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Zero Point Approximation Schemes for Monotone Vector Fields on Complete Geodesic Spaces

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ABSTRACT

In this paper, we obtain two convergence theorems to approximate zero points of a monotone vector field defined on a complete geodesic space with curvature bounded above.

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1. Introduction

In fixed point approximation theory, we have the following two iterative schemes: For a given mapping T, points y_1, z_1, u, and a sequence $\{a_n\}$, we generate sequences $\{y_n\}$ and $\{z_n\}$ by

$$y_{n+1} = a_n y_n + (1 - a_n)Ty_n;$$
$$z_{n+1} = a_n u + (1 - a_n)Tz_n$$

for $n \in \mathbb{N}$. We call a method such as $\{y_n\}$ the Mann type iteration [22], and a method such as $\{z_n\}$ the Halpern type iteration [5, 30]. In appropriate settings, such a sequence $\{y_n\}$ converges weakly to a fixed point of T, and such a sequence $\{z_n\}$ converges strongly to the closest fixed point to u. These iterations are studied in the setting of Hilbert spaces, and after that, they are generalised to geodesic spaces; see [6, 25] for instance.

On the other hand, such iterations are applied to find a zero point of a maximally monotone operator on Hilbert spaces. The proximal point algorithm is a zero point approximation scheme proved by Rockafellar [24] in 1976. Later, Kamimura and Takahashi [11] proposed two modified proximal point algorithms related to Mann's and Halpern's iterations. Even after that, these schemes are generalised to Banach spaces, and many researchers introduced several other iterative methods; see [10, 12, 27] for instance.

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As a typical application of zero point approximations, there are convex minimisation problems. That is, for a given convex function f on a set S having some convexity structure, we consider a problem to find a point $x \in S$ such that

$$f(x) = \inf f(S).$$

Recently, such a problem has been discussed in the setting of geodesic spaces. Particularly, CAT(κ) spaces are reasonable geodesic spaces, and they have effective properties to investigate convex minimisation problems. In the 1990s, Jost [7] and Mayer [23] introduced resolvent operators for convex functions in complete CAT(0) spaces. Using this resolvent, Bačák [1], Kimura and Kohsaka [15] proved approximation theorems with the canonical and two modified proximal point algorithms. Later, Kimura and Kohsaka [16, 17] investigated convex minimisation problems on CAT(1) spaces, and proved approximation theorems using resolvents dedicated to the setting of spherical surfaces. For recent related results, see [9, 13] for instance.

The notion of monotone operators has been generalised to the framework of geodesic spaces. For instance, Chaipunya, Kohsaka and Kumam [3] dealt with monotone vector fields on a CAT(0) space using tangent spaces, and the author proposed a class of monotone vector fields on a CAT(κ) space; see [28]. Furthermore, we obtained the following result:

Theorem 1.1 (Sudo [29])**.** *Let M be an admissible complete* CAT(κ) *space and A a resolvably monotone vector field on M. Let $\{r_n\}$ be a sequence of positive real numbers whose sum is divergent to ∞. For a given initial point $x_1 \in M$, generate a sequence $\{x_n\}$ of M by*

$$x_{n+1} = J_{r_n A} x_n$$

for $n \in \mathbb{N}$, where $J_{r_n A}$ is the resolvent operator of $r_n A$. Then, the following hold:

(i) *The resolvably monotone vector field A has a zero point if and only if the generated sequence $\{x_n\}$ is κ-bounded;*

(ii) *if A has a zero point and $\inf_{k \in \mathbb{N}} r_k > 0$, then the generated sequence $\{x_n\}$ Δ-converges to a zero point of A, which equals to*

$$\lim_{n\to\infty} P_{\operatorname{Zero} A} x_n.$$

Motivated by these results, in this paper, we consider the following: Let M be an admissible complete CAT(κ) space and A a resolvably monotone vector field on M. For points $y_1, z_1, u \in M$, a sequence $\{a_n\}$ of $[0,1]$ and a positive sequence $\{r_n\}$, generate sequences $\{y_n\}$ and $\{z_n\}$ by

$$\begin{aligned} y_{n+1} &= a_n y_n \oplus (1-a_n) J_{r_n A} y_n; \\ z_{n+1} &= a_n u \oplus (1-a_n) J_{r_n A} z_n \end{aligned}$$

for $n \in \mathbb{N}$, where $J_{r_n A}$ is the resolvent operator of $r_n A$. The main results of this work are about the convergence to zero points of these sequences.

2. Preliminaries

For a metric space (M, d) and $x, y \in M$, we call a mapping γ_{xy} from $[0, d(x, y)]$ to M a geodesic from x to y if $\gamma_{xy}(0) = x$, $\gamma_{xy}(d(x, y)) = y$ and

$$d(\gamma_{xy}(s), \gamma_{xy}(s')) = |s - s'|$$

for $s, s' \in [0, d(x, y)]$. Further, for $r \in]0, \infty]$, we call M a uniquely r-geodesic space if for any points $x, y \in M$ with $d(x, y) < r$, there exists a unique geodesic γ_{xy} from x to y. In this case, for $t \in [0, 1]$, we can define convex combination of x and y with a ratio t by

$$tx \oplus (1 - t)y = \gamma_{xy}((1 - t)d(x, y)).$$

We next define a function c_κ introduced by Kajimura and Kimura [8]. We define a real-valued function c_κ on $\mathbb{R}$ by

$$c_\kappa(a) = \frac{1}{2}a^2 + \sum_{n=2}^{\infty} \frac{(-\kappa)^{n-1} a^{2n}}{(2n)!} = \begin{cases} \dfrac{1 - \cos(\sqrt{\kappa}a)}{\kappa} & (\kappa > 0); \\ \dfrac{1}{2}a^2 & (\kappa = 0); \\ \dfrac{\cosh(\sqrt{-\kappa}a) - 1}{-\kappa} & (\kappa < 0) \end{cases}$$

for $a \in \mathbb{R}$. From the definition, for $a \in \mathbb{R}$, we have

$$c'_\kappa(a) = a + \sum_{n=2}^{\infty} \frac{(-\kappa)^{n-1} a^{2n-1}}{(2n-1)!} = \begin{cases} \dfrac{\sin(\sqrt{\kappa}a)}{\sqrt{\kappa}} & (\kappa > 0); \\ a & (\kappa = 0); \\ \dfrac{\sinh(\sqrt{-\kappa}a)}{\sqrt{-\kappa}} & (\kappa < 0) \end{cases}$$

and

$$c''_\kappa(a) = 1 + \sum_{n=2}^{\infty} \frac{(-\kappa)^{n-1} a^{2n-2}}{(2n-2)!} = \begin{cases} \cos(\sqrt{\kappa}a) & (\kappa > 0); \\ 1 & (\kappa = 0); \\ \cosh(\sqrt{-\kappa}a) & (\kappa < 0). \end{cases}$$

Fix $a, b \in \mathbb{R}$ arbitrarily. Then, we know the following formulae:

$$\begin{aligned} c''_\kappa(a) + \kappa c_\kappa(a) &= 1; \\ c''_\kappa(a)^2 + \kappa c'_\kappa(a)^2 &= 1. \end{aligned}$$

Additionally,

$$\begin{aligned} c'_\kappa(a + b) &= c'_\kappa(a)c''_\kappa(b) + c'_\kappa(b)c''_\kappa(a); \\ c'_\kappa(a - b) &= c'_\kappa(a)c''_\kappa(b) - c'_\kappa(b)c''_\kappa(a); \\ c''_\kappa(a + b) &= c''_\kappa(a)c''_\kappa(b) - \kappa c'_\kappa(a)c'_\kappa(b); \\ c''_\kappa(a - b) &= c''_\kappa(a)c''_\kappa(b) + \kappa c'_\kappa(a)c'_\kappa(b), \end{aligned}$$

and therefore

$$\begin{aligned}
c_\kappa'(2a) &= 2c_\kappa'(a)c_\kappa''(a);\\
c_\kappa''(2a) &= c_\kappa''(a)^2 - \kappa c_\kappa'(a)^2 = 2c_\kappa''(a)^2 - 1 = 1 - 2\kappa c_\kappa'(a)^2.
\end{aligned}$$

Moreover,

$$\begin{aligned}
\kappa c_\kappa'\left(\frac{a}{2}\right)^2 &= \frac{1 - c_\kappa''(a)}{2};\\
c_\kappa''\left(\frac{a}{2}\right)^2 &= \frac{c_\kappa''(a) + 1}{2}.
\end{aligned}$$

Further, we obtain the following formulae:

$$\begin{aligned}
c_\kappa'(a) + c_\kappa'(b) &= 2c_\kappa'\left(\frac{a+b}{2}\right)c_\kappa''\left(\frac{a-b}{2}\right);\\
c_\kappa'(a) - c_\kappa'(b) &= 2c_\kappa''\left(\frac{a+b}{2}\right)c_\kappa'\left(\frac{a-b}{2}\right);\\
c_\kappa''(a) + c_\kappa''(b) &= 2c_\kappa''\left(\frac{a+b}{2}\right)c_\kappa''\left(\frac{a-b}{2}\right);\\
c_\kappa''(a) - c_\kappa''(b) &= -2\kappa c_\kappa'\left(\frac{a+b}{2}\right)c_\kappa'\left(\frac{a-b}{2}\right)
\end{aligned}$$

and

$$\begin{aligned}
c_\kappa'(a)c_\kappa''(b) &= \frac{1}{2}\left(c_\kappa'(a+b) + c_\kappa'(a-b)\right);\\
c_\kappa''(a)c_\kappa'(b) &= \frac{1}{2}\left(c_\kappa'(a+b) - c_\kappa'(a-b)\right);\\
-\kappa c_\kappa'(a)c_\kappa'(b) &= \frac{1}{2}\left(c_\kappa''(a+b) - c_\kappa''(a-b)\right);\\
c_\kappa''(a)c_\kappa''(b) &= \frac{1}{2}\left(c_\kappa''(a+b) + c_\kappa''(a-b)\right).
\end{aligned}$$

We next define CAT(κ) spaces. Let M be a metric space. For $\kappa \in \mathbb{R}$, we define a real-valued function ϕ_κ on M^2 by

$$\phi_\kappa(x, y) = c_\kappa(d(x, y)) = \begin{cases} \dfrac{1 - \cos(\sqrt{\kappa}d(x, y))}{\sqrt{\kappa}} & (\kappa > 0);\\ \dfrac{1}{2}d(x, y)^2 & (\kappa = 0);\\ \dfrac{\cosh(\sqrt{-\kappa}d(x, y)) - 1}{\sqrt{-\kappa}} & (\kappa < 0) \end{cases}$$

for $x, y \in M$. Letting

$$D_\kappa = \begin{cases} \dfrac{\pi}{\sqrt{\kappa}} & (\kappa > 0);\\ \infty & (\kappa \leq 0), \end{cases}$$

we obtain the following:

- For $x, y \in M$, $\phi(x, y) \geq 0$;
- for $x \in M$, $\phi_\kappa(x, x) = 0$;
- if $\phi_\kappa(x, y) = 0$ for $x, y \in M$ with $d(x, y) < 2D_\kappa$, then $x = y$;
- for $x, y \in M$, $\phi_\kappa(x, y) = \phi_\kappa(y, x)$.

Further, for $t \in [0, 1]$ and $l \in [0, D_\kappa[$, let

$$(t)_l^\kappa = \begin{cases} \dfrac{c_\kappa'(tl)}{c_\kappa'(l)} & (l \neq 0); \\ t & (l = 0). \end{cases}$$

Now, we define CAT(κ) spaces. In the canonical definition, we employ geodesic triangles and their comparison triangles on the model spaces. Actually, D_κ is the space diameter of the standard model spaces. In this paper, we adopt an equivalent condition to the familiar definition of CAT(κ) spaces as follows. For a uniquely D_κ-geodesic space M, we call it a CAT(κ) space if

$$\begin{aligned} &\phi_\kappa(tx \oplus (1-t)y, z) \\ &\leq (t)_l^\kappa \phi_\kappa(x, z) + (1-t)_l^\kappa \phi_\kappa(y, z) \\ &\quad - (t)_l^\kappa \phi_\kappa(x, tx \oplus (1-t)y) - (1-t)_l^\kappa \phi_\kappa(y, tx \oplus (1-t)y) \end{aligned} \tag{2.1}$$

for $x, y, z \in M$ with $d(y, z) + d(z, x) + l < 2D_\kappa$ and $t \in [0, 1]$, where $l = d(x, y)$. We call the inequality (2.1) Stewart's inequality on a CAT(κ) space M.

Theorem 2.1 (Kimura–Kohsaka [14]). *In a* CAT(κ) *space M,*

$$\phi_\kappa(tx \oplus (1-t)y, z) \leq t\phi_\kappa(x, z) + (1-t)\phi_\kappa(y, z)$$

for $x, y, z \in M$ with $d(x, z) < D_\kappa/2$ and $d(y, z) < D_\kappa/2$, and for $t \in [0, 1]$.

For more details about this definition of CAT(κ) spaces and Stewart's inequality, see [19]. From Stewart's inequality, we obtain the following other types of inequalities.

Theorem 2.2. *Let M be a* CAT(κ) *space. Then,*

$$\phi_\kappa(tx \oplus (1-t)y, z) \leq (t)_l^\kappa \phi_\kappa(x, z) + (1-t)_l^\kappa \phi_\kappa(y, z) - \frac{(t)_{l/2}^\kappa (1-t)_{l/2}^\kappa \phi_\kappa(x, y)}{c_\kappa''(l/2)}$$

for $x, y, z \in M$ with $d(y, z) + d(z, x) + l < 2D_\kappa$ and $t \in [0, 1]$, where $l = d(x, y)$.

Proof. Let $l \in [0, D_\kappa[$ and $t \in [0, 1]$. It is sufficient to show that

$$(t)_l^\kappa c_\kappa((1-t)l) + (1-t)_l^\kappa c_\kappa(tl) = \frac{(t)_{l/2}^\kappa (1-t)_{l/2}^\kappa c_\kappa(l)}{c_\kappa''(l/2)}.$$

If $l = 0$ or $\kappa = 0$, then it is obvious. We might suppose that $l \neq 0$ and $\kappa \neq 0$. Then,

$$\begin{aligned} &(t)_l^\kappa c_\kappa((1-t)l) + (1-t)_l^\kappa c_\kappa(tl) \\ &= \frac{c_\kappa'(tl) c_\kappa((1-t)l) + c_\kappa'((1-t)l) c_\kappa(tl)}{c_\kappa'(l)} \end{aligned}$$

$$= \frac{c'_\kappa(tl)c + c'_\kappa((1-t)l) - (c'_\kappa(tl)c''_\kappa((1-t)l) + c'_\kappa((1-t)l)c''_\kappa(tl))}{\kappa c'_\kappa(l)}$$
$$= \frac{c'_\kappa(tl)c + c'_\kappa((1-t)l) - c'_\kappa(l)}{\kappa c'_\kappa(l)}.$$

By the way,

$$\begin{aligned} c'_\kappa(tl)c + c'_\kappa((1-t)l) - c'_\kappa(l) &= 2c'_\kappa\left(\frac{l}{2}\right) c''_\kappa\left(\frac{(2t-1)l}{2}\right) - 2c'_\kappa\left(\frac{l}{2}\right) c''_\kappa\left(\frac{l}{2}\right) \\ &= 2c'_\kappa\left(\frac{l}{2}\right)\left(c''_\kappa\left(\frac{(2t-1)l}{2}\right) - c''_\kappa\left(\frac{l}{2}\right)\right) \\ &= -4\kappa c'_\kappa\left(\frac{l}{2}\right) c'_\kappa\left(\frac{tl}{2}\right) c'_\kappa\left(\frac{(t-1)l}{2}\right) \\ &= 4\kappa c'_\kappa\left(\frac{l}{2}\right) c'_\kappa\left(\frac{tl}{2}\right) c'_\kappa\left(\frac{(1-t)l}{2}\right). \end{aligned}$$

Thus,

$$\begin{aligned} (t)^\kappa_l c_\kappa((1-t)l) + (1-t)^\kappa_l c_\kappa(tl) &= \frac{c'_\kappa(tl)c + c'_\kappa((1-t)l) - c'_\kappa(l)}{\kappa c'_\kappa(l)} \\ &= \frac{4c'_\kappa(l/2)c'_\kappa(tl/2)c'_\kappa((1-t)l/2)}{c'_\kappa(l)} \\ &= \frac{2c'_\kappa(tl/2)c'_\kappa((1-t)l/2)}{c''_\kappa(l/2)} \\ &= \frac{c'_\kappa(tl/2)}{c'_\kappa(l/2)} \cdot \frac{c'_\kappa((1-t)l/2)}{c'_\kappa(l/2)} \cdot \frac{2c'_\kappa(l/2)^2}{c''_\kappa(l/2)} \\ &= \frac{(t)^\kappa_{l/2}(1-t)^\kappa_{l/2} c_\kappa(l)}{c''_\kappa(l/2)}. \end{aligned}$$

Consequently, from Stewart's inequality of M, we have

$$\phi_\kappa(tx \oplus (1-t)y, z) \le (t)^\kappa_l \phi_\kappa(x,z) + (1-t)^\kappa_l \phi_\kappa(y,z) - \frac{(t)^\kappa_{l/2}(1-t)^\kappa_{l/2}\phi_\kappa(x,y)}{c''_\kappa(l/2)}$$

for $x, y, z \in M$ with $d(y,z) + d(z,x) + l < 2D_\kappa$ and $t \in [0,1]$, where $l = d(x,y)$. ∎

Lemma 2.3. *Let M be a* CAT(κ) *space. For $x, y, z \in M$ with*

$$d(y,z) + d(z,x) + d(x,y) < 2D_\kappa$$

and $t \in]0,1[$, let $l = d(x,y)$ and $b = 1 - (1-t)^\kappa_l$. Then,

$$\phi_\kappa(tx \oplus (1-t)y, z) \le (1-b)\phi_\kappa(y,z) + b \cdot \frac{c''_\kappa(tl/2)\phi_\kappa(x,z) - (1-t)^\kappa_{l/2}\phi_\kappa(x,y)}{c''_\kappa(l - tl/2)}.$$

Proof. From Theorem 2.2, we have

$$\begin{aligned} \phi_\kappa(tx \oplus (1-t)y, z) &\le (t)^\kappa_l \phi_\kappa(x,z) + (1-t)^\kappa_l \phi_\kappa(y,z) - \frac{(t)^\kappa_{l/2}(1-t)^\kappa_{l/2}\phi_\kappa(x,y)}{c''_\kappa(l/2)} \\ &= (1-b)\phi_\kappa(y,z) + (t)^\kappa_l \phi_\kappa(x,z) - \frac{(t)^\kappa_{l/2}(1-t)^\kappa_{l/2}\phi_\kappa(x,y)}{c''_\kappa(l/2)}. \end{aligned}$$

Let

$$A=\frac{(t)_l^\kappa\phi_\kappa(x,z)}{b}-\frac{(t)_{l/2}^\kappa(1-t)_{l/2}^\kappa\phi_\kappa(x,y)}{bc_\kappa''(l/2)},\tag{2.2}$$

and then

$$\phi_\kappa(tx\oplus(1-t)y,z)\leq(1-b)\phi_\kappa(y,z)+bA.$$

Thus, it is sufficient to show that

$$A=\frac{c_\kappa''(tl/2)\phi_\kappa(x,z)-(1-t)_{l/2}^\kappa\phi_\kappa(x,y)}{c_\kappa''(l-tl/2)}.$$

If $l=0$, then we immediately obtain this identity. We might assume that $l\neq 0$. We know from the equation (2.2) that

$$A=\frac{(t)_l^\kappa c_\kappa''(l/2)\phi_\kappa(x,z)-(t)_{l/2}^\kappa(1-t)_{l/2}^\kappa\phi_\kappa(x,y)}{bc_\kappa''(l/2)}.$$

By the way,

$$(t)_l^\kappa c_\kappa''\left(\frac{l}{2}\right)=\frac{c_\kappa'(tl)}{c_\kappa'(l)}\cdot c_\kappa''\left(\frac{l}{2}\right)=\frac{c_\kappa'(tl)}{2c_\kappa'(l/2)}=(t)_{l/2}^\kappa\cdot\frac{c_\kappa'(tl)}{2c_\kappa'(tl/2)}=(t)_{l/2}^\kappa c_\kappa''\left(\frac{tl}{2}\right).$$

Thus,

$$A=\frac{(t)_{l/2}^\kappa}{bc_\kappa''(l/2)}\left(c_\kappa''\left(\frac{tl}{2}\right)\phi_\kappa(x,z)-(1-t)_{l/2}^\kappa\phi_\kappa(x,y)\right).$$

Moreover,

$$\begin{aligned}bc_\kappa''\left(\frac{l}{2}\right)&=\frac{c_\kappa'(l)-c_\kappa'(l-tl)}{c_\kappa'(l)}\cdot c_\kappa''\left(\frac{l}{2}\right)=\frac{c_\kappa'(l)-c_\kappa'(l-tl)}{2c_\kappa'(l/2)}\\&=\frac{1}{c_\kappa'(l/2)}\cdot c_\kappa''\left(l-\frac{tl}{2}\right)c_\kappa'\left(\frac{tl}{2}\right)=(t)_{l/2}^\kappa c_\kappa''\left(l-\frac{tl}{2}\right).\end{aligned}$$

Consequently,

$$A=\frac{c_\kappa''(tl/2)\phi_\kappa(x,z)-(1-t)_{l/2}^\kappa\phi_\kappa(x,y)}{c_\kappa''(l-tl/2)},$$

which completes the proof. ∎

Let M be a CAT(κ) space. We say that M is admissible [16] if

$$d(x,y)<\frac{D_\kappa}{2}$$

for all $x,y\in M$. If $\kappa\leq 0$, then CAT(κ) spaces are always admissible.

Suppose that M is an admissible complete CAT(κ) space and C is a nonempty closed convex subset of M. Then, for $x\in M$, we can find a unique closest point $y_x\in C$ to x, that is,

$$d(x,y_x)=\inf_{y\in C}d(x,y);$$

see [1, 4]. We define a mapping P_C by $P_Cx=y_x$ for $x\in M$, and call it the metric projection onto C. Then, we have the following result:

Theorem 2.4 (Sudo [29]). *Let M be an admissible complete* CAT(κ) *space and C a nonempty closed convex subset of M. Let $\{x_n\}$ be a sequence of M such that*

$$d(x_{n+1}, p) \leq d(x_n, p)$$

for any $p \in C$ and $n \in \mathbb{N}$. Then, a sequence $\{P_C x_n\}$ converges to a point in C.

3. Tangent Spaces and Monotone Vector Fields

In what follows, we define tangent spaces on a CAT(κ) space. For more details, see [2, 3, 20].

Let M be an admissible CAT(κ) space. For $p, x, y \in M$, we define the Alexandrov angle A_p at p by

$$A_p(x, y) = \lim_{t \to 0+} \arccos\left(1 - \frac{d(\gamma_{px}(t), \gamma_{py}(t))^2}{2t^2}\right) \in [0, \pi]$$

if $p \neq x$ and $p \neq y$; $A_p(x, p) = A_p(p, x) = \pi/2$ if $p \neq x$; $A_p(p, p) = 0$. For more details, refer to [2, Proposition 1.14 in Chapter I.1 and Proposition 3.1 in Chapter II.3] for instance.

Let M be an admissible CAT(κ) space, and let $p \in M$. We define an equivalence relation $\sim_p$ on M by $x \sim_p y$ if

$$A_p(x, y) = 0.$$

For $x \in M$, we denote the direction from p to x by

$$[x]_p = \{z \in M \mid x \sim_p z\} = \{z \in M \mid A_p(x, z) = 0\}.$$

Notice that $[p]_p$ consists of exactly one point p. Further, we define the direction space $D_p M$ from p by

$$D_p M = M/\sim_p = \{[x]_p \mid x \in M\}.$$

Then, $(D_p M, A_p)$ is a metric space, where the distance A_p is well defined by

$$A_p([x]_p, [y]_p) = A_p(x, y)$$

for $[x]_p, [y]_p \in D_p M$. Additionally, we define an indicator function i_p from $D_p M$ into $\{0, 1\}$ by

$$i_p([x]_p) = \begin{cases} 0 & ([x]_p = [p]_p); \\ 1 & ([x]_p \neq [p]_p) \end{cases}$$

for $[x]_p \in D_p M$. We define an equivalence relation $\simeq_p$ on a Cartesian product

$$[0, \infty[\times D_p M$$

by $(r_1, [x]_p) \simeq_p (r_2, [y]_p)$ if one of the following conditions is satisfied:

- $r_1 i_p([x]_p) = r_2 i_p([y]_p) = 0$;
- $r_1 i_p([x]_p) = r_2 i_p([y]_p) > 0$ and $[x]_p = [y]_p$.

Put

$$T_pM = ([0, \infty[\times D_pM)/\simeq_p.$$

We use a notation $r[x]_p$ as $[(r, [x]_p)]_{\simeq_p} \in T_pM$, where $[(r, [x]_p)]_{\simeq_p}$ is an equivalence class of $(r, [x]_p)$ by $\simeq_p$. In particular, we denote $0[p]_p$ by 0_p. Define a bifunction d_p on T_pM by

$$d_p(r[x]_p, s[y]_p) = \sqrt{r^2 i_p([x]_p) + s^2 i_p([y]_p) - 2rs i_p([x]_p) i_p([y]_p) \cos A_p(x, y)}$$

for $r[x]_p, s[y]_p \in T_pM$, and then (T_pM, d_p) is a metric space, that is, d_p is a distance of T_pM. We call this metric space the tangent space on M at p. Let

$$TM = \bigsqcup_{p \in M} T_pM = \bigcup_{p \in M} \{(v_p, p) \mid v_p \in T_pM\},$$

and call it the tangent bundle of M. For more details about tangent spaces on geodesic spaces, see [2, 3, 20].

Let M be an admissible CAT(κ) space, and let $p \in M$. For $v_p = r[v]_p \in T_pM$ and $t \in [0, \infty[$, we denote a point $(tr)[v]_p$ of T_pM by tv_p. In particular, for $t > 0$, we denote a point $(r/t)[v]_p$ of T_pM by v_p/t. We define a canonical logarithmic mapping $\log_p$ from M to T_pM by

$$\log_p x = d(p, x)[x]_p \in T_pM$$

for $x \in M$. Similarly, we define another logarithmic mapping $\log_{\kappa,p}$ by

$$\log_{\kappa,p} x = c'_\kappa(d(p, x))[x]_p \in T_pM$$

for $x \in M$. Note that

$$\frac{d(p, x)}{c'_\kappa(d(p, x))} \log_{\kappa,p} x = \log_p x$$

for $x \in M$ with $p \neq x$. We further define a function g_p by

$$g_p(u_p, v_p) = \frac{d_p(u_p, 0_p)^2 + d_p(v_p, 0_p)^2 - d_p(u_p, v_p)^2}{2}$$

for $u_p, v_p \in T_pM$. Note that the following hold:

- $g_p(v_p, v_p) \geq 0$ for $p \in M$ and $v_p \in T_pM$;
- $g_p(u_p, v_p) = g_p(v_p, u_p)$ for $p \in M$ and $u_p, v_p \in T_pM$;
- $tg_p(u_p, v_p) = g_p(u_p, tv_p)$ for $p \in M$, $u_p, v_p \in T_pM$ and $t \geq 0$;
- $g_p(v_p, 0_p) = 0$ for $p \in M$ and $v_p \in T_pM$;
- $c'_\kappa(d(x, y))^2 = g_x(\log_{\kappa,x} y, \log_{\kappa,x} y) = g_y(\log_{\kappa,y} x, \log_{\kappa,y} x)$ for $x, y \in M$;
- $d(x, y)^2 = g_x(\log_x y, \log_x y) = g_y(\log_y x, \log_y x)$ for $x, y \in M$.

We further know the following proposition:

Theorem 3.1 (Kimura–Sudo [20]). *Let M be an admissible* CAT(κ) *space. Then,*

$$g_p(\log_{\kappa,p} x, \log_{\kappa,p} y) \geq \phi_\kappa(p,x) + c''_\kappa(d(p,x))\phi_\kappa(p,y) - \phi_\kappa(x,y)$$

for $p, x, y \in M$.

In what follows, we introduce monotone vector fields on a CAT(κ) space and their resolvent operators. For more details, refer to [28].

Let M be an admissible CAT(κ) space and A a set-valued mapping from M to a subset of the tangent bundle TM. We call A a set-valued vector field if

$$Ax \subset T_x M$$

for $x \in M$. For a set-valued vector field A on M, we denote the domain and the graph of A by

$$\begin{aligned}\operatorname{Dom} A &= \{x \in M \mid Ax \neq \emptyset\};\\ \operatorname{Gph} A &= \{(x, v_x) \in M \times TM \mid v_x \in Ax\},\end{aligned}$$

respectively. We call a point $x \in M$ a zero point of A if

$$0_x \in Ax.$$

We denote the set of all zero points of A by

$$\operatorname{Zero} A = \{x \in M \mid 0_x \in Ax\}.$$

Let A be a set-valued vector field on an admissible CAT(κ) space M. For $r > 0$, we define a set-valued vector field rA on M by

$$rAx = \{rv_x \in T_x M \mid v_x \in Ax\}$$

for $x \in M$. Note that for $r > 0$, we have

$$\begin{aligned}\operatorname{Dom}(rA) &= \operatorname{Dom} A;\\ \operatorname{Zero}(rA) &= \operatorname{Zero} A.\end{aligned}$$

We say that A is monotone if

$$g_x(\log_x y, u_x) + g_y(\log_y x, v_y) \leq 0$$

for $(x, u_x), (y, v_y) \in \operatorname{Gph} A$. We immediately obtain that A is monotone if and only if

$$g_x(\log_{\kappa,x} y, u_x) + g_y(\log_{\kappa,y} x, v_y) \leq 0$$

for $(x, u_x), (y, v_y) \in \operatorname{Gph} A$. Furthermore, if A is monotone, then so is rA for $r > 0$.

Let M be an admissible CAT(κ) space and A a set-valued vector field on M. We say that A is resolvably monotone if it is monotone, and

$$\left\{ z \in M \,\middle|\, \frac{\log_{\kappa,z} x}{r} \in Az \right\} \neq \emptyset$$

for any $r > 0$ and any $x \in M$. Suppose that A is resolvably monotone. Then, for $x \in M$ and $r > 0$, from the monotonicity of rA, a set

$$\left\{ z \in M \;\middle|\; \log_{\kappa,z} x \in rAz \right\}$$

consists of exactly one point. We denote such a unique point by $J_{rA}x$, namely,

$$\{J_{rA}x\} = \left\{ z \in M \;\middle|\; \log_{\kappa,z} x \in rAz \right\} = \left\{ z \in M \;\middle|\; \frac{\log_{\kappa,z} x}{r} \in Az \right\}$$

for $x \in M$. We call the mapping J_{rA} from M to $\operatorname{Dom} A$ the resolvent operator of rA. Note that if A is resolvably monotone, then

$$\operatorname{Zero} A = \operatorname{Fix} J_{rA}$$

for any $r > 0$, and it is closed and convex; see [28]. Here, $\operatorname{Fix} J_{rA}$ stands for the set of all fixed points of J_{rA}, that is,

$$\operatorname{Fix} J_{rA} = \{x \in M \mid J_{rA}x = x\}.$$

Theorem 3.2 (Sudo [29])**.** *Let A be a resolvably monotone vector field on an admissible* CAT(κ) *space M. Then, for fixed $r > 0$, the resolvent operator J_{rA} is geodesically nonspreading, that is,*

$$\phi_\kappa(J_{rA}x, J_{rA}y) + \phi_\kappa(J_{rA}y, J_{rA}x) \leq \phi_\kappa(J_{rA}x, y) + \phi_\kappa(J_{rA}y, x)$$

for $x, y \in M$. Furthermore, if A has a zero point, then J_{rA} is quasinonexpansive, that is,

$$d(J_{rA}x, y) \leq d(x, y)$$

for $x \in M$ and $y \in \operatorname{Fix} J_{rA} = \operatorname{Zero} A$.

4. Mann Type Proximal Point Algorithm

In this section, we show a zero point approximation theorem with the Mann type proximal point algorithm.

Let M be a metric space and $\{x_n\}$ a sequence of M. We call a point $x \in M$ an asymptotic centre of $\{x_n\}$ if

$$\limsup_{n\to\infty} d(x_n, x) = \inf_{y\in M} \limsup_{n\to\infty} d(x_n, y).$$

We further say that $\{x_n\}$ is Δ-convergent to a Δ-limit x [21] if x is a unique asymptotic centre of any subsequence of $\{x_n\}$. Assume that M is an admissible complete CAT(κ) space. We say that a sequence $\{x_n\}$ of M is κ-bounded if

$$\inf_{y\in M} \limsup_{n\to\infty} d(x_n, y) < \frac{D_\kappa}{2}.$$

We notice that the κ-boundedness is the usual one in the sense of metric spaces if $\kappa \leq 0$. Furthermore, if $\{x_n\}$ is κ-bounded, then it has a unique asymptotic centre, and it has a Δ-convergent subsequence; see [1, 4]. Moreover, we know the following:

Theorem 4.1 (Bačák [1], Kimura–Kohsaka [14]). *Let M be an admissible complete* CAT(κ) *space and $y \in M$. Then,*

$$d(x, y) \leq \liminf_{n\to\infty} d(x_n, y)$$

whenever a κ-bounded sequence $\{x_n\}$ of M is Δ-convergent to $x \in M$.

We next prove the following proposition:

Lemma 4.2. *Let A be a resolvably monotone vector field on an admissible complete* CAT(κ) *space M and let $\{r_n\}$ be a sequence of positive real numbers such that* $\inf_{k\in\mathbb{N}} r_k > 0$. *If a κ-bounded sequence $\{x_n\}$ of M satisfies that*

$$\lim_{n\to\infty} d(J_{r_nA}x_n, x_n) = 0,$$

then a unique asymptotic centre $x \in M$ of $\{x_n\}$ is a zero point of A.

Proof. Take a κ-bounded sequence $\{x_n\}$ of M and let $x \in M$ be its unique asymptotic centre. For simplicity, we denote $J_{r_nA}x_n$ by w_n for $n \in \mathbb{N}$. Then, x is a unique asymptotic centre of $\{w_n\}$. We first show this. Since

$$\begin{aligned}\inf_{y\in M}\limsup_{n\to\infty} d(w_n, y) &\leq \inf_{y\in M}\limsup_{n\to\infty}\,(d(w_n, x_n) + d(x_n, y))\\ &= \inf_{y\in M}\limsup_{n\to\infty} d(x_n, y) < \frac{D_\kappa}{2},\end{aligned}$$

the sequence $\{w_n\}$ is κ-bounded, and hence it has a unique asymptotic centre. Then, for any $w \in M$, we know that

$$\begin{aligned}\limsup_{n\to\infty} d(w_n, x) &\leq \limsup_{n\to\infty}\,(d(w_n, x_n) + d(x_n, x))\\ &= \limsup_{n\to\infty} d(x_n, x) \leq \limsup_{n\to\infty} d(x_n, w)\\ &\leq \limsup_{n\to\infty}\,(d(w_n, x_n) + d(w_n, w))\\ &= \limsup_{n\to\infty} d(w_n, w).\end{aligned}$$

It means that x is an asymptotic centre of $\{w_n\}$. Fix $n \in \mathbb{N}$ arbitrarily. Since

$$(J_Ax, \log_{\kappa,J_Ax} x), \left(w_n, \frac{\log_{\kappa,w_n} x_n}{r_n}\right) \in \operatorname{Gph} A$$

and A is monotone, we have

$$\begin{aligned}0 &\geq g_{J_Ax}(\log_{\kappa,J_Ax} w_n, \log_{\kappa,J_Ax} x) + \frac{g_{w_n}(\log_{\kappa,w_n} J_Ax, \log_{\kappa,w_n} x_n)}{r_n}\\ &\geq \phi_\kappa(J_Ax, w_n) - \phi_\kappa(w_n, x) + \frac{\phi_\kappa(w_n, J_Ax) - \phi_\kappa(J_Ax, x_n)}{r_n}\\ &\geq \phi_\kappa(J_Ax, w_n) - \phi_\kappa(w_n, x) - \frac{|\phi_\kappa(w_n, J_Ax) - \phi_\kappa(J_Ax, x_n)|}{r_n}\\ &\geq \phi_\kappa(J_Ax, w_n) - \phi_\kappa(w_n, x) - \frac{|\phi_\kappa(w_n, J_Ax) - \phi_\kappa(J_Ax, x_n)|}{\inf_{k\in\mathbb{N}} r_k},\end{aligned}$$

and therefore

$$\phi_\kappa(w_n, J_A x) \le \phi_\kappa(w_n, x) + \frac{|\phi_\kappa(w_n, J_A x) - \phi_\kappa(J_A x, x_n)|}{\inf_{k\in\mathbb{N}} r_k}. \tag{4.1}$$

Since c_κ is uniformly continuous on a compact interval and

$$|d(w_n, J_A x) - d(J_A x, x_n)| \le d(w_n, x_n) \to 0$$

as $n \to \infty$, letting $n \to \infty$ for the inequality (4.1), we obtain

$$\limsup_{n\to\infty} \phi_\kappa(w_n, J_A x) \le \limsup_{n\to\infty} \phi_\kappa(w_n, x).$$

Thus, we have $J_A x = x$, which implies that $x \in \operatorname{Zero} A$. ∎

Using this result, we finally show the following Δ-convergence theorem:

Theorem 4.3. *Let A be a resolvably monotone vector field on an admissible complete* CAT(κ) *space M, and suppose that it has a zero point. Let $\{r_n\}$ be a sequence of positive real numbers such that $\inf_{k\in\mathbb{N}} r_k > 0$, and let $\{a_n\}$ be a real sequence of $[0,1[$ such that $\sup_{k\in\mathbb{N}} a_k < 1$. For a given initial point $x_1 \in M$, generate a sequence $\{x_n\}$ of M by*

$$x_{n+1} = a_n x_n \oplus (1-a_n) J_{r_n A} x_n$$

for $n \in \mathbb{N}$. Then, the generated sequence $\{x_n\}$ Δ-converges to a zero point, which equals to

$$\lim_{n\to\infty} P_{\operatorname{Zero} A} x_n.$$

Proof. For $p \in \operatorname{Zero} A$ and $n \in \mathbb{N}$, since $J_{r_n A}$ is quasinonexpansive,

$$\begin{aligned}\phi_\kappa(x_{n+1}, p) &= \phi_\kappa(a_n x_n \oplus (1-a_n) J_{r_n A} x_n, p)\\ &\le a_n \phi_\kappa(x_n, p) + (1-a_n)\phi_\kappa(J_{r_n A} x_n, p)\\ &\le \phi_\kappa(x_n, p),\end{aligned}$$

and hence

$$d(x_{n+1}, p) \le d(x_n, p). \tag{4.2}$$

From Theorem 2.4, a sequence $\{P_{\operatorname{Zero} A} x_n\}$ converges to a zero point $x \in M$. Then, from the equation (4.2), we have

$$d(x_{n+1}, x) \le d(x_n, x)$$

for $n \in \mathbb{N}$, and therefore a real sequence $\{\phi_\kappa(x_n, x)\}$ is convergent and the generated sequence $\{x_n\}$ is κ-bounded. Furthermore, since

$$d(J_{r_n A} x_n, x) \le d(x_n, x)$$

for $n \in \mathbb{N}$, we have

$$c = \inf_{k\in\mathbb{N}} c''_\kappa(d(J_{r_k A} x_k, x)) > 0.$$

Fix $n \in \mathbb{N}$ arbitrarily. For simplicity, we denote $J_{r_n A} x_n$ by w_n. Since

$$(x, 0_x), (w_n, \log_{\kappa, w_n} x_n) \in \operatorname{Gph}(r_n A)$$

and $r_n A$ is monotone,

$$\begin{aligned} 0 &\geq g_x(\log_{\kappa,x} w_n, 0_x) + g_{w_n}(\log_{\kappa,w_n} x, \log_{\kappa,w_n} x_n) \\ &= g_{w_n}(\log_{\kappa,w_n} x, \log_{\kappa,w_n} x_n) \\ &\geq \phi_\kappa(w_n, x) + c''_\kappa(d(w_n, x))\phi_\kappa(w_n, x_n) - \phi_\kappa(x, x_n) \\ &\geq \phi_\kappa(w_n, x) + c \cdot \phi_\kappa(w_n, x_n) - \phi_\kappa(x, x_n). \end{aligned}$$

Thus,

$$\phi_\kappa(w_n, x) \leq \phi_\kappa(x_n, x) - c \cdot \phi_\kappa(w_n, x_n).$$

Therefore,

$$\begin{aligned} \phi_\kappa(x_{n+1}, x) &= \phi_\kappa(a_n x_n \oplus (1 - a_n) w_n, x) \\ &\leq a_n \phi_\kappa(x_n, x) + (1 - a_n)\phi_\kappa(w_n, x) \\ &\leq a_n \phi_\kappa(x_n, x) + (1 - a_n)\left(\phi_\kappa(x_n, x) - c \cdot \phi_\kappa(w_n, x_n)\right) \\ &\leq \phi_\kappa(x_n, x) - c(1 - a_n)\phi_\kappa(w_n, x_n), \end{aligned}$$

and hence

$$c(1 - a_n)\phi_\kappa(w_n, x_n) \leq \phi_\kappa(x_n, x) - \phi_\kappa(x_{n+1}, x).$$

Since $\sup_{k\in\mathbb{N}} a_k < 1$ and $c > 0$, letting $n \to \infty$, we have

$$\lim_{n\to\infty} d(J_{r_n A} x_n, x_n) = \lim_{n\to\infty} d(w_n, x_n) = 0.$$

Take a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ arbitrarily. Then, $\inf_{j\in\mathbb{N}} r_{n_j} > 0$ and

$$\lim_{i\to\infty} d(J_{r_{n_i} A} x_{n_i}, x_{n_i}) = 0.$$

Thus, from Lemma 4.2, a unique asymptotic centre $w \in M$ of $\{x_{n_i}\}$ is a zero point of A. Then,

$$\begin{aligned} \limsup_{i\to\infty} d(x_{n_i}, x) &\leq \limsup_{i\to\infty} \left(d(x_{n_i}, P_{\mathrm{Zero}\,A} x_{n_i}) + d(P_{\mathrm{Zero}\,A} x_{n_i}, x)\right) \\ &= \limsup_{i\to\infty} d(x_{n_i}, P_{\mathrm{Zero}\,A} x_{n_i}) \\ &\leq \limsup_{i\to\infty} d(x_{n_i}, w). \end{aligned}$$

It implies that $x = w$, and hence x is a unique asymptotic centre of $\{x_{n_i}\}$. Consequently, the generated sequence $\{x_n\}$ Δ-converges to x. ∎

5. Halpern Type Proximal Point Algorithm

In this section, we show a zero point approximation theorem with the Halpern type proximal point algorithm. We first show the following lemma:

Lemma 5.1. *Let $\{a_n\}$ be a sequence of $]0, 1[$ such that $\lim_{n\to\infty} a_n = 0$ and that $\sum_{k=1}^{\infty} a_k^2 = \infty$, and $\{l_n\}$ a bounded sequence of $[0, D_\kappa/2[$ for $\kappa \in \mathbb{R}$. Define a sequence $\{b_n\}$ of $]0, 1[$ by*

$$b_n = 1 - (1 - a_n)_{l_n}^{\kappa}$$

for $n \in \mathbb{N}$. Then, $\{b_n\}$ converges to 0, and $\sum_{k=1}^{\infty} b_k = \infty$.

Proof. We first show that $\sum_{k=1}^{\infty} b_k = \infty$. Fix $n \in \mathbb{N}$. If $l_n = 0$, then

$$b_n = 1 - (1 - a_n)_{l_n}^{\kappa} = a_n \geq a_n^2.$$

We might assume that $l_n \neq 0$. If $\kappa = 0$, then

$$b_n = a_n \geq a_n^2,$$

and hence we obtain the desired result. Similarly, if $\kappa < 0$, then

$$b_n = 1 - \frac{\sinh((1-a_n)\sqrt{-\kappa}l_n)}{\sinh(\sqrt{-\kappa}l_n)} \geq 1 - \frac{(1-a_n)\sinh(\sqrt{-\kappa}l_n)}{\sinh(\sqrt{-\kappa}l_n)} = a_n \geq a_n^2,$$

and thus we have $b_n \geq a_n^2$ for all $n \in \mathbb{N}$, when $\kappa \leq 0$. Assume that $\kappa > 0$. Although the following discussion is essentially the same as one by Kimura and Kohsaka [17, Theorem 5.1], we give a proof. Since

$$\frac{\sin((1-a_n)\sqrt{\kappa}l_n)}{\sin(\sqrt{\kappa}l_n)} \leq \sin\frac{\pi(1-a_n)}{2},$$

we obtain

$$\begin{aligned}
b_n = 1 - (1 - a_n)_{l_n}^{\kappa} &= 1 - \frac{\sin((1-a_n)\sqrt{\kappa}l_n)}{\sin(\sqrt{\kappa}l_n)} \\
&\geq 1 - \sin\frac{\pi(1-a_n)}{2} = 1 - \sin\left(\frac{\pi}{2} - \frac{\pi a_n}{2}\right) \\
&= 1 - \cos\frac{\pi a_n}{2} \geq \frac{\pi^2}{16}a_n^2.
\end{aligned}$$

Consequently, for $\kappa > 0$ and $n \in \mathbb{N}$, we obtain

$$b_n \geq \frac{\pi^2}{16}a_n^2.$$

Therefore, in any cases, $\sum_{k=1}^{\infty} b_k = \infty$.

We next show that

$$\lim_{n\to\infty} b_n = 0.$$

We immediately obtain $b_n = a_n$ for all $n \in \mathbb{N}$ if $\kappa = 0$. Suppose that $\kappa = 0$. If $l_n = 0$, then $b_n = a_n$. We might assume that $l_n \neq 0$. Moreover, if $\kappa > 0$, then

$$b_n = 1 - \frac{\sin((1-a_n)\sqrt{\kappa}l_n)}{\sin(\sqrt{\kappa}l_n)} \leq 1 - \frac{(1-a_n)\sin(\sqrt{\kappa}l_n)}{\sin(\sqrt{\kappa}l_n)} = a_n.$$

Therefore, if $\kappa \geq 0$, then $\{b_n\}$ converges to 0. We next consider the case where $\kappa < 0$. Then,

$$\begin{aligned}
b_n &= 1 - \frac{\sinh((1-a_n)\sqrt{-\kappa}l_n)}{\sinh(\sqrt{-\kappa}l_n)} \\
&= 1 - \frac{\sinh(\sqrt{-\kappa}l_n)\cosh(a_n\sqrt{-\kappa}l_n) - \sinh(a_n\sqrt{-\kappa}l_n)\cosh(\sqrt{-\kappa}l_n)}{\sinh(\sqrt{-\kappa}l_n)} \\
&= 1 - \cosh(a_n\sqrt{-\kappa}l_n) + \frac{\sinh(a_n\sqrt{-\kappa}l_n)\cosh(\sqrt{-\kappa}l_n)}{\sinh(\sqrt{-\kappa}l_n)}
\end{aligned}$$

$$\le \frac{\sinh(a_n\sqrt{-\kappa}l_n)\cosh(\sqrt{-\kappa}l_n)}{\sinh(\sqrt{-\kappa}l_n)}$$
$$\le \frac{a_n\sinh(\sqrt{-\kappa}l_n)\cosh(\sqrt{-\kappa}l_n)}{\sinh(\sqrt{-\kappa}l_n)} = a_n\cosh(\sqrt{-\kappa}l_n).$$

Consequently, for $\kappa < 0$ and for all $n \in \mathbb{N}$, we obtain

$$0 \le b_n \le a_n\cosh(\sqrt{-\kappa}l_n).$$

Since $\{l_n\}$ is bounded, $\{b_n\}$ converges to 0. ∎

Furthermore, we have the following result, which is effective for the Halpern type iteration:

Theorem 5.2 (Kimura–Saejung [18], Saejung–Yotkaew [26])**.** *Let $\{s_n\}$ be a nonnegative real sequence and $\{t_n\}$ a real sequence. Let $\{b_n\}$ be a real sequence of $]0,1]$ such that $\sum_{k=1}^{\infty} b_k = \infty$. Suppose that*

$$s_{n+1} \le (1-b_n)s_n + b_n t_n$$

for $n \in \mathbb{N}$ and that

$$\limsup_{i\to\infty} t_{n_i} \le 0$$

for every subsequence $\{s_{n_i}\}$ of $\{s_n\}$ satisfying that

$$\liminf_{i\to\infty}(s_{n_i+1} - s_{n_i}) \ge 0.$$

Then, the sequence $\{s_n\}$ converges to 0.

Using these results, we finally show the following convergence theorem:

Theorem 5.3. *Let A be a resolvably monotone vector field on an admissible complete* CAT(κ) *space M, and suppose that it has a zero point. Let $\{r_n\}$ be a sequence of positive real numbers such that $\inf_{k\in\mathbb{N}} r_k > 0$, and let $\{a_n\}$ be a real sequence of $]0,1[$ such that $\lim_{n\to\infty} a_n = 0$ and that $\sum_{k=1}^{\infty} a_k^2 = \infty$. For a given anchor point and a given initial point $u, x_1 \in M$, generate a sequence $\{x_n\}$ of M by*

$$x_{n+1} = a_n u \oplus (1-a_n)J_{r_nA}x_n$$

for $n \in \mathbb{N}$. Then, the generated sequence $\{x_n\}$ converges to $P_{\mathrm{Zero}\,A}u$.

Proof. Let $p = P_{\mathrm{Zero}\,A}u$. For $n \in \mathbb{N}$, since J_{r_nA} is quasinonexpansive,

$$\begin{aligned}\phi_\kappa(x_{n+1}, p) &= \phi_\kappa(a_n u \oplus (1-a_n)J_{r_nA}x_n, p)\\ &\le a_n\phi_\kappa(u,p) + (1-a_n)\phi_\kappa(J_{r_nA}x_n, p)\\ &\le a_n\phi_\kappa(u,p) + (1-a_n)\phi_\kappa(x_n, p),\end{aligned}$$

and hence

$$d(J_{r_nA}x_n, p) \le d(x_n, p) \le \max\{d(u,p), d(x_1,p)\} < \frac{D_\kappa}{2}. \tag{5.1}$$

Thus, $\{x_n\}$ is κ-bounded. Fix $n \in \mathbb{N}$ arbitrarily. For simplicity, we denote $J_{r_nA}x_n$ by w_n, and $d(u, w_n)$ by l_n. Notice that

$$c = \inf_{k\in\mathbb{N}} c''_\kappa(d(w_n, p)) > 0$$

from the inequality (5.1). Let

$$b_n = 1 - (1 - a_n)_{l_n}^{\kappa} \in \left]0, 1\right[.$$

We know that $\sum_{k=1}^{\infty} b_k = \infty$ and that

$$\lim_{n\to\infty} (1 - a_n)_{l_n}^{\kappa} = 1$$

from Lemma 5.1. Furthermore, from Lemma 2.3, we have

$$\begin{aligned}
&\phi_\kappa(x_{n+1}, p)\\
&= \phi_\kappa(a_n u \oplus (1 - a_n) w_n, p)\\
&\leq (1 - b_n)\phi_\kappa(w_n, p) + b_n \cdot \frac{c''_\kappa(a_n l_n/2)\phi_\kappa(u, p) - (1 - a_n)_{l_n/2}^{\kappa}\phi_\kappa(u, w_n)}{c''_\kappa(l_n - a_n l_n/2)}\\
&\leq (1 - b_n)\phi_\kappa(x_n, p) + b_n \cdot \frac{c''_\kappa(a_n l_n/2)\phi_\kappa(u, p) - (1 - a_n)_{l_n/2}^{\kappa}\phi_\kappa(u, w_n)}{c''_\kappa(l_n - a_n l_n/2)}.
\end{aligned}$$

Let

$$s_n = \phi_\kappa(x_n, p).$$

Then, $\{s_n\}$ is a nonnegative real sequence. Further, let

$$t_n = \frac{c''_\kappa(a_n l_n/2)\phi_\kappa(u, p) - (1 - a_n)_{l_n/2}^{\kappa}\phi_\kappa(u, w_n)}{c''_\kappa(l_n - a_n l_n/2)}. \tag{5.2}$$

Then, we have

$$s_{n+1} \leq (1 - b_n)s_n + b_n t_n.$$

Finally, we show that $\{s_n\}$ converges to 0 using Theorem 5.2. Take a subsequence $\{s_{n_i}\}$ of $\{s_n\}$ such that

$$\liminf_{i\to\infty} (s_{n_i+1} - s_{n_i}) \geq 0,$$

and show that

$$\limsup_{i\to\infty} t_{n_i} \leq 0.$$

Then, since $\{a_n\}$ converges to 0, we have

$$\begin{aligned}
0 \leq \liminf_{i\to\infty} (s_{n_i+1} - s_{n_i}) &= \liminf_{i\to\infty} (\phi_\kappa(x_{n_i+1}, p) - \phi_\kappa(x_{n_i}, p))\\
&= \liminf_{i\to\infty} (\phi_\kappa(a_{n_i} u \oplus (1 - a_{n_i}) w_{n_i}, p) - \phi_\kappa(x_{n_i}, p))\\
&\leq \liminf_{i\to\infty} (a_{n_i}\phi_\kappa(u, p) + (1 - a_{n_i})\phi_\kappa(w_{n_i}, p) - \phi_\kappa(x_{n_i}, p))\\
&= \liminf_{i\to\infty} (\phi_\kappa(w_{n_i}, p) - \phi_\kappa(x_{n_i}, p))\\
&\leq \limsup_{i\to\infty} (\phi_\kappa(w_{n_i}, p) - \phi_\kappa(x_{n_i}, p)) \leq 0.
\end{aligned}$$

Thus,

$$\lim_{i\to\infty} |\phi_\kappa(w_{n_i}, p) - \phi_\kappa(x_{n_i}, p)| = 0.$$

Fix $i \in \mathbb{N}$ arbitrarily. Since

$$(p, 0_p), (w_{n_i}, \log_{\kappa, w_{n_i}} x_{n_i}) \in \mathrm{Gph}(r_{n_i} A)$$

and A is monotone,

$$\begin{aligned} 0 &\geq g_p(\log_{\kappa,p} w_{n_i}, 0_p) + g_{w_{n_i}}(\log_{\kappa, w_{n_i}} p, \log_{\kappa, w_{n_i}} x_{n_i}) \\ &= g_{w_{n_i}}(\log_{\kappa, w_{n_i}} p, \log_{\kappa, w_{n_i}} x_{n_i}) \\ &\geq \phi_\kappa(w_{n_i}, p) + c''_\kappa(d(w_{n_i}, p))\phi_\kappa(w_{n_i}, x_{n_i}) - \phi_\kappa(p, x_{n_i}) \\ &\geq \phi_\kappa(w_{n_i}, p) + c \cdot \phi_\kappa(w_{n_i}, x_{n_i}) - \phi_\kappa(p, x_{n_i}). \end{aligned}$$

It implies that

$$\phi_\kappa(w_{n_i}, x_{n_i}) \leq \frac{\phi_\kappa(x_{n_i}, p) - \phi_\kappa(w_{n_i}, p)}{c} \to 0$$

as $i \to \infty$. Therefore,

$$\lim_{i\to\infty} d(J_{r_{n_i} A} x_{n_i}, x_{n_i}) = \lim_{i\to\infty} d(w_{n_i}, x_{n_i}) = 0,$$

and then

$$\liminf_{i\to\infty} d(u, w_{n_i}) = \liminf_{i\to\infty} d(u, x_{n_i}).$$

Take a subsequence $\{y_j\}$ of $\{x_{n_i}\}$ such that

$$\lim_{j\to\infty} d(u, y_j) = \liminf_{i\to\infty} d(u, x_{n_i})$$

and that $\{y_j\}$ Δ-converges to $y \in M$. From Lemma 4.2, we have $y \in \mathrm{Zero}\, A$. Thus, Theorem 4.1 yields that

$$\liminf_{i\to\infty} d(u, w_{n_i}) = \liminf_{i\to\infty} d(u, x_{n_i}) = \lim_{j\to\infty} d(u, y_j) \geq d(u, y) \geq d(u, p) \tag{5.3}$$

since $\{y_j\}$ Δ-converges to y and $p = P_{\mathrm{Zero}\, A} u$. Therefore, since

$$\lim_{i\to 0} a_{n_i} = 0,$$

from the equation (5.2),

$$\begin{aligned} \limsup_{i\to\infty} t_{n_i} &= \limsup_{i\to\infty} \frac{c''_\kappa(a_{n_i} l_{n_i}/2)\phi_\kappa(u, p) - (1 - a_{n_i})^\kappa_{l_{n_i}/2}\phi_\kappa(u, w_{n_i})}{c''_\kappa(l_{n_i} - a_{n_i} l_{n_i}/2)} \\ &= \limsup_{i\to\infty} \frac{\phi_\kappa(u, p) - \phi_\kappa(u, w_{n_i})}{c''_\kappa(l_{n_i})} \\ &= \frac{1}{\liminf_{i\to\infty} c''_\kappa(l_{n_i})} \left(c_\kappa(d(u, p)) - c_\kappa\left(\liminf_{i\to\infty} d(u, w_{n_i}) \right) \right). \end{aligned}$$

We know that

$$\frac{1}{\liminf_{i\to\infty} c''_\kappa(l_{n_i})} \in [0, \infty]$$

for any $\kappa \in \mathbb{R}$. Thus, from the inequality (5.3), we obtain

$$\limsup_{i\to\infty} t_{n_i} = \frac{1}{\liminf_{i\to\infty} c_\kappa''(l_{n_i})}\left(c_\kappa(d(u,p)) - c_\kappa\left(\liminf_{i\to\infty} d(u, w_{n_i})\right)\right) \le 0.$$

Consequently, from Theorem 5.2, we have

$$\lim_{n\to\infty} \phi_\kappa(x_n, p) = \lim_{n\to\infty} s_n = 0,$$

which means that the generated sequence $\{x_n\}$ converges to $P_{\mathrm{Zero}\,A}u$. ∎

6. Conclusion

In this work, we obtained approximation theorems with two modified proximal point algorithms. Particularly, Theorem 6.1 is a strong convergence theorem, unlike Theorem 1.1 and 4.3. However, the assumption of a coficient sequence $\{a_n\}$ is

(a) $\lim_{n\to\infty} a_n = 0$;

(b) $\sum_{k=1}^{\infty} a_k^2 = \infty$.

In a related result by Kimura and Kohsaka [15], it is enough to assume that (a) and

(c) $\sum_{k=1}^{\infty} a_k = \infty$.

We know that if a sequence $\{a_n\}$ of $]0,1[$ satisfies the condition (b), then it satisfies the condition (c). In fact, we obtain divergence of the sum of $\{b_n\}$ in Lemma 5.1 even if we only suppose the condition (c) in the case where $\kappa \le 0$. Thus, in the same way as Theorem 6.1, we obtain the following result:

Proposition 6.1. *Let M be a complete* CAT(κ) *space for $\kappa \le 0$, and suppose that A and $\{r_n\}$ are the same as Theorem 6.1. Let $\{a_n\}$ be a real sequence of $]0,1[$ such that $\lim_{n\to\infty} a_n = 0$ and that $\sum_{k=1}^{\infty} a_k = \infty$. Define a sequence $\{x_n\}$ in the same way as Theorem 6.1. Then, the generated sequence $\{x_n\}$ converges to $P_{\mathrm{Zero}\,A}u$.*

However, we cannot obtain a result such as the above proposition in the case where $\kappa > 0$ so far.

Competing Interests

The author declares that there are no competing interests.

References

[1] M. Bačák, Convex Analysis and Optimization in Hadamard spaces, De Gruyter, Berlin, 2014.

[2] M.R. Bridson and A. Haefliger, Metric Spaces of Non-positive Curvature, Springer-Verlag, Berlin, 1999.

[3] P. Chaipunya, F. Kohsaka and P. Kumam, Monotone vector fields and generation of nonexpansive semigroups in complete CAT(0) spaces, Numer. Funct. Anal. Optim., 42 (2021), 989–1018.

[4] R. Espínola and A. Fernández-León, CAT(k)-spaces, weak convergence and fixed points, J. Math. Anal. Appl., 353 (2009), 410–427.

[5] B. Halpern, Fixed points of nonexpansive mappings, Bull. Amer. Math. Soc., 73 (1967), 957–961.

[6] J.S. He, D.H. Fang, G. López and C. Li, Mann's algorithm for nonexpansive mappings in CAT(κ) spaces, Nonlinear Anal., 75 (2012), 445–452.

[7] J. Jost, Convex functionals and generalized harmonic maps into spaces of nonpositive curvature, Comment. Math. Helv., 70 (1995), 659–673.

[8] T. Kajimura and Y. Kimura, A vicinal mapping on geodesic spaces, In Proceedings of International Conference on Nonlinear Analysis and Convex Analysis & International Conference on Optimization: Techniques and Applications –I– (Hakodate, Japan, 2019) (Y. Kimura, M. Muramatsu, W. Takahashi, and A. Yoshise, eds.), pages 183–195, 2021.

[9] T. Kajimura and Y. Kimura, The proximal point algorithm with a general perturbation on geodesic spaces, Results Appl. Math., 27 (2025), Paper No. 100618, 1–9.

[10] S. Kamimura, F. Kohsaka and W. Takahashi, Weak and strong convergence theorems for maximal monotone operators in a Banach space, Set-Valued Anal., 12 (2004), 417–429.

[11] S. Kamimura and W. Takahashi, Approximating solutions of maximal monotone operators in Hilbert spaces, J. Approx. Theory Appl., 106 (2000), 226–240.

[12] S. Kamimura and W. Takahashi, Weak and strong convergence of solutions to accretive operator inclusions and applications, Set-Valued Anal., 8 (2000), 361–374.

[13] C. Khaofong, P. Saipara, S. Srathonglang and A. Padcharoen, New modified proximal point algorithm for solving minimization and common fixed point problem over CAT(κ) spaces, WSEAS Transactions on Mathematics, 23 (2024), 87–97.

[14] Y. Kimura and F. Kohsaka, Spherical nonspreadingness of resolvents of convex functions in geodesic spaces, J. Fixed Point Theory Appl., 18 (2016), 93–115.

[15] Y. Kimura and F. Kohsaka, Two modified proximal point algorithm for convex functions in Hadamard spaces, Linear Nonlinear Anal., 2 (2016), 69–86.

[16] Y. Kimura and F. Kohsaka, The proximal point algorithm in geodesic spaces with curvature bounded above, Linear Nonlinear Anal., 3 (2017), 133–148.

[17] Y. Kimura and F. Kohsaka, Two modified proximal point algorithms in geodesic spaces with curvature bounded above, Rend. Circ. Mat. Palermo, II, 68 (2019), 83–104.

[18] Y. Kimura and S. Saejung, Strong convergence for a common fixed point of two different generalizations of cutter operators, Linear Nonlinear Anal., 1 (2015), 53–65.

[19] Y. Kimura and S. Sudo, New type parallelogram laws in Banach spaces and geodesic spaces with curvature bounded above, Arab. J. Math., 12 (2023), 389–412.

[20] Y. Kimura and S. Sudo, Tangent spaces and a metric on geodesic spaces, RIMS Kôkyûroku, 2240 (2023), 7–19.

[21] T.C. Lim, Remarks on some fixed point theorems, Proc. Amer. Math. Soc., 60 (1976), 179–182.

[22] W.R. Mann, Mean value method in iteration, Proc. Amer. Math. Soc., 4 (1953), 506–510.

[23] U.F. Mayer, Gradient flows on nonpositively curved metric spaces and harmonic maps, Comm. Anal. Geom., 6 (1998), 199–253.

[24] R.T. Rockafellar, Monotone operators and the proximal point algorithm, SIAM J. Control Optimization, 14 (1976), 877–898.

[25] S. Saejung, Halpern's iteration in CAT(0) spaces, Fixed Point Theory Appl., 2010 (2010), Article number: 471781, 1–13.

[26] S. Saejung and P. Yotkaew, Approximation of zeros of inverse strongly monotone operators in Banach spaces, Nonlinear Anal., 75 (2012), 742–750.

[27] M.V. Solodov and B.F. Svaiter, Forcing strong convergence of proximal point iterations in a Hilbert space, Math. Program., 87 (2000), 189–202.

[28] S. Sudo, Monotone vector fields on a geodesic space with curvature bounded above by a general real number, Lett. Nonlinear Anal. Appl., 3 (2025), 5–27.

[29] S. Sudo, The proximal point algorithm for monotone vector fields on complete geodesic spaces, Nonlinear Convex Anal. & Optim., 4 (2025), 1–15.

[30] R. Wittmann, Approximation of fixed points of nonexpansive mappings, Arch. Math., 58 (1992), 486–491.