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On the Rus-Hicks-Rhoades Contraction Principle

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ABSTRACT

Recently the Banach contractions are extended to the weak contractions or the Rus-Hicks-Rhoades (RHR) maps, for which we collected lots of examples and studied in our previous works [18, 20, 21, 22]. Our aim in the present article is to introduce the weak contraction principle or the RHR contraction principle (Theorem P) which is an extension of the traditional Banach contraction principle. We also improve some known theorems closely related to Theorem P.

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1. Introduction

The metric fixed point theory originates from Banach [1] in 1922 on the study of the Banach contraction $f : X \to X$ on a normed vector space X. Later it was extended to a selfmap f on a complete metric space (X, d) satisfying

$$d(fx, fy) \leq \alpha\, d(x, y) \text{ with } \alpha \in [0, 1)$$

for any $x,\ y \in X$. Since then there have appeared several hundreds of contractive type conditions and almost one thousand spaces extending or modifying complete metric spaces.

One of such extended contractive type conditions was due to Rus [25] in 1973 and Hicks-Rhoades [9] in 1979 as follows:

$$d(fx, f^2x) \leq \alpha\, d(x, fx) \ \text{ for every } x \in X,$$

where $\alpha \in [0, 1)$. Such f is called a *weak contraction* or a *Rus-Hicks-Rhoades map* or an *RHR map*, and it has a large number of closely related mapping classes. An RHR map was also called a graphic contraction, iterative contraction, weakly contraction, Banach mapping, ...; see Berinde-Petrușel-Rus [2].

Recently, we noticed in [21, 18, 22, 20] that the RHR map has an interesting long history. It extends the Banach contraction [1] in 1922, but we found that it is also close to its multi-valued versions due to Nadler [14] in 1969 and Covitz-Nadler [6] in 1970. The aim of our [20] was to trace such history of the Rus-Hicks-Rhoades theorem, and to show its grown-up

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versions or equivalents or closely related theorems. Such theorems are too many and could be called its relatives.

In our previous article [19], we showed that the so-called Suzuki type maps in 2008 are RHR maps and the proofs of results of Suzuki and his colleagues can be simplified within a few lines based on our recent works on quasi-metric spaces. Moreover, in [22], we trace the history of a large number of RHR maps in ordered fixed point theory, and new proofs are also given for known theorems.

In the present article, our aim is to introduce the RHR contraction principle (Theorem P) which is a proper extension of the traditional Banach contraction principle. Moreover, we improve several known theorems closely related to Theorem P.

This article is organized as follows: Section 2 is for preliminaries on quasi-metric spaces and a basic fixed point theorem. In Section 3, we reprove our basic RHR contraction principle (Theorem P). From its proof, we add a particular form of the Caristi type fixed point theorem. Section 4 deals with an extended form of the Banach contraction principle. Consequently the usual Banach principle holds for orbitally complete quasi-metric spaces. In Sections 5 to 11, we are concerned with the works of Banach [1], Park and Rhoades [23], Park [16], Suzuki [26] and related works, Miñana and Valero [13], Park [18], Fierro and Pizzaro [7], respectively. In each section, we introduce the main results of each paper and discuss the possibility to improve them based on our Theorem P. Finally, Section 12 devotes to Epilogue.

2. Preliminaries on Quasi-metric Spaces

We recall the following:

Definition 2.1. A *quasi-metric* on a nonempty set X is a function $q : X \times X \to [0, \infty)$ satisfying the following conditions for all $x, y, z \in X$:

(a) (self-distance) $q(x, y) = q(y, x) = 0 \iff x = y$;

(b) (triangle inequality) $q(x, z) \leq q(x, y) + q(y, z)$.

A *metric* on a set X is a quasi-metric satisfying

(c) (symmetry) $q(x, y) = q(y, x)$ for all $x, y \in X$.

For quasi-metric spaces, the convergence of a sequence, (right) Cauchy sequences, completeness, orbits, and orbital continuity are routinely defined as follows; see Jleli-Samet [10].

Definition 2.2. [10]

(1) A sequence (x_n) in X *converges* to $x \in X$ if

$$\lim_{n\to\infty} q(x_n, x) = \lim_{n\to\infty} q(x, x_n) = 0.$$

(2) A sequence (x_n) is *left-Cauchy* if for every $\varepsilon > 0$, there is a positive integer $N = N(\varepsilon)$ such that $q(x_n, x_m) < \varepsilon$ for all $n > m > N$.

(3) A sequence (x_n) is *right-Cauchy* if for every $\varepsilon > 0$, there is a positive integer $N = N(\varepsilon)$ such that $q(x_n, x_m) < \varepsilon$ for all $m > n > N$.

(4) A sequence (x_n) is *Cauchy* if for every $\varepsilon > 0$ there is positive integer $N = N(\varepsilon)$ such that $q(x_n, x_m) < \varepsilon$ for all $m, n > N$; that is (x_n) is a *Cauchy sequence* if it is left and right Cauchy.

Definition 2.3. [10]

(1) (X, q) is *left-complete* if every left-Cauchy sequence in X is convergent;

(2) (X, q) is *right-complete* if every right-Cauchy sequence in X is convergent;

(3) (X, q) is *complete* if every Cauchy sequence in X is convergent.

Definition 2.4. Let (X, q) be a quasi-metric space and $f : X \to X$ a selfmap. The *orbit* of f at $x \in X$ is the set
$$O_f(x) = \{x, fx, \cdots, f^n x, \cdots\}.$$
The space (X, q) is said to be *f-orbitally complete* if every right-Cauchy sequence in $O_f(x)$ is convergent in X. A selfmap f of X is said to be *orbitally continuous* at $x_0 \in X$ if
$$\lim_{n\to\infty} f^n x = x_0 \implies \lim_{n\to\infty} f^{n+1} x = fx_0$$
for any $x \in X$.

Note that every complete metric space is f-orbitally complete for all maps $f : X \to X$. There exists a f-orbitally complete metric space but it is not complete. Moreover, there exists an orbitally continuous map but it is not continuous.

Every quasi-metric induces a metric, that is, if (X, q) is a quasi-metric space, then the function $d : X \times X \to [0, \infty)$ defined by
$$d(x, y) = \max\{q(x, y), q(y, x)\}$$
is a metric on X; see Jleli-Samet [10].

The following was given in [17, 18, 19]:

Theorem 2.5. *A selfmap $f : X \to X$ of a quasi-metric space (X, q) has a fixed point $z \in X$ if and only if z is a fixed point of the selfmap f of the induced metric space (X, d).*

Proof. If $z = fz$ in (X, q), then
$$d(z, fz) = \max\{q(z, fz), q(fz, z)\} = 0,$$
and hence $d(z, fz) = 0$. The converse is true for $d = q$. ∎

3. The Rus-Hicks-Rhoades Contraction Principle

In this section, we re-examine the proofs of the Banach contraction principle or the Rus-Hicks-Rhoades contraction principle for a quasi-metric space (X, q) with a selfmap $T : X \to X$ such that X is T-orbitally complete.

Lemma 3.1. *Let (X, q) be a quasi-metric space, $T : X \to X$ a selfmap, and $\varphi : X \to [0, \infty)$ any function such that*

$$q(x, T(x)) \leq \varphi(x) - \varphi(T(x)) \quad \text{for some} \quad x \in M.$$

Then $\{T^n(x)\}$ is a right-Cauchy sequence.

Proof. If we fix $x \in X$ and take $m > n \in \mathbb{N}$, we obtain

$$q(T^n(x), T^{m+1}(x)) \leq \sum_{i=n}^{m} q(T^i(x), T^{i+1}(x)) \leq \varphi(T^n(x)) - \varphi(T^{m+1}(x)).$$

(Notice that the last inequality comes from cancelation in the telescoping sum.) In particular by taking $n = 1$ and letting $m \to \infty$ we conclude that

$$\sum_{i=1}^{\infty} q(T^i(x), T^{i+1}(x)) \leq \varphi(T(x)) < \infty.$$

This implies that $\{T^n(x)\}$ is a right-Cauchy sequence. ∎

The following in Park [17] will be called *the weak contraction principle* or *the Rus-Hicks-Rhoades (RHR) contraction principle*:

Theorem P. *Let (X, q) be a quasi-metric space and let $T : X \to X$ be an RHR map; that is,*

$$q(T(x), T^2(x)) \leq \alpha\, q(x, T(x)) \text{ for every } x \in X,$$

where $0 < \alpha < 1$ and X is T-orbitally complete. Then

(i) for each $x \in X$, there exists a point $x_0 \in X$ such that

$$\lim_{n\to\infty} T^n(x) = x_0,$$

$$q(T^n(x), x_0) \leq \frac{\alpha^n}{1-\alpha} q(x, T(x)), \quad n = 1, 2, \cdots,$$

$$q(T^n(x), x_0) \leq \frac{\alpha}{1-\alpha} q(T^{n-1}(x), T^n(x)), \quad n = 1, 2, \cdots,$$

and

(ii) x_0 is a fixed point of T if and only if T is orbitally continuous at x_0.

This was proved in [17] by analyzing a typical proof of the Banach contraction principle given by Art Kirk ([11, Theorem 2.2]). We reprove this for the completeness:

Proof. Step 1. *For each $x \in X$, $\{T^n(x)\}$ is right Cauchy*: Adding $q(x, T(x))$ to both sides of the inequality $q(T(x), T^2(x)) \leq \alpha\, q(x, T(x))$ yields

$$q(x, T(x)) + q(T(x), T^2(x)) \leq q(x, T(x)) + \alpha\, q(x, T(x))$$

which can be rewritten

$$q(x, T(x)) - \alpha\, q(x, T(x)) \leq q(x, T(x)) - q(T(x), T^2(x)).$$

This in turn is equivalent to

$$q(x, T(x)) \leq (1-\alpha)^{-1}[q(x, T(x)) - q(T(x), T^2(x))].$$

Now define the function $\varphi : X \to [0, \infty)$ by setting $\varphi(x) = (1-\alpha)^{-1} q(x, T(x))$, for $x \in X$. This gives us the basic inequality

$$q(x, T(x)) \leq \varphi(x) - \varphi(T(x)), \quad x \in X.$$

Therefore $\{T^n(x)\}$ is a right-Cauchy sequence by Lemma 3.1.

Step 2. *T-orbital completeness*: Since X is T-orbitally complete, for any $x \in X$ there exists $x_0 \in X$ such that

$$\lim_{n\to\infty} T^n(x) = x_0.$$

Step 3. *Orbital continuity at x_0*: If T is orbitally continuous at x_0, then

$$x_0 = \lim_{n\to\infty} T^n(x) = \lim_{n\to\infty} T^{n+1}(x) = T(x_0).$$

Thus x_0 is a fixed point of T. Conversely, if x_0 is fixed, then clearly T is orbitally continuous at x_0.

Step 4. *Convergence for $\{T^n(x)\}$*: The last part of Kirk's original proof in [11] is added for completeness. Returning to the inequality

$$q(T^n(x), T^{m+1}(x)) \leq \varphi(T^n(x)) - \varphi(T^{m+1}(x)),$$

upon letting $m \to \infty$ we see that

$$q(T^n(x), x_0) \leq \varphi(T^n(x)) = (1-\alpha)^{-1} q(T^n(x), T^{n+1}(x)).$$

Since $(1-\alpha)^{-1} q(T^n(x), T^{n+1}(x)) \leq \frac{\alpha^n}{1-\alpha} q(x, T(x))$, we obtain

$$q(T^n(x), x_0) \leq \frac{\alpha^n}{1-\alpha} q(x, T(x)).$$

This provides an estimate on the rate of convergence for the sequence $\{T^n(x)\}$ which depends only on $q(x, T(x))$. ∎

Remark 3.2.

(1) In a quasi-metric space, the condition (a) is consistent with

$$q(T(x), T^2(x)) \leq \alpha\, q(x, T(x))$$

when x is a fixed point of T. The triangle inequality (b) is used in Step 2.

(2) In the above lengthy proof, the symmetry (c) of a metric is not used. Moreover, it is enough to assume the (orbital) continuity at x_0 only. For example, it is well-known that the Kannan map is continuous at its fixed point only.

(3) Hicks and Rhoades [9] in 1979 obtained Theorem P for a complete metric space with the following conclusion instead of our (ii) in Theorem P:

(iii) z is a fixed point of T if and only if $G(x) := q(x, Tx)$ is T-orbitally lower semicontinuous at z.

This is not claiming the existence of fixed point; see our (ii).

(4) In Step 1 of the proof of Theorem P, we obtained the basic inequality

$$q(x, T(x)) \leq \varphi(x) - \varphi(T(x)), \quad x \in M.$$

If $\varphi : X \to [0, \infty)$ is lower semicontinuous from above and bounded from below, by applying the Caristi fixed point theorem [3], we can obtain fixed points. This leads to Step 4 directly. Therefore, the orbital continuity in Theorem P(ii) holds in this case.

Example 3.3. Let $X = [0, 2] \subset \mathbb{R}$ with the usual metric q and

$$T(x) = \begin{cases} 1 & \text{if } x \in [0, 1] \\ x & \text{if } x \in [1, 2]. \end{cases}$$

Then $q(T(x), T^2(x)) \leq \alpha\, q(x, T(x))$ for some $\alpha \in [0, 1)$. In fact,

$$0 = q(1, 1) \leq \alpha\, q(x, 1) \quad \text{for } x \in [0, 1],$$

$$0 = q(x, x) \leq \alpha\, q(x, x) \quad \text{for } x \in [1, 2].$$

This example has 'many' fixed points of T. Note that T is an RHR map and not a Banach contraction.

From the proof of Theorem P, we have the following:

Theorem 3.4. *Let (X, q) be a quasi-metric space and let $T : X \to X$ be a map satisfying*

$$q(x, T(x)) \leq \varphi(x) - \varphi(T(x)), \quad x \in X,$$

for a real-valued function $\varphi : X \to [0, \infty)$ such that

$$\varphi(x) = (1 - \alpha)^{-1} q(x, T(x)) \text{ with } 0 < \alpha < 1,$$

and X is T-orbitally complete. Then

(i) for each $x \in X$, there exists a point $x_0 \in X$ such that

$$\lim_{n\to\infty} T^n(x) = x_0,$$

$$q(T^n(x), x_0) \leq \frac{\alpha^n}{1 - \alpha} q(x, T(x)), \quad n = 1, 2, \cdots .$$

and

(ii) $T : X \to X$ is orbitally continuous at $x_0 \in X$ in (i) if and only if x_0 is a fixed point of T.

This is a particular form of the Caristi type fixed point theorem.

The original Rus-Hicks-Rhoades theorem can be extended to the following consequence of the Caristi type fixed point theorem; see [17, Theorem 6.3].

Theorem 3.5. *Let T be a continuous selfmap of a complete quasi-metric space (M, q) satisfying*

$$q(T(x), T^2(x)) \leq \alpha\, q(x, T(x)) \quad \text{for every } x \in M,$$

where $0 < \alpha < 1$. Then T has a fixed point and the statement (i) of Theorem P holds.

Proof. Completeness implies T-orbitally completeness. Then Theorem 3.5 follows from Theorem P. ∎

4. The Extended Banach Contraction Principle

The following consequence of Theorem P in Park [17] properly extends the Banach contraction principle:

Theorem Q. *Let (X, q) be a quasi-metric space and let $T : X \to X$ be an improved Banach contraction, that is, for each $x \in X$, there exists a $y \in X$ such that*

$$q(T(x), T(y)) \leq \alpha\, q(x, y) \text{ where } 0 < \alpha < 1. \tag{q}$$

(i) If X is T-orbitally complete, then, for each $x \in X$, there exists a point $x_0 \in X$ such that

$$\lim_{n\to\infty} T^n(x) = x_0$$

and

$$q(T^n(x), x_0) \leq \frac{\alpha^n}{1-\alpha} q(x, T(x)), \quad n = 1, 2, \cdots,$$

$$q(T^n(x), x_0) \leq \frac{\alpha}{1-\alpha} q(T^{n-1}(x), T^n(x)), \quad n = 1, 2, \cdots.$$

(ii) x_0 is the unique fixed point of T if and only if T is orbitally continuous at x_0.

The Banach contraction principle appeared in thousands of publications should be improved as in Theorem Q.

The following extends the standard Banach contraction principle formulated by Art Kirk [11, Theorem 2.2]:

Theorem 4.1. *Let (X, q) be a quasi-metric space and let $T : X \to X$ be a contraction, that is,*

$$q(T(x), T(y)) \leq \alpha\, q(x, y) \text{ for every } x, y \in X,$$

with $0 < \alpha < 1$. If (X, q) is T-orbitally complete, then T has a unique fixed point $x_0 \in X$. Moreover, for each $x \in X$,

$$\lim_{n\to\infty} T^n(x) = x_0$$

and, in fact, for each $x \in X$,

$$q(T^n(x), x_0) \leq \frac{\alpha^n}{1-\alpha} q(x, T(x)), \quad n = 1, 2, \cdots.$$

Proof. All things follow from Theorem Q except the following in the proof of Kirk ([11, Theorem 2.2]):

Step 5. *Uniqueness of fixed point*: In order to see that x_0 is the only fixed point of T, suppose $T(y) = y$. Then by what we have shown in Theorem 3.2

$$x_0 = \lim_{n\to\infty} T^n(y) = y.$$

This completes the proof. ∎

Moreover, under the hypothesis of Theorem 4.1, Steps 1-4 of the proof of Theorem P hold. Therefore, Theorem P implies all conclusions of the Banach contraction principle except the uniqueness of the fixed point.

5. Banach [1] in 1922

From now on, we try to improve some known theorems closely related to Theorem P. The traditional Banach contraction principle is a particular form of Theorem Q when X is a metric space and (q) holds for all $x, y \in X$. It appears in thousands of publications and should be corrected or replaced by Theorem Q.

The origin of the Banach contraction is the following due to Banach [1] in 1922:

Theorem 5.1 (Banach). *If $U(X)$ be a continuous operator on E, the counter-domain of $U(X)$ is contained in E_1. There exists a number $0 < M < 1$ which implies, for every X' and X'', the inequality*

$$||U(X') - U(X'')|| \leq M||X' - X''||.$$

Then there exists an element X such that $X = U(X)$.

Here E and E_1 is a normed space and its complete subset, resp.

Now we have the following:

Banach's Theorem 5.1 $\Longrightarrow$ The usual Banach contraction principle $\Longrightarrow$
Theorem 4.1 $\Longrightarrow$ Theorem Q $\Longrightarrow$ Theorem P.

6. Park and Rhoades [23] in 1980

In [23] we established several fixed point theorems involving hypotheses weak enough to include a large number of known theorems as special cases. However, we have to correct certain things as follows:

Let f be a selfmap of a topological space X. A function $G : X \to [0, \infty)$ is said to be *f-orbitally lower semicontinuous* at a point $p \in X$ if, for every $x_0 \in X$, $x_{n_k} \to p$ implies $G(p) \leq \liminf_k G(x_{n_k})$ where $\{x_{n_k}\}_{k=1}^{\infty}$ is a subsequence of $\{x_n\}_{n=0}^{\infty}$, which is defined by $x_{n+1} = f(x_n)$, i.e. $\{x_n\}_{n=0}^{\infty} = O_f(x_0)$.

A function $G : X \to [0, \infty)$ is said to be f-orbitally lower semicontinuous *from above* at a point $p \in X$ if, for every $x_0 \in X$, $x_n \to p$ implies $G(p) \leq \lim_n G(x_n)$, where $\{x_n\}_{n=0}^{\infty}$ is defined by $x_{n+1} = fx_n$, i.e. $\{x_n\}_{n=0}^{\infty} = O_f(x_0)$.

In 2001, Kirk and Saliga [12] introduced this concept and applied it to extend the Caristi fixed point theorem.

Now we apply this concept to improve Park-Rhoades [23, Theorem 1]:

Theorem 6.1. *Let d be a nonnegative real valued function defined on $X \times X$ such that, for any $x, y \in X$, $d(x,y) = d(y,x) = 0$ if and only if $x = y$. If there exists a point $u \in X$ such that $\lim_n d(f^n u, f^{n+1}u) = 0$, and if $\{f^n u\}$ converges to the limit $p \in X$, then p is a fixed point of f if and only if $G(x) = d(x, fx)$ is f-orbitally lower semicontinuous from above at p.*

Proof. Suppose $\{f^n u\}$ converges to a fixed point p of X. Then $0 = G(p) \le \lim_n G(f^n u)$.

Conversely, if G is f-orbitally lower semicontinuous from above at p, then

$$0 = \lim_n d(f^n u, f^{n+1}u) \ge d(p, fp).$$

∎

The following improves [23, Corollary 1]:

Corollary 6.2. *Let (X,d) be a metric space, $f : X \to X$, and $\varphi : X \to [0,\infty)$ such that, there exists a point $u \in X$ with $d(x, fx) \le \varphi(x) - \varphi(fx)$ for each $x \in O_f(u)$, and $\overline{O_f(u)}$ is complete. Then*

(i) $\lim_n f^n u = p$ exists, and

(ii) p is a fixed point of f if and only if $G(x) = d(x, fx)$ is f-orbitally lower semicontinuous from above at p.

The following improves [23, Theorem 2]:

Theorem 6.3. *Let f be a selfmap of a metric space (X,d) satisfying:*

(i) $\delta(O_f(x)) < \infty$ for each $x \in X$, where δ denotes the diameter.

(ii) There exists a $u \in X$ such that $\overline{O_f(u)}$ has a cluster point $p \in X$.

(iii) There exists a function $\varphi : [0,\infty) \to [0,\infty)$ which is nondecreasing, continuous from the right and satisfies $\phi(t) < t$ for each $t > 0$ and the inequality

$$d(fx, f^2x) \le \varphi(\delta(O_f(x)) \quad \textit{for each} \quad x \in X.$$

Then p is the unique fixed point of f and $\lim_n f^n u = p$.

Proof. Define $\rho_n = \delta(O(f^n u))$. From (i), ρ_n is finite for each n. Since $\rho_{n+1} \le \rho_n$ for each n, $\{\rho_n\}$ converges to some $\rho \ge 0$.

For each $j > i \ge n+1$, from (iii),

$$d(f^i u, f^j u) \le \varphi(\delta(O_f(f^{i-1}u) \cup O_f(f^{i-2}u))) \le \varphi(\delta(O_f(f^n u)) = \varphi(\rho_n),$$

so that $\rho_{n+1} \le \varphi(\rho_n)$ for each n. Since φ is continuous from the right, we have $\rho \le \varphi(\rho)$, which implies $\rho = 0$. Therefore $\{f^n u\}$ is Cauchy, and $f^n u \to p$ by (ii).

For each $\varepsilon > 0$ there exists an integer N such that $n \ge N$ implies $d(f^n u, p) < \varepsilon$.

For any integers $m > 0$ and $n. > N$, from (iii) it follows

$$d(p, f^m p) \le d(p, f^{n+1}u) + d(f(f^{m-1}p), f^2(f^{n-1}u))$$

$$\leq d(p, f^{n+1}u + \varphi(\delta(O(f^{m-1}p) \cup O(f^{n-1}u)))$$
$$\leq \varepsilon + \varphi(\max\{2\varepsilon, \delta(O_f(p)) + \varepsilon\}).$$

From the Lemma of Hegedüs [8], $\delta(O_f(p)) = \sup_m d(p, f^m p)$, so that we have

$$\delta(O_f(p)) \leq \varepsilon + \varphi(\max\{2\varepsilon, \delta(O_f(p)) + \varepsilon\}).$$

Since ε is arbitrary, $\delta(O_f(p)) \leq \varphi(\delta(O_f(p)))$, so that $O_f(p) = 0$, which implies $\delta(O_f(p)) = 0$. Therefore $p = fp$.

Uniqueness follows from (iii). ∎

In [23], this is extended to 2-metric spaces.

7. Park [16] in 1980

The following improved version of [16, Theorem 2] is an extension of the Banach contraction principle:

Theorem 7.1. *Let f be a selfmap of a metric space (X, d). If there exists a point $u \in X$ and a $\lambda \in [0, 1)$ such that $\overline{O_f(u)}$ is complete and*

$$d(fx, f^2x) \leq \lambda d(x, fx) \tag{*}$$

holds for any $x \in O_f(u)$, then $\{f^i u\}$ converges to some $\xi \in X$, and

$$d(f^i u, \xi) \leq \frac{\lambda^i}{1-\lambda} d(u, fu) \text{ for } i > 1.$$

Further, if f is orbitally continuous at ξ or if $(*)$ holds for any $x \in \overline{O(u)}$, then ξ is fixed under f.

Proof. Since $d(f^i u, f^{i+1}u) \leq \lambda (f^{i-1}u, f^i u)$, we have

$$d(f^i u, f^{i+1}u) \leq \lambda^i d(u, fu) \text{ for } i > 1.$$

For any $i, j \geq 1$, we have

$$\begin{aligned} d(f^i u, f^{i+j}u) &\leq d(f^i u, f^{i+1}u) + \cdots + d(f^{i+j-1}u, f^{i+j}u) \\ &\leq d(f^i u, f^{i+1}u) \cdot (1 + \lambda + \cdots + \lambda^{j-1}) \\ &= \frac{1-\lambda^j}{1-\lambda} d(f^i u, f^{i+1}u) \\ &\leq \frac{1}{1-\lambda} d(f^i u, f^{i+1}u) \\ &\leq \frac{\lambda^i}{1-\lambda} d(u, fu). \end{aligned}$$

This shows that, $\{f^i u\}$ is Cauchy and converges to some $\xi \in X$. By letting $j \to \infty$ in the above inequality, we have

$$d(f^i u, \xi) \leq \frac{\lambda^i}{1-\lambda} d(u, fu) \text{ for } i \geq 1.$$

Suppose f is orbitally continuous at ξ. Then $f^i u \to \xi$ implies $f^{i+1}u \to f\xi$. This shows that $\xi = f\xi$. Suppose (*) holds for any $x \in \overline{O}(u)$. Then

$$d(f^{i+l}u, f\xi) \le \lambda\, d(f^i u, \xi)$$

for any i. This implies $\xi = f\xi$. ∎

In [16], we showed that many works of Pal-Maiti, Rhoades, Wong, Ćirić, Fisher, Jaggi, and Taskovitz are consequences of Theorem 7.1.

8. Suzuki [26] in 2008 and Related Works

From Theorem P, we have the following simplified form:

Theorem 8.1. *Let (X, q) be a quasi-metric space and $T : X \to X$ be a continuous RHR map, that is,*

$$q(Tx, T^2x) \le \alpha\, q(x, Tx) \text{ for every } x \in X,$$

for $0 < \alpha < 1$. If X is T-orbitally complete, then

(i) for each $x \in X$, there exists $x_0 \in X$ satisfying $\lim_{n\to\infty} T^n(x) = x_0$, and

(ii) x_0 is a fixed point of T.

The following consequence of Theorem 8.1 generalizes the Banach contraction principle as shown by Suzuki [26, Theorem] in 2008:

Theorem 8.2 (Suzuki)**.** *Let (X, d) be a complete metric space and T be a mapping on X. Define a nonincreasing function θ from $[0, 1)$ onto $(1/2, 1]$ by*

$$\theta(r) = \begin{cases} 1 & \text{if } 0 \le r \le (\sqrt{5}-1)/2, \\ (1-r)r^{-2} & \text{if } (\sqrt{5}-1)/2 \le r \le 2^{-1/2}, \\ (1+r)^{-1} & \text{if } 2^{-1/2} \le r < 1. \end{cases}$$

Assume there exists $r \in [0, 1)$ such that

$$\theta(r)d(x, Tx) \le d(x, y) \text{ implies } d(Tx, Ty) \le r\, d(x, y)$$

for all $x, y \in X$. Then there exists a unique fixed point z of T. Moreover $\lim_n T^n x = z$ for all $x \in X$.

This means T is an RHR map and a Picard operator, hence Theorem 8.2 for quasi-metric spaces follows from Theorem 8.1. Note that uniqueness is clear. From now on, T is called the Suzuki type as its many followers used it.

The following consequence of Theorem 8.1 is motivated by Suzuki [25, Corollary 1]:

Corollary 8.3. *Let T be a selfmap of a complete quasi-metric space (M, q).*

(1) For any function θ from $[0, 1)$ onto $[0, 1]$, there exists $r \in [0, 1)$ such that

$$\theta(r)q(x, Tx) \le q(x, y) \quad \text{implies} \quad q(Tx, Ty) \le r\, q(x, y)$$

for all $x, y \in M$. Then there exists a unique fixed point z of T. Moreover $\lim_n T^n x = z$ for all $x \in M$.

(2) *There exists $r \in (0,1)$ such that every selfmap T on M satisfying the following has a fixed point:*

$$\frac{1}{10,000} q(x, Tx) \leq q(x, y) \quad \textit{implies} \quad q(Tx, Ty) \leq r\, q(x, y)$$

for all $x, y \in M$.

Proof. For $y = Tx$, we have $\delta(Tx, T^2x) \leq r\,\delta(x, Tx)$ for all $x \in M$. Then we can apply Theorem 6.1 For the uniqueness, if we have two different fixed points x, y, then we have the contradiction:

$$\delta(x, y) = \delta(Tx, Ty) \leq r\,\delta(x, y).$$

∎

Example 8.4. Pant et al. [15, Theorem 2.1] in 2021 considered the map $f : X \to X$ satisfying

$$d(fx, fy) \leq [\varphi(x) - \varphi(fx)] + [\varphi(y) - \varphi(fy)]$$

for all $x, y \in X$, where (X, d) is a complete metric space.

For $y = fx$, their condition reduces to

$$\delta(x, f^2x) \leq \varphi(x) - \varphi(fx)), \quad x \in X.$$

Hence our argument in Remark 3.2 (3) works for f.

Example 8.5. There have appeared too many variants of the Suzuki type maps. We state only one of them motivated from Chandra-Joshi-Joshi [4] in 2022.

Let (M, δ) be a quasi-metric space, and $T : M \to M$. Then for all $x, y \in X$, we denote

$$m(Tx, Ty) = a\,\delta(x, y) + b \max\{\delta(x, Tx), \delta(y, Ty)\} + c[\delta(x, Ty) + \delta(y, Tx)],$$

where a, b and c are non-negative reals such that $a + b + 2c = r$ with $r \in [0, 1)$. Now, we consider the following generalized contractive condition

$$\theta(r) \min\{q(x, Tx), q(x, Ty)\} \leq q(x, y) \text{ implies } q(Tx, Ty) \leq m(Tx, Ty),$$

where $\theta : [0, 1) \to (1/2, 1]$ is as defined in Theorem 8.2. It is remarkable that this condition is a generalization of the condition (22) and several other conditions mentioned in the Transaction Paper of Billy E. Rhoades [24].

The following improves Chandra-Joshi-Joshi [4, Theorem 4]:

Corollary 8.6. *Let (M, q) be a complete quasi-metric space, and $T : M \to M$. Assume that there exists $r \in [0, 1)$ such that the requirement (1) of Corollary 8.3 is satisfied for each $x, y \in M$. Then T has a unique fixed point $z \in M$. Moreover, $\lim_{n\to\infty} T^n x = z$ for all $x \in M$.*

As in the original proof in [4], we have

$$q(Tx, T^2x) \leq r\, q(x, Tx), \quad \forall x \in M.$$

Therefore the conclusion follows from Theorem 8.1.

Remark 8.7. Our Corollary 8.6 and Chandra-Joshi-Joshi [4, Theorem 4] follow from Theorem 8.1 In fact, the beginning few lines of its proof in [4] shows

$$d(Tx, T^2x) \leq r\, d(x, Tx) \ \ \forall\, x \in X.$$

Therefore Theorem 8.1 works.

9. Miñana and Valero [13] in 2019

Miñana and Valero [13] stated: We show that the existence of fixed points for the most part in the aforesaid G-metric fixed point results is guaranteed by a very general celebrated result by Park, even when the G-contractive condition is reduced to a quasi-metric one which is not considered as a contractive condition in any celebrated fixed point result.

Taking into account the exposed facts about G-metric spaces and quasi-metric spaces, we are able to show that most fixed point results obtained in G-metric spaces can be deduced from a fixed point result stated in quasi-metric spaces obtained by Park in [16]. To this end, let us recall such a result.

Theorem 9.1. *Let (X, τ) be a topological space, let $d : X \times X \to [0, \infty[$ be a continuous mapping, such that $d(x, y) = 0 \iff x = y$, and let $f : X \to X$ be a mapping. Suppose that there exist x, $x_0 \in X$, such that the following conditions hold:*

1. $\lim_{n\to\infty} d(f^n(x_0), f^{n+1}(x_0)) = 0$;
2. $(f^n(x_0))_{n\in\mathbb{N}}$ *converges to x with respect to τ;*
3. *f is orbitally continuous at x with respect to τ.*

Then $x \in Fix(f) = \{y \in X : f(y) = y\}$.

It must be stressed that Park's original version of the preceding result was stated for lower semicontinuous mappings d. However, we have focused our attention on continuous ones, because it is enough for our announced purpose.

10. Park [18] in 2023

Here we add new information related to the above theorems: Kirk-Saliga [12] in 2001 and Chen-Cho-Yang [5] in 2002 introduced the following concept: We say that $\varphi : M \to \mathbb{R}$ for a metric space M is *lower semicontinuous from above* if given any sequence $\{x_n\}$ in M, the conditions $\lim_n x_n = x$ and $\{\varphi(x_n)\} \downarrow r \Rightarrow \varphi(x) \leq r$.

This concept can be applied to improve many facts, for example, Theorems 7.1 and 8.1 in the present paper.

11. Fierro and Pizzaro [7] in 2023

In this note, Fierro and Pizzaro [7] prove a fixed point existence theorem for set-valued functions by extending the usual Banach orbital condition concept for single valued mappings. As they show, this result applies to various types of set-valued contractions existing in the literature.

Let (X, d) be a complete metric space. We denote by $B(X)$ the family of all bounded sets of X and by $C(X)$ the family of all nonempty and closed subsets of X. In what follows, $CB(X) = C(X) \cap B(X)$ and $B(A, r) = \bigcup_{a \in A} B(a, r)$, for each $A \in B(X)$ and $r > 0$.

Let $T : X \to CB(X)$ be a multimap, $x \in X$ and B be a subset of X. We denote $T(B) = \bigcup_{y \in B} Ty$ and for each $n \in \mathbb{N}$, $T^{n+1}x = T(T^n x)$, with $T^0 x = \{x\}$. The *orbit* of x under T is defined as

$$O(x, T) = \bigcup_{n=0}^{\infty} T^n x.$$

Let $x_0 \in X$. A function $G : X \to \mathbb{R}$ is said to be (x_0, T)-*orbitally lower semicontinuous* at $x^* \in X$, if for any sequence $\{x_n\}_{n \in \mathbb{N}}$ in $O_T(x_0)$ converging to x^*, we have $G(x^*) \leq \liminf G(x_n)$. In the sequel, $G^T : X \to \mathbb{R}$ stands for the function defined as $G^T(x) = d(x, Tx)$ and for $\xi : X \to X$, we denote $G^\xi = G^{\{\xi\}}$.

Given a multimap $T : X \to CB(X)$, $x_0 \in X$, and $k \in [0, 1)$, we say T satisfies the *multivalued Banach orbital* (MBO) condition at x_0 with constant k, whenever for all $x \in O_T(x_0)$, $\inf_{y \in Tx} d(y, Ty) \leq k\, d(x, Tx)$, and that, T satisfies the *strong multivalued Banach orbital* (SMBO) condition at x_0 with constant k, whenever for all $x \in O_T(x_0)$, $\sup_{y \in Tx} d(y, Ty) \leq k\, d(x, Tx)$.

Note that *MBO conditions are more closer to the RHR maps than the Banach contractions*.

The following improves the main result of Fierro-Pizzaro [7, Theorem 3.1]:

Theorem 11.1. *Let (X, d) be a quasi-metric space and $T : X \to CB(X)$ be a multimap satisfying the MBO condition at $x_0 \in X$ with constant k. Then, there exist $x^* \in X$ and a sequence $\{x_n\}_{n \in \mathbb{N}}$ converging to x^* such that, for all $n \in \mathbb{N}$, $x_{n+1} \in Tx_n$, and the following two conditions hold:*

(i) $d(x_n, Tx_n) \leq d(x_n, x_{n+1}) \leq k^n d(x_0, Tx_0)$ and

(ii) $d(x^*, Tx_n) \leq \frac{k^n}{1-k} d(x_0, Tx_0)$, for all $n \in \mathbb{N}$.

Moreover, the following conditions are equivalent:

(iii) $x^* \in Tx^*$,

(iv) G_T is (x_0, T)-orbitally lower semicontinuous [from above] at x^*, and

(v) the function $h : X \to \mathbb{R}$, defined by $h(x) = d(x, Tx)$, is lower semicontinuous [from above] at x^*.

Remark 11.2.

(1) the origin of this theorem is Park-Rhoades [23] and others;

(2) this holds for quasi-metric spaces; and

(3) the lower semicontinuity can be extended to the one from right.

(4) The CB(X) can be extended to C(X) as in Covitz-Nadler [6].

12. Epilogue

From 2022, we have applied our 2023 Metatheorem and its old versions to nearly one hundred theorems and obtained almost one thousand new theorems for the mathematical community. While we were seeking on applications of Metatheorem, we found that the Metatheorem can be very useful to the fixed point theory.

In metric fixed point theory, there have been appeared hundreds of contractive type conditions and almost one thousand artificial metric type spaces. Recently, the present author found that many contractive type conditions are the weak contractions or the Rus-Hicks-Rhoades type maps. Moreover, even the very popular Banach contraction principle is inadequately stated. Further, we found many incorrect statements and unnecessarily lengthy proofs of them for artificial metric type spaces.

However, we are not going to make any new contractive conditions or any new artificial spaces. Our main aim to study in metric fixed point theory is to improve known facts and to correct inadequate statements.

Competing Interests

The author declares that there are no competing interests.

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