreover, we perform numerical experiments on some test problems to depict the efficiency of the proposed algorithm in comparison with the ones proposed in [20] and [33]. Additionally, we apply the proposed algorithm in signal recovery problems, and the quality of the recovered signal proves that the proposed algorithm is more efficient than some existing methods.

The remaining part of this paper is organized as follows. In the next section, we introduce the proposed algorithm, some important definitions and prove global convergence. Then, in section 3, we present some numerical experiments of the proposed algorithm and compare it performance with two existing ones. This is followed by a section where we show the application of the proposed algorithm in signal recovery. In the last section, we give the conclusion of the work.

## 2. Algorithm and Convergence Analysis

We begin this section by defining the projection operator as follows:

**Definition 2.1.** Let $\Lambda \subset \mathbb{R}^n$ be a nonempty closed convex set. Then for any $x \in \mathbb{R}^n$, its projection onto $\Lambda$, denoted by $P_\Lambda(x)$, is defined by

$$P_\Lambda(x) = \arg\min\{\|x - y\| \ : y \in \Lambda\}.$$

The projection operator $P_\Lambda$ has the properties

$$\|P_\Lambda(x) - P_\Lambda(y)\| \leq \|x - y\|, \quad \forall x, y \in \mathbb{R}^n, \tag{2.1}$$

and

$$\|P_\Lambda(x) - y\| \leq \|x - y\|, \quad \forall y \in \Lambda. \tag{2.2}$$

In this work, we define a new search direction as follows:

$$d_k = \begin{cases} -J(x_k), & \text{if } k = 0, \\ -\tau_k J(x_k) + \frac{\|J(x_k)\|^2}{-J(x_{k-1})^T d_{k-1}} s_{k-1}, & \text{if } k \geq 1, \end{cases} \tag{2.3}$$

where $s_{k-1} = \alpha_{k-1} d_{k-1}$, and the parameter $\tau_k$ is obtained such that the direction (2.3) satisfies

$$J(x_k)^T d_k \leq -c\|J(x_k)\|^2, \tag{2.4}$$

which is an important property in establishing the global convergence.

Observe that from (2.3), when $k = 0$, $J(x_k)^T d_k = -\|J(x_k)\|^2$, thus, (2.4) is satisfied with $c = 1$. However, when $k \geq 1$,

$$\begin{aligned} J(x_k)^T d_k &= -\tau_k \|J(x_k)\|^2 - \frac{\|J(x_k)\|^2 J(x_k)^T s_{k-1}}{J(x_{k-1})^T d_{k-1}} \\ &= -(\tau_k + \frac{J(x_k)^T s_{k-1}}{J(x_{k-1})^T d_{k-1}})\|J(x_k)\|^2 \end{aligned} \tag{2.5}$$

To satisfy (2.4), we only need $\tau_k + \frac{J(x_k)^T s_{k-1}}{J(x_{k-1})^T d_{k-1}} \geq c$, $c > 0$. That is, $\tau_k \geq c - \frac{J(x_k)^T s_{k-1}}{J(x_{k-1})^T d_{k-1}}$. In this work, we choose

$$\tau_k = c - \frac{J(x_k)^T s_{k-1}}{J(x_{k-1})^T d_{k-1}}. \tag{2.6}$$

Thus, it is not difficult to see that $\forall k \geq 1$, multiplying (2.3) by $J(x_k)^T$ and substituting $\tau_k = c - \frac{J(x_k)^T s_{k-1}}{J(x_{k-1})^T d_{k-1}}$ gives

$$J(x_k)^T d_k = -c\|J(x_k)\|^2. \tag{2.7}$$

Furthermore, taking absolute value from (2.7), we get $|J(x_k)^T d_k| = c\|J(x_k)\|^2, \ \ \forall k$ and consequently,

$$|J(x_{k-1})^T d_{k-1}| = c\|J(x_{k-1})\|^2. \tag{2.8}$$

We now give the steps of our proposed algorithm as follows:

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**Algorithm 1:** Spectral Conjugate Gradient Projection Method (SCD)

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**Input :** Choose initial point $x_0 \in \Lambda$, $\gamma \in (0, 2), \sigma \in (0, 1)$, $\kappa \in (0, 1]$, $c > 0$, $Tol > 0$ and $\beta \in (0, 1)$. Set $k := 0$

**Step 1:** If $\|J(x_k)\| \leq Tol$, stop, otherwise go to **Step 2**.

**Step 2:** Compute $d_k$ using equation (2.3).

**Step 3:** Compute the step size $\alpha_k = \max\{\kappa\beta^i \ : i = 0, 1, 2, \cdots\}$ such that

$$-J(x_k + \kappa\beta^i d_k)^T d_k \geq \sigma\kappa\beta^i \|d_k\|^2. \tag{2.9}$$

**Step 4:** Set $z_k = x_k + \alpha_k d_k$. If $z_k \in \Lambda$ and $\|J(z_k)\| = 0$, stop. Else compute

$$x_{k+1} = P_\Lambda[x_k - \gamma\zeta_k J(z_k)],$$

where

$$\zeta_k = \frac{J(z_k)^T(x_k - z_k)}{\|J(z_k)\|^2}.$$

**Step 5:** Let $k = k + 1$ and go to **Step 1**.

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In order to establish the global convergence of the proposed algorithm, we assumed the following:

$(Q_1)$ The mapping $J$ is monotone.

$(Q_2)$ The mapping $J$ is Lipschitz continuous, that is there exists a positive constant $L$ such that

$$\|J(x) - J(y)\| \leq L\|x - y\|, \ \forall x, y \in \mathbb{R}^n.$$

$(Q_3)$ The solution set of (1.1), denoted by $\Lambda$, is nonempty.

**Lemma 2.2.** *Suppose that assumptions ($Q_1$)-($Q_3$) hold, then the sequences $\{x_k\}$ and $\{z_k\}$ generated by Algorithm 1 are bounded. Moreover, we have*

$$\lim_{k\to\infty} \|x_k - z_k\| = 0, \tag{2.10}$$

*and*

$$\lim_{k\to\infty} \|x_{k+1} - x_k\| = 0. \tag{2.11}$$

*Proof.* Let $\bar{x}$ be the solution of (1.1). Then, by monotonicity of the mapping $J$, we get

$$\begin{aligned} \langle J(z_k),\ x_k - \bar{x}\rangle &= \langle J(z_k),\ x_k - z_k + z_k - \bar{x}\rangle \\ &= \langle J(z_k),\ x_k - z_k\rangle + \langle J(z_k) - J(\bar{x}),\ z_k - \bar{x}\rangle \\ &\geq \langle J(z_k),\ x_k - z_k\rangle. \end{aligned} \tag{2.12}$$

Using $x_{k+1}$ definition from **Step 4**, equation (2.2) and (2.12) we obtain

$$\begin{aligned} \|x_{k+1} - \bar{x}\|^2 &= \|P_\Lambda[x_k - \gamma\zeta_k J(z_k)] - \bar{x}\|^2 \leq \|x_k - \gamma\zeta_k J(z_k) - \bar{x}\|^2 \\ &= \|x_k - \bar{x}\|^2 - 2\gamma\zeta_k J(z_k)^T(x_k - \bar{x}) + \gamma^2\zeta_k^2\|J(z_k)\|^2 \\ &= \|x_k - \bar{x}\|^2 - 2\gamma\frac{J(z_k)^T(x_k - z_k)}{\|J(z_k)\|^2}J(z_k)^T(x_k - \bar{x}) + \gamma^2\left(\frac{J(z_k)^T(x_k - z_k)}{\|J(z_k)\|}\right)^2 \\ &\leq \|x_k - \bar{x}\|^2 - 2\gamma\frac{J(z_k)^T(x_k - z_k)}{\|J(z_k)\|^2}J(z_k)^T(x_k - z_k) + \gamma^2\left(\frac{J(z_k)^T(x_k - z_k)}{\|J(z_k)\|}\right)^2 \\ &= \|x_k - \bar{x}\|^2 - \gamma(2-\gamma)\left(\frac{J(z_k)^T(x_k - z_k)}{\|J(z_k)\|}\right)^2 \\ &\leq \|x_k - \bar{x}\|^2 - \gamma(2-\gamma)\frac{\sigma^2\|x_k - z_k\|^4}{\|J(z_k)\|^2}. \end{aligned} \tag{2.13}$$

This shows that the sequence $\{\|x_k - \bar{x}\|\}$ is a decreasing sequence, and hence $\{x_k\}$ is bounded. In addition, combining this with continuity of $J$, we can find $n_1 > 0$ such that

$$\|J(x_k)\| \leq n_1. \tag{2.14}$$

Since $(J(x_k) - J(z_k))^T(x_k - z_k) \geq 0$, using Cauchy-Schwarz inequality, we obtain

$$\|J(x_k)\|\|x_k - z_k\| \geq J(x_k)^T(x_k - z_k) \geq J(z_k)^T(x_k - z_k) \geq \sigma\|x_k - z_k\|^2,$$

where the above inequality follows from the definition of the line search and setting $z_k = x_k + \alpha_k d_k$ which gives

$$J(z_k)^T(x_k - z_k) = -\alpha_k J(z_k)d_k \geq \sigma\alpha_k^2\|d_k\|^2 = \sigma\|x_k - z_k\|^2.$$

Therefore,

$$\sigma\|x_k - z_k\| \leq \|J(x_k)\| \leq n_1,$$

showing that $\{z_k\}$ is bounded.

Again, using the continuity of $J$, we can find another constant $n_2 > 0$ such that $\|J(z_k)\| \leq n_2$ for all $k \geq 0$ this, together with (2.13) give us

$$\gamma(2-\gamma)\frac{\sigma^2}{n_2^2}\|x_k - z_k\|^4 \leq \|x_k - \bar{x}\|^2 - \|x_{k+1} - \bar{x}\|^2, \tag{2.15}$$

adding (2.15) for $k = 0, 1, 2, \cdots$, we have

$$\gamma(2-\gamma)\frac{\sigma^2}{n_2^2}\sum_{k=0}^{\infty}\|x_k - z_k\|^4 \le \sum_{k=0}^{\infty}(\|x_k - \bar{x}\|^2 - \|x_{k+1} - \bar{x}\|^2) \le \|x_0 - \bar{x}\|^2, \tag{2.16}$$

which implies

$$\lim_{k\to\infty}\|x_k - z_k\| = 0.$$

This and the definition of $z_k$ implies

$$\lim_{k\to\infty}\alpha_k\|d_k\| = 0. \tag{2.17}$$

From the definition of projection operator, we get

$$\begin{aligned}\lim_{k\to\infty}\|x_{k+1} - x_k\| &= \lim_{k\to\infty}\|P_\Lambda[x_k - \gamma\zeta_k J(z_k)] - x_k\| \\ &\le \lim_{k\to\infty}\|x_k - \gamma\zeta_k J(z_k) - x_k\| \\ &\le \gamma\lim_{k\to\infty}\|\zeta_k J(z_k)\| \\ &\le \gamma\lim_{k\to\infty}\|x_k - z_k\| \\ &= 0.\end{aligned} \tag{2.18}$$

∎

**Lemma 2.3.** *Suppose assumptions $(Q_1)$-$(Q_3)$ hold, and the sequences $\{x_k\}$ and $\{z_k\}$ are generated by Algorithm 1. Then*

$$\alpha_k \ge \max\left\{\kappa, \frac{c\beta\|J(x_k)\|^2}{(L+\sigma)\|d_k\|^2}\right\}. \tag{2.19}$$

*Proof.* From the line search (2.9), if $\alpha_k \neq \kappa$, then $\alpha_k' = \alpha_k\beta^{-1}$ does not satisfy (2.9), that is,

$$-J(x_k + \alpha_k' d_k)^T d_k < \sigma\alpha_k'\|d_k\|^2.$$

Using (2.7) and assumption $(Q_2)$, we have

$$\begin{aligned}c\|J(x_k)\|^2 &= -J(x_k)^T d_k \\ &= (J(x_k + \alpha_k' d_k) - J(x_k))^T d_k - J(x_k + \alpha_k' d_k)^T d_k \\ &\le \alpha_k'(L+\sigma)\|d_k\|^2.\end{aligned}$$

Replacing $\alpha_k' = \alpha_k\beta^{-1}$ and solving for $\alpha_k$ gives the required result. ∎

**Theorem 2.4.** *Suppose that assumptions $(Q_1)$-$(Q_3)$ hold, and let the sequence $\{x_k\}$ be generated by Algorithm 1, then*

$$\liminf_{k\to\infty}\|J(x_k)\| = 0. \tag{2.20}$$

*Proof.* Suppose by contradiction the relation (2.20) is not satisfied, then there exist a positive constant $r_1$ such that $\forall k \geq 0$,

$$\|J(x_k)\| \geq r_1. \tag{2.21}$$

From (2.7) and (2.21), we have that $\forall k \geq 0$,

$$\|d_k\| \geq cr_1. \tag{2.22}$$

From (2.3), (2.6), (2.8), (2.14) and (2.21), we have

$$\begin{aligned} \|d_k\| &= \left\| - \left( c - \frac{J(x_k)^T s_{k-1}}{J(x_{k-1})^T d_{k-1}} \right) J(x_k) - \frac{\|J(x_k)\|^2 s_{k-1}}{J(x_{k-1})^T d_{k-1}} \right\| \\ &\leq c\|J(x_k)\| + 2\frac{\|J(x_k)\|^2 \|s_{k-1}\|}{c\|J(x_{k-1})\|^2} \\ &\leq cn_1 + \frac{2n_1^2 \alpha_{k-1}\|d_{k-1}\|}{cr_1^2}. \end{aligned} \tag{2.23}$$

Equation (2.17) implies $\forall \epsilon_0 > 0$ there exist $k_0$ such that $\alpha_{k-1}\|d_{k-1}\| \leq \epsilon_0 \ \forall k > k_0$. Therefore, choosing $\epsilon_0 = r_1^2$ and $N = \max\{\|d_0\|, \|d_1\|, \|d_2\|, \cdots, \|d_{k_0}\|, M_1\}$ where $M_1 = cn_1 + \frac{2n_1^2}{c}$, we have $\|d_k\| \leq N$. Multiplying both sides of (2.19) with $\|d_k\|$ we get

$$\begin{aligned} \alpha_k\|d_k\| &\geq \max\left\{\kappa, \frac{c\beta\|J(x_k))\|^2}{(L+\sigma)\|d_k\|^2}\right\}\|d_k\| \\ &\geq \max\left\{\kappa cr_1\ , \ \frac{c\beta r_1^2}{(L+\sigma)N}\right\}. \end{aligned}$$

Taking the limit as $k \to \infty$ on both sides, we get

$$\lim_{k\to\infty} \alpha_k\|d_k\| > 0. \tag{2.24}$$

This contradicts (2.17). Hence,

$$\liminf_{k\to\infty} \|J(x_k)\| = 0. \tag{2.25}$$

∎

## 3. Numerical Experiments

In this section, we present the numerical experiments of our proposed SCD algorithm in comparison with two existing algorithms, specifically, the PDY algorithm proposed by Liu and Feng [20], and the algorithm (which we called LLY for simplicity) proposed by Zheng et al. [33]. All codes are written on Matlab R2019b and are run on a PC of corei3-4005U processor, 4 GB RAM and 1.70 GHZ CPU.

In PDY and LLY algorithms, we fixed the parameters as reported in the respective papers [20] and [33]. However, in our SCD algorithm, we choose $\beta = 0.6$, $\sigma = 0.0001$, $\kappa = 1$, $c = 1$ and $\gamma = 1.8$. We perform the experiments on seven test problems with eight initial points. These problems are tested on five different dimensions: $n = 1000$, $n = 5000$, $n = 10000$, $n = 50000$ and $n = 100000$. We used $\|J(x_k)\| \leq 10^{-5}$ as a stopping criteria.We now state the test problems considered for the experiment, where the function $J$ is taken as

$J(x) = (j_1(x),\ j_2(x),\ \dots,\ j_n(x))^T$.

**Problem 1** [18] Exponential Function.

$$j_1(x) = e^{x_1} - 1,$$
$$j_i(x) = e^{x_i} + x_i - 1,\ \text{for } i = 2, 3, \dots, n,$$
$$\text{and } \Lambda = \mathbb{R}^n_+.$$

**Problem 2** [18] Modified Logarithmic Function.

$$j_i(x) = \ln(x_i + 1) - \frac{x_i}{n},\ \text{for } i = 2, 3, \dots, n,$$
$$\text{and } \Lambda = \{x \in \mathbb{R}^n : \sum_{i=1}^{n} x_i \le n, x_i > -1, i = 1, 2, \dots, n\}.$$

**Problem 3** [18] Strictly Convex Function I.

$$j_i(x) = e^{x_i} - 1,\ \text{for } i = 1, 2, \dots, n,$$
$$\text{and } \Lambda = \mathbb{R}^n_+.$$

**Problem 4**

$$j_i(x) = \frac{i}{n} e^{x_i} - 1,\ \text{for } i = 1, 2, \dots, n,$$
$$\text{and } \Lambda = \mathbb{R}^n_+.$$

**Problem 5** [7] Tridiagonal Exponential Function.

$$j_1(x) = x_1 - e^{\cos(h(x_1 + x_2))},$$
$$j_i(x) = x_i - e^{\cos(h(x_{i-1} + x_i + x_{i+1}))},\ \text{for } i = 2, \dots, n-1,$$
$$j_n(x) = x_n - e^{\cos(h(x_{n-1} + x_n))},$$
$$h = \frac{1}{n+1} \text{ and } \Lambda = \mathbb{R}^n_+.$$

**Problem 6** [32] Nonsmooth Function.

$$j_i(x) = x_i - \sin|x_i - 1|,\ i = 1, 2, 3, \dots, n.$$
$$\text{and } \Lambda = \{x \in \mathbb{R}^n : \sum_{i=1}^{n} x_i \le n, x_i \ge -1, i = 1, 2, \dots, n\}.$$

**Problem 7** Pursuit-Evasion problem.

$$j_i(x) = \sqrt{8} x_1 - 1,\ i = 1, 2, 3, \dots, n.$$
$$\text{and } \Lambda = \mathbb{R}^n_+.$$

The results of the experiments are tabulated in the following tables where ITER denotes the number of iterations, FVAL denotes the number of function evaluation, TIME denotes the CPU time and NORM denotes the norm of the function when an approximate solution is obtained. From the tables, it can be observed that all the three algorithms solved the seven test problems considered. However, our proposed SCD algorithm proved to be more efficient by solving most of the problems with less ITER, FVAL and TIME.

**Table 1.** Numerical Results of the **SCD**, **PDY** and **LLY** Algorithms on Problem 1 with given initial points and dimensions

| | | SCD | | | | PDY | | | | LLY | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| DIMENSION | INITIAL POINT | ITER | FVAL | TIME | NORM | ITER | FVAL | TIME | NORM | ITER | FVAL | TIME | NORM |
| 1000 | x1 | 1 | 3 | 0.0023 | 0.00E+00 | 24 | 71 | 0.0552 | 7.80E-06 | 16 | 47 | 0.1044 | 6.27E-06 |
| | x2 | 1 | 3 | 0.0026 | 0.00E+00 | 21 | 62 | 0.0250 | 8.99E-06 | 14 | 41 | 0.0233 | 9.50E-06 |
| | x3 | 9 | 27 | 0.0065 | 5.95E-06 | 27 | 80 | 0.0211 | 7.48E-06 | 18 | 54 | 0.0327 | 7.62E-06 |
| | x4 | 10 | 30 | 0.0121 | 2.20E-07 | 26 | 77 | 0.0269 | 8.98E-06 | 17 | 50 | 0.0301 | 8.58E-06 |
| | x5 | 10 | 30 | 0.0083 | 2.13E-07 | 22 | 65 | 0.0326 | 5.66E-06 | 17 | 50 | 0.0348 | 8.19E-06 |
| | x6 | 5 | 15 | 0.0048 | 5.31E-07 | 37 | 110 | 0.0416 | 8.03E-06 | 22 | 65 | 0.0286 | 6.98E-06 |
| | x7 | 10 | 30 | 0.0111 | 2.20E-07 | 26 | 77 | 0.0267 | 8.98E-06 | 17 | 50 | 0.0214 | 8.58E-06 |
| | x8 | 10 | 30 | 0.0097 | 1.99E-07 | 22 | 65 | 0.0216 | 5.67E-06 | 17 | 50 | 0.0187 | 8.25E-06 |
| 5000 | x1 | 1 | 3 | 0.1281 | 0.00E+00 | 24 | 71 | 0.0649 | 7.57E-06 | 16 | 47 | 0.1032 | 5.24E-06 |
| | x2 | 1 | 3 | 0.0051 | 0.00E+00 | 21 | 62 | 0.2491 | 9.56E-06 | 15 | 44 | 0.0707 | 8.28E-06 |
| | x3 | 9 | 27 | 0.0214 | 5.95E-06 | 27 | 80 | 0.0522 | 7.48E-06 | 18 | 54 | 0.1663 | 7.62E-06 |
| | x4 | 10 | 30 | 0.1192 | 4.67E-07 | 25 | 74 | 0.0595 | 7.95E-06 | 18 | 53 | 0.6757 | 7.35E-06 |
| | x5 | 10 | 30 | 0.0356 | 4.63E-07 | 23 | 68 | 0.1996 | 6.33E-06 | 18 | 53 | 0.0589 | 7.26E-06 |
| | x6 | 5 | 15 | 0.0151 | 5.36E-07 | 37 | 110 | 0.1605 | 8.02E-06 | 22 | 65 | 0.5717 | 7.00E-06 |
| | x7 | 10 | 30 | 0.4784 | 4.67E-07 | 25 | 74 | 0.0927 | 7.95E-06 | 18 | 53 | 0.1644 | 7.35E-06 |
| | x8 | 10 | 30 | 0.0508 | 4.57E-07 | 23 | 68 | 0.0572 | 6.34E-06 | 18 | 53 | 0.0556 | 7.27E-06 |
| 10000 | x1 | 1 | 3 | 0.0882 | 0.00E+00 | 24 | 71 | 0.6136 | 8.67E-06 | 16 | 47 | 0.1069 | 5.63E-06 |
| | x2 | 1 | 3 | 0.0119 | 0.00E+00 | 22 | 65 | 0.1816 | 6.24E-06 | 16 | 47 | 0.2273 | 4.62E-06 |
| | x3 | 9 | 27 | 0.1883 | 5.95E-06 | 27 | 80 | 0.1676 | 7.48E-06 | 18 | 54 | 1.0679 | 7.62E-06 |
| | x4 | 10 | 30 | 0.0738 | 6.56E-07 | 25 | 74 | 0.1600 | 7.11E-06 | 19 | 56 | 0.1169 | 4.09E-06 |
| | x5 | 10 | 30 | 0.0415 | 6.53E-07 | 23 | 68 | 0.1979 | 8.96E-06 | 19 | 56 | 0.4385 | 4.06E-06 |
| | x6 | 5 | 15 | 0.2782 | 5.37E-07 | 37 | 110 | 1.1591 | 8.02E-06 | 22 | 65 | 0.5424 | 7.00E-06 |
| | x7 | 10 | 30 | 0.0558 | 6.56E-07 | 25 | 74 | 0.2842 | 7.11E-06 | 19 | 56 | 1.4820 | 4.09E-06 |
| | x8 | 10 | 30 | 0.0568 | 6.49E-07 | 23 | 68 | 0.1104 | 8.96E-06 | 19 | 56 | 0.0980 | 4.06E-06 |
| 50000 | x1 | 1 | 3 | 0.1613 | 0.00E+00 | 73 | 218 | 2.4883 | 9.05E-06 | 16 | 47 | 1.2844 | 9.30E-06 |
| | x2 | 1 | 3 | 0.0400 | 0.00E+00 | 23 | 68 | 0.4548 | 5.67E-06 | 17 | 50 | 0.4646 | 4.07E-06 |
| | x3 | 9 | 27 | 0.1168 | 5.95E-06 | 27 | 80 | 0.5052 | 7.48E-06 | 18 | 54 | 0.6553 | 7.62E-06 |
| | x4 | 10 | 30 | 0.9183 | 1.46E-06 | 25 | 74 | 0.4628 | 7.52E-06 | 19 | 56 | 0.5190 | 9.09E-06 |
| | x5 | 10 | 30 | 0.1708 | 1.46E-06 | 25 | 74 | 0.6266 | 5.01E-06 | 19 | 56 | 0.4309 | 9.08E-06 |
| | x6 | 5 | 15 | 0.0923 | 5.37E-07 | 37 | 110 | 0.6462 | 8.02E-06 | 22 | 65 | 1.5992 | 7.00E-06 |
| | x7 | 10 | 30 | 0.2276 | 1.46E-06 | 25 | 74 | 0.4413 | 7.52E-06 | 19 | 56 | 0.5693 | 9.09E-06 |
| | x8 | 10 | 30 | 0.3395 | 1.45E-06 | 25 | 74 | 1.2421 | 5.01E-06 | 19 | 56 | 0.3643 | 9.08E-06 |
| 100000 | x1 | 1 | 3 | 0.0782 | 0.00E+00 | 163 | 488 | 5.5216 | 9.42E-06 | 17 | 50 | 1.0319 | 4.97E-06 |
| | x2 | 1 | 3 | 0.0638 | 0.00E+00 | 23 | 68 | 0.6252 | 7.60E-06 | 17 | 50 | 0.7126 | 5.76E-06 |
| | x3 | 9 | 27 | 0.9063 | 5.95E-06 | 27 | 80 | 0.9798 | 7.48E-06 | 18 | 54 | 0.6714 | 7.62E-06 |
| | x4 | 10 | 30 | 0.5563 | 2.06E-06 | 73 | 218 | 2.9124 | 9.73E-06 | 20 | 59 | 0.8634 | 5.08E-06 |
| | x5 | 10 | 30 | 0.4564 | 2.06E-06 | 60 | 179 | 1.8583 | 8.93E-06 | 20 | 59 | 1.1861 | 5.07E-06 |
| | x6 | 5 | 15 | 0.1634 | 5.37E-07 | 37 | 110 | 1.1625 | 8.02E-06 | 22 | 65 | 0.8905 | 7.00E-06 |
| | x7 | 10 | 30 | 0.2648 | 2.06E-06 | 73 | 218 | 2.3206 | 9.73E-06 | 20 | 59 | 0.8795 | 5.08E-06 |
| | x8 | 10 | 30 | 1.4084 | 2.06E-06 | 60 | 179 | 1.7355 | 8.93E-06 | 20 | 59 | 0.9313 | 5.07E-06 |

**Table 2.** Numerical Results of the **SCD**, **PDY** and **LLY** Algorithms on Problem 2 with given initial points and dimensions

| | | SCD | | | | PDY | | | | LLY | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| DIMENSION | INITIAL POINT | ITER | FVAL | TIME | NORM | ITER | FVAL | TIME | NORM | ITER | FVAL | TIME | NORM |
| 1000 | x1 | 7 | 20 | 0.0887 | 3.54E-06 | 5 | 9 | 0.0165 | 3.60E-08 | 9 | 25 | 0.0173 | 9.20E-06 |
| | x2 | 6 | 17 | 0.0108 | 9.88E-06 | 3 | 5 | 0.0068 | 5.17E-07 | 7 | 19 | 0.0114 | 2.66E-06 |
| | x3 | 7 | 19 | 0.0106 | 7.53E-06 | 18 | 50 | 0.0253 | 5.39E-06 | 9 | 25 | 0.0101 | 4.43E-06 |
| | x4 | 9 | 25 | 0.0161 | 4.48E-06 | 22 | 60 | 0.0337 | 7.42E-06 | 28 | 82 | 0.0440 | 3.94E-06 |
| | x5 | 9 | 25 | 0.0175 | 4.48E-06 | 22 | 60 | 0.0206 | 7.42E-06 | 28 | 82 | 0.0445 | 3.94E-06 |
| | x6 | 8 | 22 | 0.0142 | 5.70E-06 | 19 | 53 | 0.0319 | 7.68E-06 | 21 | 61 | 0.0281 | 4.81E-06 |
| | x7 | 9 | 25 | 0.0363 | 4.48E-06 | 22 | 60 | 0.0583 | 7.42E-06 | 28 | 82 | 0.1102 | 3.94E-06 |
| | x8 | 9 | 25 | 0.0156 | 4.49E-06 | 25 | 66 | 0.0886 | 6.32E-06 | 28 | 82 | 0.2167 | 3.75E-06 |
| 5000 | x1 | 7 | 20 | 0.1270 | 8.48E-06 | 5 | 9 | 0.2202 | 6.26E-09 | 10 | 28 | 0.1341 | 3.18E-06 |
| | x2 | 7 | 19 | 0.1724 | 9.33E-06 | 3 | 5 | 0.0160 | 1.75E-07 | 7 | 19 | 0.0248 | 5.70E-06 |
| | x3 | 7 | 19 | 0.0422 | 7.72E-06 | 18 | 50 | 0.0524 | 5.37E-06 | 9 | 25 | 0.0287 | 3.98E-06 |
| | x4 | 9 | 26 | 0.2498 | 2.08E-06 | 23 | 66 | 0.0857 | 5.26E-06 | 30 | 88 | 1.0625 | 3.86E-06 |
| | x5 | 9 | 26 | 0.2010 | 2.08E-06 | 23 | 66 | 0.4698 | 5.26E-06 | 30 | 88 | 0.1174 | 3.86E-06 |
| | x6 | 8 | 22 | 0.2322 | 6.45E-06 | 19 | 53 | 0.6032 | 7.43E-06 | 21 | 61 | 0.3431 | 6.36E-06 |
| | x7 | 9 | 26 | 0.0440 | 2.08E-06 | 23 | 66 | 0.0918 | 5.26E-06 | 30 | 88 | 0.3690 | 3.86E-06 |
| | x8 | 9 | 26 | 0.0336 | 2.08E-06 | 23 | 66 | 0.0886 | 5.26E-06 | 30 | 88 | 0.1348 | 3.87E-06 |
| 10000 | x1 | 8 | 22 | 0.2472 | 4.84E-06 | 5 | 9 | 0.6819 | 3.62E-09 | 10 | 28 | 0.0891 | 4.48E-06 |
| | x2 | 7 | 20 | 0.5338 | 2.65E-06 | 3 | 5 | 0.1175 | 1.21E-07 | 7 | 19 | 0.0432 | 8.03E-06 |
| | x3 | 7 | 19 | 0.1134 | 7.74E-06 | 18 | 50 | 0.6410 | 5.37E-06 | 10 | 28 | 1.0279 | 4.40E-06 |
| | x4 | 9 | 26 | 0.3269 | 2.95E-06 | 23 | 66 | 0.7092 | 7.43E-06 | 30 | 88 | 0.2219 | 5.48E-06 |
| | x5 | 9 | 26 | 0.1460 | 2.95E-06 | 23 | 66 | 0.2032 | 7.43E-06 | 30 | 88 | 0.8150 | 5.48E-06 |
| | x6 | 8 | 22 | 0.2239 | 6.55E-06 | 19 | 53 | 0.1378 | 7.40E-06 | 21 | 61 | 0.1314 | 6.51E-06 |
| | x7 | 9 | 26 | 0.1494 | 2.95E-06 | 23 | 66 | 0.3822 | 7.43E-06 | 30 | 88 | 0.1683 | 5.48E-06 |
| | x8 | 9 | 26 | 0.6483 | 2.96E-06 | 23 | 66 | 0.5253 | 7.43E-06 | 30 | 88 | 0.1853 | 5.49E-06 |
| 50000 | x1 | 8 | 23 | 0.4542 | 2.18E-06 | 26 | 77 | 1.0290 | 7.75E-06 | 10 | 28 | 0.7978 | 9.97E-06 |
| | x2 | 7 | 20 | 0.3154 | 5.97E-06 | 3 | 5 | 0.0983 | 6.32E-08 | 8 | 22 | 0.2194 | 2.86E-06 |
| | x3 | 7 | 19 | 0.6218 | 7.76E-06 | 18 | 50 | 0.3373 | 5.36E-06 | 11 | 31 | 0.7054 | 8.67E-06 |
| | x4 | 9 | 26 | 0.4564 | 6.64E-06 | 24 | 69 | 1.4849 | 8.30E-06 | 32 | 94 | 2.7029 | 4.59E-06 |
| | x5 | 9 | 26 | 1.1128 | 6.64E-06 | 24 | 69 | 0.4811 | 8.30E-06 | 32 | 94 | 0.9953 | 4.59E-06 |
| | x6 | 8 | 22 | 0.1500 | 6.63E-06 | 19 | 53 | 0.5591 | 7.37E-06 | 21 | 61 | 0.5683 | 6.63E-06 |
| | x7 | 9 | 26 | 0.5885 | 6.64E-06 | 24 | 69 | 0.5904 | 8.30E-06 | 32 | 94 | 1.5171 | 4.59E-06 |
| | x8 | 9 | 26 | 0.1896 | 6.64E-06 | 24 | 69 | 1.0501 | 8.30E-06 | 32 | 94 | 1.0207 | 4.59E-06 |
| 100000 | x1 | 8 | 23 | 0.8410 | 3.08E-06 | 63 | 188 | 2.8714 | 7.77E-06 | 11 | 31 | 0.6148 | 2.26E-06 |
| | x2 | 7 | 20 | 0.3791 | 8.45E-06 | 3 | 5 | 0.1158 | 5.40E-08 | 8 | 22 | 0.6978 | 4.04E-06 |
| | x3 | 7 | 19 | 0.7152 | 7.76E-06 | 18 | 50 | 1.7273 | 5.36E-06 | 13 | 37 | 0.8367 | 3.15E-06 |
| | x4 | 9 | 26 | 0.5317 | 9.39E-06 | 26 | 77 | 1.1706 | 5.19E-06 | 32 | 94 | 2.1078 | 6.49E-06 |
| | x5 | 9 | 26 | 0.5580 | 9.39E-06 | 26 | 77 | 1.2016 | 5.19E-06 | 32 | 94 | 2.1205 | 6.49E-06 |
| | x6 | 8 | 22 | 0.6292 | 6.64E-06 | 19 | 53 | 0.9325 | 7.36E-06 | 21 | 61 | 1.2519 | 6.65E-06 |
| | x7 | 9 | 26 | 1.6351 | 9.39E-06 | 26 | 77 | 1.2891 | 5.19E-06 | 32 | 94 | 2.1439 | 6.49E-06 |
| | x8 | 9 | 26 | 0.4745 | 9.39E-06 | 26 | 77 | 1.1348 | 5.19E-06 | 32 | 94 | 2.1427 | 6.49E-06 |

**Table 3.** Numerical Results of the **SCD**, **PDY** and **LLY** Algorithms on Problem 3 with given initial points and dimensions

| | | SCD | | | | PDY | | | | LLY | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| DIMENSION | INITIAL POINT | ITER | FVAL | TIME | NORM | ITER | FVAL | TIME | NORM | ITER | FVAL | TIME | NORM |
| 1000 | x1 | 1 | 3 | 0.0162 | 0.00E+00 | 20 | 59 | 0.0298 | 7.37E-06 | 21 | 62 | 0.0354 | 6.57E-06 |
| | x2 | 1 | 3 | 0.0018 | 0.00E+00 | 19 | 56 | 0.0182 | 5.45E-06 | 8 | 23 | 0.0138 | 2.78E-06 |
| | x3 | 1 | 3 | 0.0062 | 2.22E-16 | 16 | 48 | 0.0131 | 5.62E-06 | 25 | 75 | 0.0253 | 8.99E-06 |
| | x4 | 15 | 44 | 0.0069 | 8.85E-06 | 20 | 59 | 0.0233 | 8.76E-06 | 22 | 65 | 0.0222 | 5.38E-06 |
| | x5 | 15 | 44 | 0.0070 | 8.85E-06 | 20 | 59 | 0.0414 | 8.76E-06 | 22 | 65 | 0.0406 | 5.38E-06 |
| | x6 | 13 | 38 | 0.0636 | 7.75E-06 | 17 | 50 | 0.0235 | 6.76E-06 | 17 | 50 | 0.0187 | 9.63E-06 |
| | x7 | 15 | 44 | 0.0124 | 8.85E-06 | 20 | 59 | 0.0125 | 8.76E-06 | 22 | 65 | 0.0197 | 5.38E-06 |
| | x8 | 15 | 44 | 0.0125 | 8.94E-06 | 20 | 59 | 0.0120 | 8.77E-06 | 22 | 65 | 0.0217 | 5.42E-06 |
| 5000 | x1 | 1 | 3 | 0.0222 | 0.00E+00 | 21 | 62 | 0.2091 | 8.24E-06 | 22 | 65 | 0.3812 | 7.28E-06 |
| | x2 | 1 | 3 | 0.0041 | 0.00E+00 | 20 | 59 | 0.1648 | 6.10E-06 | 8 | 23 | 0.0397 | 6.21E-06 |
| | x3 | 1 | 3 | 0.0049 | 2.22E-16 | 16 | 48 | 0.0373 | 5.62E-06 | 25 | 75 | 0.9425 | 8.99E-06 |
| | x4 | 16 | 47 | 0.0427 | 6.99E-06 | 21 | 62 | 0.0509 | 9.80E-06 | 23 | 68 | 1.7854 | 5.99E-06 |
| | x5 | 16 | 47 | 0.2445 | 6.99E-06 | 21 | 62 | 0.5293 | 9.80E-06 | 23 | 68 | 0.0538 | 5.99E-06 |
| | x6 | 13 | 38 | 0.0480 | 7.75E-06 | 17 | 50 | 0.0411 | 6.76E-06 | 17 | 50 | 0.0412 | 9.63E-06 |
| | x7 | 16 | 47 | 0.0308 | 6.99E-06 | 21 | 62 | 0.0425 | 9.80E-06 | 23 | 68 | 0.0838 | 5.99E-06 |
| | x8 | 16 | 47 | 0.0378 | 7.01E-06 | 21 | 62 | 0.4519 | 9.80E-06 | 23 | 68 | 0.4262 | 6.00E-06 |
| 10000 | x1 | 1 | 3 | 0.0104 | 0.00E+00 | 22 | 65 | 0.0911 | 5.83E-06 | 23 | 68 | 0.0913 | 5.11E-06 |
| | x2 | 1 | 3 | 0.1023 | 0.00E+00 | 20 | 59 | 0.0625 | 8.62E-06 | 8 | 23 | 0.0547 | 8.79E-06 |
| | x3 | 1 | 3 | 0.0114 | 2.22E-16 | 16 | 48 | 0.0557 | 5.62E-06 | 25 | 75 | 1.2883 | 8.99E-06 |
| | x4 | 16 | 47 | 0.0475 | 9.89E-06 | 22 | 65 | 0.3407 | 6.93E-06 | 23 | 68 | 0.0952 | 8.47E-06 |
| | x5 | 16 | 47 | 0.0495 | 9.89E-06 | 22 | 65 | 0.9084 | 6.93E-06 | 23 | 68 | 0.1117 | 8.47E-06 |
| | x6 | 13 | 38 | 0.0592 | 7.75E-06 | 17 | 50 | 0.0576 | 6.76E-06 | 17 | 50 | 0.0766 | 9.63E-06 |
| | x7 | 16 | 47 | 0.0475 | 9.89E-06 | 22 | 65 | 0.0776 | 6.93E-06 | 23 | 68 | 0.3569 | 8.47E-06 |
| | x8 | 16 | 47 | 0.1758 | 9.90E-06 | 22 | 65 | 0.7398 | 6.93E-06 | 23 | 68 | 0.1712 | 8.48E-06 |
| 50000 | x1 | 1 | 3 | 0.0287 | 0.00E+00 | 57 | 170 | 1.7362 | 8.87E-06 | 24 | 71 | 1.5008 | 5.67E-06 |
| | x2 | 1 | 3 | 0.0194 | 0.00E+00 | 21 | 62 | 0.2404 | 9.64E-06 | 9 | 26 | 0.2185 | 3.14E-06 |
| | x3 | 1 | 3 | 0.0236 | 2.22E-16 | 16 | 48 | 0.1844 | 5.62E-06 | 25 | 75 | 0.5480 | 8.99E-06 |
| | x4 | 17 | 50 | 0.4714 | 7.79E-06 | 56 | 167 | 1.0036 | 7.97E-06 | 24 | 71 | 0.6973 | 9.40E-06 |
| | x5 | 17 | 50 | 0.7310 | 7.79E-06 | 56 | 167 | 1.2997 | 7.97E-06 | 24 | 71 | 0.6250 | 9.40E-06 |
| | x6 | 13 | 38 | 0.2250 | 7.75E-06 | 17 | 50 | 0.3479 | 6.76E-06 | 17 | 50 | 0.8318 | 9.63E-06 |
| | x7 | 17 | 50 | 0.7120 | 7.79E-06 | 56 | 167 | 1.7737 | 7.97E-06 | 24 | 71 | 0.9931 | 9.40E-06 |
| | x8 | 17 | 50 | 0.2993 | 7.79E-06 | 56 | 167 | 0.7052 | 7.97E-06 | 24 | 71 | 0.6573 | 9.40E-06 |
| 100000 | x1 | 1 | 3 | 0.0709 | 0.00E+00 | 126 | 377 | 3.1485 | 9.15E-06 | 24 | 71 | 0.6936 | 8.01E-06 |
| | x2 | 1 | 3 | 0.0773 | 0.00E+00 | 22 | 65 | 0.4317 | 6.82E-06 | 9 | 26 | 0.5340 | 4.45E-06 |
| | x3 | 1 | 3 | 0.2186 | 2.22E-16 | 16 | 48 | 0.3266 | 5.62E-06 | 25 | 75 | 1.4667 | 8.99E-06 |
| | x4 | 17 | 51 | 0.7640 | 6.06E-06 | 57 | 170 | 1.9869 | 8.46E-06 | 25 | 74 | 1.7901 | 6.59E-06 |
| | x5 | 17 | 51 | 0.3707 | 6.06E-06 | 57 | 170 | 1.3907 | 8.46E-06 | 25 | 74 | 1.2399 | 6.59E-06 |
| | x6 | 13 | 38 | 0.3836 | 7.75E-06 | 17 | 50 | 0.3276 | 6.76E-06 | 17 | 50 | 0.5267 | 9.63E-06 |
| | x7 | 17 | 51 | 0.4190 | 6.06E-06 | 57 | 170 | 2.3809 | 8.46E-06 | 25 | 74 | 1.2321 | 6.59E-06 |
| | x8 | 17 | 51 | 0.6001 | 6.06E-06 | 57 | 170 | 1.3800 | 8.46E-06 | 25 | 74 | 1.5502 | 6.59E-06 |

**Table 4.** Numerical Results of the **SCD**, **PDY** and **LLY** Algorithms on Problem 4 with given initial points and dimensions

| | | SCD | | | | PDY | | | | LLY | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| DIMENSION | INITIAL POINT | ITER | FVAL | TIME | NORM | ITER | FVAL | TIME | NORM | ITER | FVAL | TIME | NORM |
| 1000 | x1 | 11 | 32 | 0.0329 | 4.63E-06 | 25 | 74 | 0.0168 | 5.22E-06 | 23 | 68 | 0.0270 | 9.92E-06 |
| | x2 | 16 | 44 | 0.0171 | 9.38E-06 | 25 | 68 | 0.0149 | 7.57E-06 | 28 | 79 | 0.0285 | 8.26E-06 |
| | x3 | 31 | 89 | 0.0455 | 9.48E-06 | 24 | 65 | 0.0402 | 5.51E-06 | 26 | 73 | 0.0334 | 3.80E-06 |
| | x4 | 15 | 42 | 0.0148 | 6.35E-06 | 24 | 65 | 0.0214 | 8.15E-06 | 24 | 67 | 0.0297 | 3.74E-06 |
| | x5 | 17 | 50 | 0.0185 | 7.34E-06 | 28 | 83 | 0.0436 | 5.81E-06 | 26 | 76 | 0.1502 | 5.06E-06 |
| | x6 | 30 | 87 | 0.0350 | 7.98E-06 | 25 | 67 | 0.0415 | 6.04E-06 | 30 | 85 | 0.0413 | 5.71E-06 |
| | x7 | 15 | 42 | 0.0115 | 6.35E-06 | 24 | 65 | 0.0289 | 8.15E-06 | 24 | 67 | 0.0251 | 3.74E-06 |
| | x8 | 17 | 50 | 0.0155 | 6.93E-06 | 28 | 83 | 0.0216 | 5.78E-06 | 26 | 76 | 0.0445 | 5.04E-06 |
| 5000 | x1 | 11 | 32 | 0.0300 | 4.34E-06 | 27 | 80 | 0.8318 | 5.74E-06 | 25 | 74 | 0.0834 | 9.78E-06 |
| | x2 | 13 | 36 | 0.1954 | 8.58E-07 | 26 | 70 | 0.0767 | 5.44E-06 | 28 | 79 | 0.0832 | 7.83E-06 |
| | x3 | 13 | 36 | 0.0910 | 2.86E-06 | 26 | 70 | 0.1027 | 8.59E-06 | 20 | 55 | 1.0185 | 2.06E-06 |
| | x4 | 30 | 86 | 0.4084 | 9.99E-06 | 24 | 65 | 0.2713 | 8.94E-06 | 25 | 70 | 0.0703 | 8.60E-06 |
| | x5 | 18 | 53 | 0.1808 | 5.69E-06 | 30 | 89 | 0.1591 | 6.71E-06 | 27 | 79 | 0.0681 | 7.40E-06 |
| | x6 | 18 | 50 | 0.4918 | 7.40E-06 | 30 | 79 | 0.2369 | 6.40E-06 | 26 | 73 | 0.7855 | 8.49E-06 |
| | x7 | 30 | 86 | 0.0633 | 9.99E-06 | 24 | 65 | 0.6722 | 8.94E-06 | 25 | 70 | 0.0644 | 8.60E-06 |
| | x8 | 18 | 53 | 0.0559 | 5.64E-06 | 30 | 89 | 0.3491 | 6.70E-06 | 27 | 79 | 0.1182 | 7.39E-06 |
| 10000 | x1 | 11 | 32 | 0.0422 | 5.65E-06 | 28 | 83 | 0.1617 | 5.41E-06 | 26 | 77 | 0.3622 | 8.92E-06 |
| | x2 | 12 | 32 | 0.3794 | 7.21E-06 | 26 | 70 | 0.1132 | 5.55E-06 | 25 | 70 | 0.4465 | 9.01E-06 |
| | x3 | 12 | 32 | 0.0459 | 6.50E-06 | 26 | 70 | 0.9694 | 5.32E-06 | 29 | 82 | 0.3014 | 8.43E-06 |
| | x4 | 31 | 90 | 0.1584 | 7.88E-06 | 25 | 68 | 0.1102 | 5.39E-06 | 30 | 86 | 1.9858 | 5.65E-06 |
| | x5 | 18 | 53 | 0.3939 | 8.01E-06 | 31 | 92 | 0.4441 | 6.42E-06 | 28 | 82 | 0.2502 | 5.95E-06 |
| | x6 | 20 | 56 | 0.0879 | 9.91E-06 | 27 | 73 | 0.1085 | 6.00E-06 | 29 | 82 | 0.2371 | 3.01E-06 |
| | x7 | 31 | 90 | 0.0993 | 7.88E-06 | 25 | 68 | 0.3265 | 5.39E-06 | 30 | 86 | 1.1008 | 5.65E-06 |
| | x8 | 18 | 53 | 0.5774 | 7.98E-06 | 31 | 92 | 0.1272 | 6.42E-06 | 28 | 82 | 0.1615 | 5.95E-06 |
| 50000 | x1 | 11 | 33 | 0.1342 | 2.36E-06 | 75 | 224 | 1.4491 | 9.99E-06 | 28 | 83 | 2.0269 | 9.26E-06 |
| | x2 | 12 | 32 | 0.1422 | 6.04E-06 | 34 | 101 | 1.2540 | 5.92E-06 | 32 | 92 | 0.5892 | 5.34E-06 |
| | x3 | 12 | 32 | 0.7745 | 5.20E-06 | 34 | 101 | 0.6702 | 5.01E-06 | 31 | 88 | 0.9940 | 4.35E-06 |
| | x4 | 18 | 50 | 0.2091 | 9.23E-06 | 27 | 74 | 0.4584 | 6.13E-06 | 31 | 89 | 0.5563 | 6.21E-06 |
| | x5 | 19 | 56 | 0.4123 | 6.27E-06 | 80 | 239 | 2.1233 | 8.32E-06 | 31 | 91 | 2.1652 | 5.42E-06 |
| | x6 | 12 | 32 | 0.4123 | 7.47E-06 | 33 | 98 | 0.4356 | 7.79E-06 | 32 | 91 | 0.6577 | 6.18E-06 |
| | x7 | 18 | 50 | 0.4731 | 9.23E-06 | 27 | 74 | 0.3987 | 6.13E-06 | 31 | 89 | 0.7467 | 6.21E-06 |
| | x8 | 19 | 56 | 0.2843 | 6.27E-06 | 80 | 239 | 1.7214 | 8.32E-06 | 31 | 91 | 0.8834 | 5.42E-06 |
| 100000 | x1 | 11 | 33 | 0.2290 | 3.31E-06 | 78 | 233 | 2.4460 | 8.43E-06 | 29 | 86 | 1.8326 | 8.61E-06 |
| | x2 | 12 | 32 | 0.5847 | 7.74E-06 | 35 | 104 | 0.9774 | 5.62E-06 | 32 | 92 | 1.3910 | 6.64E-06 |
| | x3 | 12 | 32 | 0.3136 | 6.73E-06 | 34 | 101 | 1.0017 | 9.54E-06 | 32 | 92 | 1.4917 | 7.45E-06 |
| | x4 | 18 | 51 | 0.4862 | 5.77E-06 | 33 | 98 | 1.1617 | 8.46E-06 | 30 | 86 | 1.2599 | 6.09E-06 |
| | x5 | 19 | 56 | 1.3369 | 8.88E-06 | 82 | 245 | 2.3141 | 9.35E-06 | 31 | 92 | 1.2728 | 6.75E-06 |
| | x6 | 12 | 32 | 0.3084 | 7.97E-06 | 34 | 101 | 0.9639 | 7.42E-06 | 32 | 92 | 1.2935 | 7.42E-06 |
| | x7 | 18 | 51 | 0.3937 | 5.77E-06 | 33 | 98 | 1.1212 | 8.46E-06 | 30 | 86 | 1.5763 | 6.09E-06 |
| | x8 | 19 | 56 | 0.4793 | 8.87E-06 | 82 | 245 | 3.9044 | 9.35E-06 | 31 | 92 | 1.3877 | 6.75E-06 |

**Table 5.** Numerical Results of the **SCD**, **PDY** and **LLY** Algorithms on Problem 5 with given initial points and dimensions

| | | **SCD** | | | | **PDY** | | | | **LLY** | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| DIMENSION | INITIAL POINT | ITER | FVAL | TIME | NORM | ITER | FVAL | TIME | NORM | ITER | FVAL | TIME | NORM |
| 1000 | x1 | 7 | 20 | 0.0890 | 5.73E-06 | 23 | 68 | 0.1312 | 6.47E-06 | 16 | 47 | 0.0781 | 6.49E-06 |
| | x2 | 7 | 20 | 0.0116 | 8.72E-06 | 23 | 68 | 0.0344 | 9.86E-06 | 16 | 47 | 0.0283 | 6.84E-06 |
| | x3 | 7 | 20 | 0.0125 | 9.05E-06 | 24 | 71 | 0.0322 | 5.11E-06 | 15 | 44 | 0.0300 | 6.34E-06 |
| | x4 | 7 | 20 | 0.0123 | 7.46E-06 | 23 | 68 | 0.0261 | 8.42E-06 | 15 | 44 | 0.0232 | 8.75E-06 |
| | x5 | 7 | 20 | 0.0126 | 7.46E-06 | 23 | 68 | 0.1559 | 8.42E-06 | 15 | 44 | 0.0274 | 8.75E-06 |
| | x6 | 7 | 20 | 0.0088 | 9.03E-06 | 24 | 71 | 0.0258 | 5.10E-06 | 15 | 44 | 0.0249 | 6.98E-06 |
| | x7 | 7 | 20 | 0.0107 | 7.46E-06 | 23 | 68 | 0.1476 | 8.42E-06 | 15 | 44 | 0.0412 | 8.75E-06 |
| | x8 | 7 | 20 | 0.0119 | 7.45E-06 | 23 | 68 | 0.0293 | 8.42E-06 | 15 | 44 | 0.0228 | 8.74E-06 |
| 5000 | x1 | 7 | 21 | 0.0271 | 2.55E-06 | 24 | 71 | 0.2577 | 7.24E-06 | 12 | 35 | 0.1706 | 2.77E-06 |
| | x2 | 7 | 21 | 0.0616 | 3.88E-06 | 25 | 74 | 0.1040 | 5.52E-06 | 12 | 35 | 0.1578 | 4.56E-06 |
| | x3 | 7 | 21 | 0.0304 | 4.03E-06 | 25 | 74 | 0.1218 | 5.73E-06 | 12 | 35 | 1.1935 | 3.25E-06 |
| | x4 | 7 | 21 | 0.2711 | 3.32E-06 | 24 | 71 | 0.1051 | 9.43E-06 | 12 | 35 | 0.1123 | 5.18E-06 |
| | x5 | 7 | 21 | 0.0268 | 3.32E-06 | 24 | 71 | 0.1291 | 9.43E-06 | 12 | 35 | 0.0652 | 5.18E-06 |
| | x6 | 7 | 21 | 0.0999 | 4.03E-06 | 25 | 74 | 0.0876 | 5.72E-06 | 12 | 35 | 0.0821 | 1.75E-06 |
| | x7 | 7 | 21 | 0.0388 | 3.32E-06 | 24 | 71 | 0.2387 | 9.43E-06 | 12 | 35 | 0.0664 | 5.18E-06 |
| | x8 | 7 | 21 | 0.1139 | 3.32E-06 | 24 | 71 | 0.1726 | 9.43E-06 | 12 | 35 | 0.0552 | 5.17E-06 |
| 10000 | x1 | 7 | 21 | 0.0726 | 3.60E-06 | 25 | 74 | 0.2084 | 5.12E-06 | 11 | 32 | 2.6108 | 9.22E-06 |
| | x2 | 7 | 21 | 0.2388 | 5.49E-06 | 60 | 179 | 0.4705 | 8.35E-06 | 12 | 35 | 0.1240 | 8.05E-06 |
| | x3 | 7 | 21 | 0.0838 | 5.70E-06 | 60 | 179 | 0.4566 | 8.67E-06 | 12 | 35 | 0.1646 | 2.72E-06 |
| | x4 | 7 | 21 | 0.2351 | 4.69E-06 | 59 | 176 | 0.9589 | 9.51E-06 | 15 | 44 | 0.2307 | 2.15E-06 |
| | x5 | 7 | 21 | 0.0911 | 4.69E-06 | 59 | 176 | 0.3671 | 9.51E-06 | 15 | 44 | 1.4382 | 2.15E-06 |
| | x6 | 7 | 21 | 0.2526 | 5.70E-06 | 60 | 179 | 0.4100 | 8.67E-06 | 12 | 35 | 0.1004 | 3.33E-06 |
| | x7 | 7 | 21 | 0.0543 | 4.69E-06 | 59 | 176 | 0.7330 | 9.51E-06 | 15 | 44 | 0.1264 | 2.15E-06 |
| | x8 | 7 | 21 | 0.0482 | 4.69E-06 | 59 | 176 | 0.4571 | 9.51E-06 | 15 | 44 | 0.2378 | 2.15E-06 |
| 50000 | x1 | 7 | 21 | 0.5848 | 8.06E-06 | 61 | 182 | 2.6156 | 9.19E-06 | 11 | 32 | 2.0915 | 4.97E-06 |
| | x2 | 8 | 23 | 0.2971 | 4.91E-06 | 134 | 401 | 3.3776 | 9.92E-06 | 11 | 32 | 0.7214 | 7.74E-06 |
| | x3 | 8 | 23 | 0.4958 | 5.10E-06 | 135 | 404 | 3.2328 | 9.01E-06 | 11 | 32 | 0.5964 | 7.43E-06 |
| | x4 | 8 | 23 | 0.5933 | 4.20E-06 | 133 | 398 | 3.0908 | 9.69E-06 | 12 | 35 | 0.6424 | 2.81E-06 |
| | x5 | 8 | 23 | 0.3562 | 4.20E-06 | 133 | 398 | 3.3884 | 9.69E-06 | 12 | 35 | 0.6049 | 2.81E-06 |
| | x6 | 8 | 23 | 0.2888 | 5.10E-06 | 135 | 404 | 3.3171 | 9.01E-06 | 11 | 32 | 0.3994 | 6.89E-06 |
| | x7 | 8 | 23 | 0.2346 | 4.20E-06 | 133 | 398 | 3.2587 | 9.69E-06 | 12 | 35 | 0.6985 | 2.81E-06 |
| | x8 | 8 | 23 | 0.1696 | 4.20E-06 | 133 | 398 | 3.3766 | 9.69E-06 | 12 | 35 | 1.7276 | 2.81E-06 |
| 100000 | x1 | 8 | 23 | 0.5196 | 4.56E-06 | 134 | 401 | 6.5738 | 9.21E-06 | 11 | 32 | 1.6355 | 2.69E-06 |
| | x2 | 8 | 23 | 0.3381 | 6.95E-06 | 283 | 848 | 20.1453 | 9.68E-06 | 11 | 32 | 1.2017 | 4.13E-06 |
| | x3 | 8 | 23 | 0.5179 | 7.21E-06 | 284 | 851 | 14.4649 | 9.42E-06 | 11 | 32 | 1.1021 | 4.20E-06 |
| | x4 | 8 | 23 | 0.9699 | 5.93E-06 | 136 | 407 | 7.4457 | 9.18E-06 | 11 | 32 | 1.0706 | 6.59E-06 |
| | x5 | 8 | 23 | 0.5388 | 5.93E-06 | 136 | 407 | 6.8313 | 9.18E-06 | 11 | 32 | 0.8234 | 6.59E-06 |
| | x6 | 8 | 23 | 0.5458 | 7.21E-06 | 284 | 851 | 13.6980 | 9.42E-06 | 11 | 32 | 1.2348 | 4.13E-06 |
| | x7 | 8 | 23 | 0.4051 | 5.93E-06 | 136 | 407 | 6.5345 | 9.18E-06 | 11 | 32 | 1.2016 | 6.59E-06 |
| | x8 | 8 | 23 | 0.8464 | 5.93E-06 | 136 | 407 | 6.6888 | 9.18E-06 | 11 | 32 | 0.8019 | 6.59E-06 |

**Table 6.** Numerical Results of the **SCD**, **PDY** and **LLY** Algorithms on Problem 6 with given initial points and dimensions

| | | **SCD** | | | | **PDY** | | | | **LLY** | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| DIMENSION | INITIAL POINT | ITER | FVAL | TIME | NORM | ITER | FVAL | TIME | NORM | ITER | FVAL | TIME | NORM |
| 1000 | x1 | 10 | 30 | 0.0430 | 6.68E-06 | 6 | 17 | 0.0103 | 6.75E-07 | 6 | 17 | 0.0164 | 3.52E-06 |
| | x2 | 10 | 29 | 0.0109 | 4.57E-06 | 6 | 17 | 0.0087 | 3.28E-06 | 6 | 17 | 0.0259 | 6.62E-06 |
| | x3 | 10 | 29 | 0.0144 | 4.79E-06 | 24 | 71 | 0.1287 | 5.35E-06 | 20 | 59 | 0.0222 | 9.01E-06 |
| | x4 | 11 | 32 | 0.0121 | 3.96E-06 | 23 | 68 | 0.0292 | 9.21E-06 | 20 | 59 | 0.0293 | 4.39E-06 |
| | x5 | 11 | 32 | 0.0139 | 3.96E-06 | 23 | 68 | 0.0303 | 9.21E-06 | 20 | 59 | 0.0206 | 4.39E-06 |
| | x6 | 10 | 29 | 0.0132 | 4.87E-06 | 20 | 59 | 0.0714 | 9.10E-06 | 17 | 50 | 0.0216 | 6.74E-06 |
| | x7 | 11 | 32 | 0.0080 | 3.96E-06 | 23 | 68 | 0.0376 | 9.21E-06 | 20 | 59 | 0.0279 | 4.39E-06 |
| | x8 | 11 | 32 | 0.0122 | 3.97E-06 | 23 | 68 | 0.0274 | 9.23E-06 | 20 | 59 | 0.0179 | 4.40E-06 |
| 5000 | x1 | 11 | 32 | 0.0895 | 4.87E-06 | 6 | 17 | 0.9856 | 1.51E-06 | 6 | 17 | 0.0242 | 7.86E-06 |
| | x2 | 10 | 30 | 0.0597 | 6.69E-06 | 6 | 17 | 0.0333 | 7.33E-06 | 7 | 20 | 0.8547 | 8.34E-07 |
| | x3 | 10 | 30 | 0.1727 | 7.00E-06 | 18 | 53 | 0.0545 | 9.70E-06 | 17 | 50 | 0.1656 | 4.70E-06 |
| | x4 | 11 | 32 | 0.0399 | 8.86E-06 | 25 | 74 | 0.0773 | 5.83E-06 | 20 | 59 | 0.0785 | 9.82E-06 |
| | x5 | 11 | 32 | 0.1033 | 8.86E-06 | 25 | 74 | 0.1119 | 5.83E-06 | 20 | 59 | 0.0682 | 9.82E-06 |
| | x6 | 10 | 30 | 0.0278 | 7.03E-06 | 21 | 62 | 0.0946 | 7.52E-06 | 19 | 56 | 0.5985 | 6.90E-06 |
| | x7 | 11 | 32 | 0.1430 | 8.86E-06 | 25 | 74 | 0.1078 | 5.83E-06 | 20 | 59 | 0.0861 | 9.82E-06 |
| | x8 | 11 | 32 | 0.0289 | 8.86E-06 | 25 | 74 | 0.0929 | 5.84E-06 | 20 | 59 | 0.0803 | 9.83E-06 |
| 10000 | x1 | 11 | 32 | 0.0476 | 6.89E-06 | 6 | 17 | 0.0389 | 2.13E-06 | 7 | 20 | 0.1593 | 6.27E-07 |
| | x2 | 10 | 30 | 0.0469 | 9.46E-06 | 7 | 20 | 0.0532 | 6.62E-07 | 7 | 20 | 0.4635 | 1.18E-06 |
| | x3 | 10 | 30 | 0.1110 | 9.91E-06 | 16 | 47 | 0.1259 | 2.57E-06 | 17 | 50 | 0.0965 | 4.92E-06 |
| | x4 | 11 | 33 | 0.0534 | 8.20E-06 | 25 | 74 | 0.4258 | 8.25E-06 | 21 | 62 | 0.1299 | 6.03E-06 |
| | x5 | 11 | 33 | 0.1411 | 8.20E-06 | 25 | 74 | 0.9868 | 8.25E-06 | 21 | 62 | 1.1701 | 6.03E-06 |
| | x6 | 10 | 30 | 0.0433 | 9.92E-06 | 25 | 74 | 0.2755 | 7.73E-06 | 21 | 62 | 0.1347 | 7.80E-06 |
| | x7 | 11 | 33 | 0.1191 | 8.20E-06 | 25 | 74 | 0.3708 | 8.25E-06 | 21 | 62 | 0.3222 | 6.03E-06 |
| | x8 | 11 | 33 | 0.0481 | 8.20E-06 | 25 | 74 | 0.1590 | 8.25E-06 | 21 | 62 | 0.6405 | 6.03E-06 |
| 50000 | x1 | 12 | 35 | 0.4852 | 3.28E-06 | 27 | 80 | 0.4618 | 7.69E-06 | 7 | 20 | 0.2612 | 1.40E-06 |
| | x2 | 11 | 32 | 0.7793 | 6.89E-06 | 7 | 20 | 0.5416 | 1.48E-06 | 7 | 20 | 0.2092 | 2.64E-06 |
| | x3 | 11 | 32 | 0.2076 | 7.22E-06 | 21 | 62 | 0.4876 | 6.85E-06 | 17 | 50 | 0.7491 | 6.82E-06 |
| | x4 | 12 | 35 | 0.1999 | 5.97E-06 | 26 | 77 | 0.5229 | 9.81E-06 | 22 | 65 | 0.6362 | 5.85E-06 |
| | x5 | 12 | 35 | 0.3531 | 5.97E-06 | 26 | 77 | 0.6792 | 9.81E-06 | 22 | 65 | 2.1945 | 5.85E-06 |
| | x6 | 11 | 32 | 0.1696 | 7.22E-06 | 20 | 59 | 1.3486 | 8.01E-06 | 18 | 53 | 1.5017 | 7.51E-06 |
| | x7 | 12 | 35 | 0.5930 | 5.97E-06 | 26 | 77 | 1.2159 | 9.81E-06 | 22 | 65 | 1.0642 | 5.85E-06 |
| | x8 | 12 | 35 | 0.4212 | 5.97E-06 | 26 | 77 | 0.4800 | 9.81E-06 | 22 | 65 | 0.4915 | 5.85E-06 |
| 100000 | x1 | 12 | 35 | 0.3894 | 4.64E-06 | 28 | 83 | 0.9476 | 5.79E-06 | 7 | 20 | 1.4489 | 1.98E-06 |
| | x2 | 11 | 32 | 1.1153 | 9.75E-06 | 28 | 83 | 1.5076 | 5.43E-06 | 7 | 20 | 0.4727 | 3.73E-06 |
| | x3 | 11 | 33 | 0.3290 | 6.68E-06 | 28 | 83 | 1.0226 | 7.06E-06 | 17 | 50 | 1.2814 | 8.73E-06 |
| | x4 | 12 | 35 | 0.9861 | 8.45E-06 | 27 | 80 | 1.0002 | 6.81E-06 | 22 | 65 | 1.2254 | 8.27E-06 |
| | x5 | 12 | 35 | 0.2797 | 8.45E-06 | 27 | 80 | 0.7974 | 6.81E-06 | 22 | 65 | 1.2500 | 8.27E-06 |
| | x6 | 11 | 33 | 1.0946 | 6.68E-06 | 28 | 83 | 1.8733 | 7.06E-06 | 18 | 53 | 0.8855 | 7.57E-06 |
| | x7 | 12 | 35 | 0.4250 | 8.45E-06 | 27 | 80 | 0.9973 | 6.81E-06 | 22 | 65 | 1.3610 | 8.27E-06 |
| | x8 | 12 | 35 | 0.6744 | 8.45E-06 | 27 | 80 | 1.0637 | 6.81E-06 | 22 | 65 | 1.3076 | 8.27E-06 |

**Table 7.** Numerical Results of the **SCD**, **PDY** and **LLY** Algorithms on Problem 7 with given initial points and dimensions

| | | SCD | | | | PDY | | | | LLY | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| DIMENSION | INITIAL POINT | ITER | FVAL | TIME | NORM | ITER | FVAL | TIME | NORM | ITER | FVAL | TIME | NORM |
| 1000 | x1 | 7 | 21 | 0.0285 | 5.66E-06 | 13 | 38 | 0.0125 | 6.75E-06 | 9 | 26 | 0.0098 | 4.32E-06 |
| | x2 | 7 | 20 | 0.0051 | 8.66E-06 | 12 | 35 | 0.0072 | 9.04E-06 | 9 | 26 | 0.0093 | 1.69E-06 |
| | x3 | 7 | 21 | 0.0139 | 3.09E-06 | 13 | 38 | 0.0152 | 3.69E-06 | 9 | 26 | 0.0185 | 2.36E-06 |
| | x4 | 7 | 21 | 0.0073 | 2.83E-06 | 13 | 38 | 0.0096 | 3.38E-06 | 9 | 26 | 0.0088 | 2.16E-06 |
| | x5 | 7 | 21 | 0.0075 | 2.83E-06 | 13 | 38 | 0.0119 | 3.38E-06 | 9 | 26 | 0.0103 | 2.16E-06 |
| | x6 | 7 | 21 | 0.0056 | 3.05E-06 | 13 | 38 | 0.0113 | 3.64E-06 | 9 | 26 | 0.0090 | 2.33E-06 |
| | x7 | 7 | 21 | 0.0053 | 2.83E-06 | 13 | 38 | 0.0128 | 3.38E-06 | 9 | 26 | 0.0062 | 2.16E-06 |
| | x8 | 7 | 21 | 0.0075 | 2.84E-06 | 13 | 38 | 0.0106 | 3.38E-06 | 9 | 26 | 0.0070 | 2.16E-06 |
| 5000 | x1 | 8 | 23 | 0.2014 | 4.92E-06 | 14 | 41 | 0.0237 | 4.42E-06 | 9 | 26 | 0.0186 | 9.66E-06 |
| | x2 | 7 | 21 | 0.0851 | 4.96E-06 | 13 | 38 | 0.1749 | 5.92E-06 | 9 | 26 | 0.0445 | 3.79E-06 |
| | x3 | 7 | 21 | 0.0335 | 6.92E-06 | 13 | 38 | 0.0585 | 8.25E-06 | 9 | 26 | 0.0193 | 5.28E-06 |
| | x4 | 7 | 21 | 0.0290 | 6.34E-06 | 13 | 38 | 0.0320 | 7.56E-06 | 9 | 26 | 0.0971 | 4.84E-06 |
| | x5 | 7 | 21 | 0.0105 | 6.34E-06 | 13 | 38 | 0.0343 | 7.56E-06 | 9 | 26 | 0.0494 | 4.84E-06 |
| | x6 | 7 | 21 | 0.0918 | 6.89E-06 | 13 | 38 | 0.0269 | 8.22E-06 | 9 | 26 | 0.4760 | 5.26E-06 |
| | x7 | 7 | 21 | 0.0347 | 6.34E-06 | 13 | 38 | 0.1156 | 7.56E-06 | 9 | 26 | 0.0774 | 4.84E-06 |
| | x8 | 7 | 21 | 0.0706 | 6.34E-06 | 13 | 38 | 0.0825 | 7.56E-06 | 9 | 26 | 0.0292 | 4.84E-06 |
| 10000 | x1 | 8 | 23 | 0.0256 | 6.96E-06 | 14 | 41 | 0.3006 | 6.25E-06 | 10 | 29 | 0.0773 | 1.98E-06 |
| | x2 | 7 | 21 | 0.0321 | 7.02E-06 | 13 | 38 | 0.0394 | 8.37E-06 | 9 | 26 | 0.0928 | 5.36E-06 |
| | x3 | 7 | 21 | 0.0219 | 9.79E-06 | 14 | 41 | 0.0437 | 3.42E-06 | 9 | 26 | 0.6629 | 7.47E-06 |
| | x4 | 7 | 21 | 0.2105 | 8.96E-06 | 14 | 41 | 0.0442 | 3.13E-06 | 9 | 26 | 0.1134 | 6.84E-06 |
| | x5 | 7 | 21 | 0.0215 | 8.96E-06 | 14 | 41 | 0.1545 | 3.13E-06 | 9 | 26 | 0.0547 | 6.84E-06 |
| | x6 | 7 | 21 | 0.0213 | 9.77E-06 | 14 | 41 | 0.0483 | 3.41E-06 | 9 | 26 | 0.2624 | 7.46E-06 |
| | x7 | 7 | 21 | 0.0936 | 8.96E-06 | 14 | 41 | 0.0731 | 3.13E-06 | 9 | 26 | 0.9003 | 6.84E-06 |
| | x8 | 7 | 21 | 0.2350 | 8.96E-06 | 14 | 41 | 0.1291 | 3.13E-06 | 9 | 26 | 0.3378 | 6.84E-06 |
| 50000 | x1 | 8 | 24 | 0.1129 | 3.99E-06 | 41 | 122 | 1.0051 | 6.97E-06 | 10 | 29 | 0.5945 | 4.42E-06 |
| | x2 | 8 | 23 | 0.3362 | 6.11E-06 | 14 | 41 | 0.1410 | 5.48E-06 | 10 | 29 | 0.2369 | 1.73E-06 |
| | x3 | 8 | 23 | 0.0782 | 8.51E-06 | 14 | 41 | 0.1532 | 7.65E-06 | 10 | 29 | 1.5064 | 2.42E-06 |
| | x4 | 8 | 23 | 0.0707 | 7.80E-06 | 14 | 41 | 0.3986 | 7.00E-06 | 10 | 29 | 0.2640 | 2.21E-06 |
| | x5 | 8 | 23 | 0.0840 | 7.80E-06 | 14 | 41 | 0.5471 | 7.00E-06 | 10 | 29 | 0.4523 | 2.21E-06 |
| | x6 | 8 | 23 | 0.2947 | 8.51E-06 | 14 | 41 | 0.2980 | 7.64E-06 | 10 | 29 | 1.2022 | 2.42E-06 |
| | x7 | 8 | 23 | 0.0799 | 7.80E-06 | 14 | 41 | 0.5583 | 7.00E-06 | 10 | 29 | 0.2554 | 2.21E-06 |
| | x8 | 8 | 23 | 0.3384 | 7.80E-06 | 14 | 41 | 0.2536 | 7.00E-06 | 10 | 29 | 0.3382 | 2.21E-06 |
| 100000 | x1 | 8 | 24 | 0.2210 | 5.64E-06 | 41 | 122 | 1.3457 | 9.86E-06 | 10 | 29 | 0.2277 | 6.25E-06 |
| | x2 | 8 | 23 | 0.3563 | 8.64E-06 | 14 | 41 | 0.2428 | 7.75E-06 | 10 | 29 | 0.4301 | 2.45E-06 |
| | x3 | 8 | 24 | 0.1647 | 3.09E-06 | 15 | 44 | 0.9632 | 3.17E-06 | 10 | 29 | 0.9843 | 3.42E-06 |
| | x4 | 8 | 24 | 0.2206 | 2.82E-06 | 14 | 41 | 0.3214 | 9.90E-06 | 10 | 29 | 0.6343 | 3.13E-06 |
| | x5 | 8 | 24 | 0.1516 | 2.82E-06 | 14 | 41 | 0.5326 | 9.90E-06 | 10 | 29 | 0.4172 | 3.13E-06 |
| | x6 | 8 | 24 | 0.1568 | 3.08E-06 | 15 | 44 | 0.3448 | 3.17E-06 | 10 | 29 | 0.3376 | 3.42E-06 |
| | x7 | 8 | 24 | 0.4245 | 2.82E-06 | 14 | 41 | 0.3345 | 9.90E-06 | 10 | 29 | 1.2244 | 3.13E-06 |
| | x8 | 8 | 24 | 0.1547 | 2.82E-06 | 14 | 41 | 0.2437 | 9.90E-06 | 10 | 29 | 0.3299 | 3.13E-06 |

Moreover, using the Dolan and Morè performance profile [12], we plot the graphs of the three algorithms in order to visualize their performance. The performance is shown in Figures 1, 2 and 3.

It can be observed from Figure 1 and 2 that the SCD algorithm outperformed the PDY and LLY algorithms by solving around 90% of the problems with less number of iterations and function evaluations. Furthermore, with regards to time, Figure 3 shows that the SCD algorithm solved around 70% of the problems with less time. Therefore, we can conclude that the proposed algorithm is more efficient than the PDY and LLY algorithms in terms of number of iterations, function evaluations as well as CPU time.

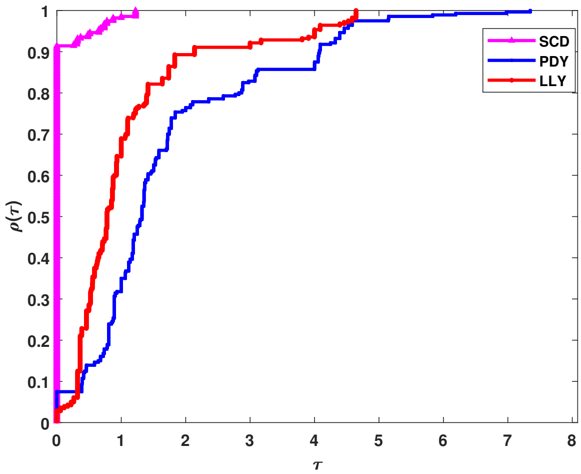


**Fig. 1.** Performance profile on number of iterations

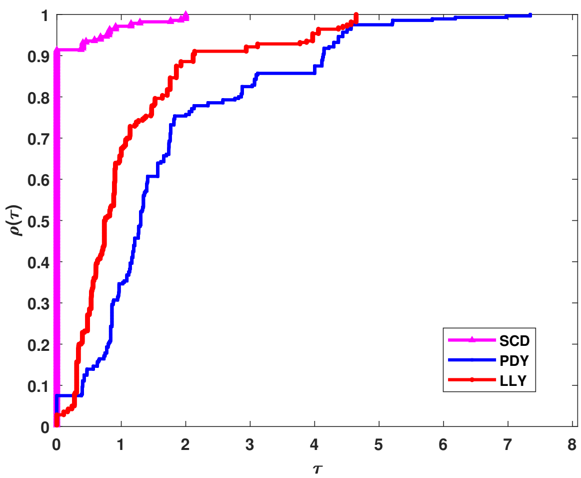


**Fig. 2.** Performance profile on function evaluations

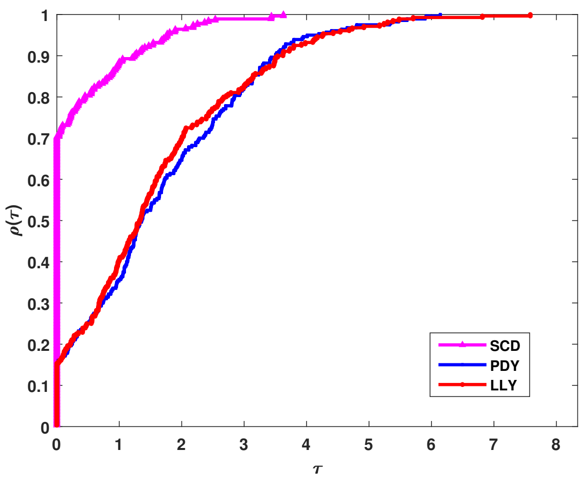


**Fig. 3.** Performance profile on CPU time

## 4. Application in Compressive Sensing

The problem of sparse signal reconstruction involves solving minimization of an objective function:

$$\min_x \frac{1}{2}\|Mx - y\|_2^2 + \rho\|x\|_1, \tag{4.1}$$

where $x \in R^n$, $q \in R^m$, $M \in R^{m\times n}$ $(m << n)$ is a linear operator, $\rho \geq 0$, $\|x\|_2$ is the Euclidean norm of $x$ and $\|x\|_1 = \sum_{i=1}^n |x_i|$ is the $\ell_1-$norm of $x$.

This problem is of interest to many researchers in signal processing. Some of the popular methods for solving (4.1) can be found in [13, 16, 5, 14, 27]. First, a reformulation of problem (4.1) into a quadratic problem was given by Figueiredo et al. [14]. They expressed $x \in \mathbb{R}^n$ in two parts

$$x = b - y, \qquad b \geq 0, \quad y \geq 0,$$

where $b_i = (x_i)_+$, $y_i = (-x_i)_+$ for all $i = 1, 2, ..., n$, and $(.)_+ = \max\{0, .\}$. Also, we have $\|x\|_1 = e_n^T b + e_n^T y$, where $e_n = (1, 1, ..., 1)^T \in R^n$. From this reformulation, equation (4.1) can be written as

$$\min_{b,y} \frac{1}{2}\|q - M(b - y)\|_2^2 + \rho e_n^T b + \rho e_n^T y, \qquad b \geq 0, \quad y \geq 0, \tag{4.2}$$

from [14], equation (4.2) can be written as

$$\min_z \frac{1}{2} z^T E z + c^T z, \qquad \text{such that} \quad z \geq 0, \tag{4.3}$$

where $z = \begin{pmatrix} b \\ y \end{pmatrix}$, $\quad c = \omega e_{2n} + \begin{pmatrix} -a \\ a \end{pmatrix}$, $\quad a = M^T q$, $\quad E = \begin{pmatrix} M^T M & -M^T M \\ -M^T M & M^T M \end{pmatrix}$.
It can be observed that $E$ is a positive semi-definite showing that Problem (4.3) is quadratic programming problem.

Moreover, a translation of (4.3) into a linear variable problem, equivalently, a linear complementary problem was given by Xiao et al [30], and the variable $z$ solves the linear complementary problem provided that it solves the nonlinear equation:

$$M(z) = \min\{z, Ez + c\} = 0, \tag{4.4}$$

where $M$ is a vector-valued function. The mapping $M(z)$ is continuous and monotone as shown in [29, 23]. This implies problem (4.1) is equivalent to problem (1.1). Hence, the proposed SCD algorithm for solving (1.1) can be applicable to solve (4.1).

In this section, our proposed SCD algorithm is applied to restore a signal of length n from k observations. The performance of the SCD is compared with two existing methods, specifically, the CGD and PCG algorithms proposed in [21] and [30] respectively. We choose the parameters in the SCD algorithm as follows: $\beta = 10^{-5}$, $\alpha = 0.03$, $\sigma = 0.1$, $\kappa = 1$ and $c = 1$. However, in the CGD and PCG algorithms, the parameters are maintained as reported in the respective papers [21] and [30]. We run each of the three algorithms with same initial point and continuation technique on parameter $\mu$. The convergence behviour of the algorithms is observed to obtain a solution with similar accuracy. For initialization, we used $x_0 = M^T y$ and the iterations were stopped when the inequality

$$\left| \frac{f(x_k) - f(x_{k-1})}{f(x_{k-1})} \right| < 10^{-5}.$$

is satisfied. To understand the quality of the restoration, mean squared error (MSE) is used as a metric. The MSE is given as:

$$MSE = \frac{1}{n} \|\widehat{x} - x\|^2,$$

In the experiment, M is a Gaussian matrix with $n = 2^{12}$, $k = 2^{10}$, the original signal contains $2^7$ nonzero elements. We recover the original signal $\widehat{x}$ from $y$ by $y = 2^{10}$ observations, and set $f(x) = \frac{1}{2}\|Mx - y\|_2^2 + \rho\|x\|_1$. Where $y = M\widehat{x} + \omega$, and $\omega$ is the Gaussian noise distributed as $N(0, 10^{-4})$.

Figures 4 and 5 depicts the perfomance of all the three methods. All the three successfully restored the signal. However, it can be observed that our SCD method has some advantages over the CGD and PCG methods. These advantages include lesser MSE, number of iterations and CPU time. In addition, four graphs are plotted to demonstrate the convergence behaviour of the algorithms. In this case also, our proposed SCD method is reported to have faster convergence rate in comparison with the other two. Thus, our proposed SCD method can efficiently restore a sparse signal with less error, in fewer iterations as well as CPU time.

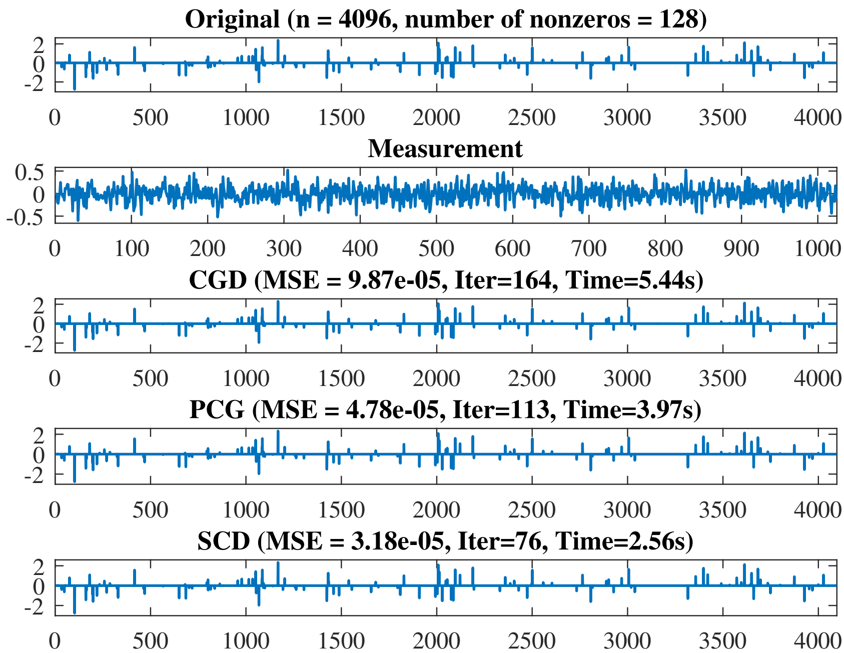


**Fig. 4.** From top to bottom: the original signal, the measurement, and the recovery signals by CGD, PCG and SCD methods.

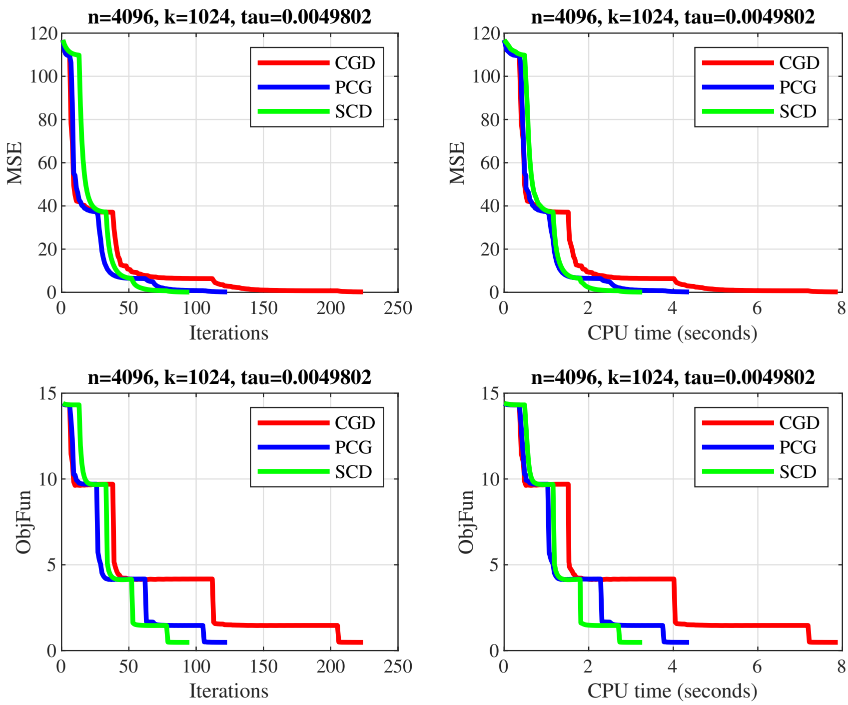


**Fig. 5.** Comparison of CGD, PCG and SCD methods based on MSE, number of iterations, objective function and CPU time.

## 5. Conclusion

In this paper, a derivative-free algorithm for solving systems of nonlinear monotone equations is proposed. The algorithm combined the well-known conjugate gradient method with a projection method. The global convergence of the method is established by assuming that the operator under study is Lipschitz continuous and monotone. Numerical experiments are performed to show the efficiency of the algorithm in comparison with some existing works, specifically, the algorithms proposed in [20] and [33]. The numerical results reported in this work have proved that the proposed algorithm is more efficient than the ones proposed in [20] and [33]. Furthermore, the proposed algorithm is applied in signal recovery problems, and the recovery result is compared with the PCG and CGD methods proposed in [21] and [30] respectively. The proposed algorithm is shown to recover the distorted signal with less MSE, number of iterations as well as CPU time, proving its effectiveness over the compared algorithms.

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