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Interpolative Reich-Istratescu-type Hybrid Contractions in Modular Metric Spaces

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ABSTRACT

In this article, some novel contraction mappings pertaining to modular metric spaces are presented. The conditions that must be met for such operators to possess invariant points are described. Through a nontrivial case, the validity of the derived results is proven.

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1. Introduction

A fixed point result proposed by Banach [2] laid the groundwork for metric fixed point theory. According to Banach, if the self-map Ω, defined over a complete metric space (X, ϱ), satisfies the contraction inequality, that is, if there exists a constant $\kappa \in [0, 1)$ such that $\varrho(\Omega\xi, \Omega\zeta) \leq \kappa\varrho(\xi, \zeta)$ for all $\xi, \zeta \in X$, then it has a unique fixed point. Kannan [8], who considered the contraction $\varrho(\Omega\xi, \Omega\zeta) \leq \kappa[\varrho(\xi, \Omega\zeta) + \varrho(\zeta, \Omega\zeta)$, for all $\xi, \zeta \in X$, where $\kappa \in [0, \frac{1}{2})$, shows this result has a new fixed point. This result has a new fixed point. The Banach contraction principle has since been extended in a number of ways (see [25],[12],[27], and numerous others for additional information). However, the creation of new spaces and the associated fixed point theorem is one of the current advancements in the field of non-linear functional analysis. Nakano [19] introduced modular space as a metric space generalization, and Koshi [13] and Yamamuro [28] went on to study it in great detail. Additionally, Chistyakov [3, 4] introduced the idea of modular metric spaces in 2008 and constructed a novel structure employing the attributes of modular spaces. The physical interpretation of Chistyakov's recently developed modular is one of its main motivations. Specifically, a metric defined on a non-empty set indicates the distance between any two locations in the set, while a modular

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within a set connects the elements to a non-negative, occasionally indefinitely valued field of velocities. Mongkolkeha et al. [16] studied and proved new fixed point existence theorems for contraction mappings in modular metric spaces. Hardy-Rogers contractions have fixed points, as demonstrated by Surajit et al. [30], who also provided numerical examples to back up their conclusions. Additionally, [29] demonstrates that a new class of generalized F-contraction has fixed points, and the results are subsequently used to solve existence problems for integral equations. A mixed G-monotone mapping on a modular metric space with a graph was created by Yogita and Shishir [31], who also provided fixed point theorems for this novel class of mappings. Additionally, fixed point theorems for such contractions were introduced, along with generalized presic type w-contractive mappings and highly w-contractive mappings in a modular metric space [1]. Additionally, in order to eliminate the possibility of triangle inequality and non-zero self-distance discrepancies, [5] invented the concept of partial modular metrics and gave certain fixed point theorems for four self-maps. The Uniform Limit Theorem for Metric Modular Spaces and Baire's Theorem are two well-known metric space conclusions that were established by Somaye et al. [7]. They also defined Hausdorff topology on modular spaces with metric. In addition, Rahimpoor [23] shows several fixed point theorems for partial order sets in modular metric spaces using the mixed monotone mapping feature. A few fixed point outcomes of modular metric spaces with cyclic weak φ-contractions that are ω-compact and ω-complete, respectively, were also reported in [24]. Additionally, three pairs of weakly commuting self-maps that share a single fixed point [20] extended the concept of weakly commuting mappings in modular metric spaces to the context of modular ω^G-metric spaces. Furthermore, Doru and Ariana [6] showed that several types of convex contractions have fixed points in the setting of JS-metric spaces. Moreover, [21] introduced the notion of new hybrid generalized weakly contractive mappings in a complete metric spaces and prove the existence and unique common fixed point for this mappings. In addition, [26] introduced the notion of generalized weakly quasi contractive operators in metric-like space and investigates the existence and uniqueness of these operators' fixed points.

We note from the literature that there is a lack of focus on fixed point results of modular contraction types and associated applications. This article establishes fixed point conclusions of several kinds of modular contraction inequality in light of this gap. An example is provided to bolster the theories developed by the findings reported here. In particular, the work of Karapinar et. al [10] is considered in modular metric space.

2. Preliminaries

First, we provide an overview of the basic concepts in both modular metric spaces and modular spaces (for further information, see [14, 17, 3, 4]).

Definition 2.1. Suppose that X is a vector space over $\mathbb{R}(or\ \mathbb{C})$. A modular is a functional $\rho : X \to [0, \infty]$ such that for arbitrary $\xi, \zeta \in X$, the following three requirements are met:

$(A1)$ $\rho(\xi) = \xi \Leftrightarrow \xi = 0$;

$(A2)$ $\rho(\alpha\xi) = \rho(\xi)$ $\forall$ scalar α with $|\alpha| = 1$;

$(A3)$ $\rho(\alpha\xi + \beta\zeta) \leq \rho(\xi) + \rho(\zeta)$ whenever $\alpha, \beta \geq 0$ and $\alpha + \beta = 1$.

Moreover, Chistyakov [3] claimed that if condition (A 3) is replaced by $\rho(\alpha\xi + \beta\zeta) \leq \rho^s(\xi) + \rho^s(\zeta)$, for $\alpha, \beta \geq 0$, $\alpha^s + \beta^s = 1$ with $s \in (0, 1]$, the modular ρ will thus be referred

to as s-convex modular, and if $s = 1$, it will be referred to as a convex modular. If ρ is a modular in X, then the set defined by $X_\rho = \{\xi \in X : \rho(\lambda\xi) \to 0\}$ as $\lambda \to 0^+$ is called a modular space. X_ρ is a vector subspace of X. Hence, it can be equipped with an F-norm defined by setting $\|\xi_\rho\| = \inf\{\lambda > 0 : \rho(\frac{\xi}{\lambda}) \leq \lambda\}$, $\xi \in X_\rho$.
In addition, if ρ is convex, then the modular space X_ρ coincides with $X^*_\rho = \{\xi \in X : \exists\lambda = \lambda(\xi) > 0$ such that $\rho(\lambda\xi) < \infty\}$ and the functional $\|\xi_\rho\| = \inf\{\lambda > 0 : \rho(\frac{\xi}{\lambda}) \leq 1\}$, is an ordinary norm on X^*_ρ which is equivalence to $\|\xi_\rho\|$ (see [18]).

The function $\omega : (0, \infty) \times X \times X \to [0, \infty]$ will be expressed as $\omega_\lambda(\xi, \zeta) = \omega(\lambda, \xi, \zeta)$ for all $\lambda > 0$ and $\xi, \zeta \in X$ due to the disparity of the inputs.

Definition 2.2. [3, Definition 2.1] Consider the nonempty set X. For any $\xi, \zeta, \gamma \in X$, a function $\omega : (0, \infty) \times X \times X \to [0, \infty]$ is considered a metric modular on X, if the following conditions hold:

(i) $\omega_\lambda(\xi, \zeta) = 0$ for all $\lambda > 0$ if and only if $\xi = \zeta$;

(ii) $\omega_\lambda(\xi, \zeta) = \omega_\lambda(\zeta, \xi)$ for all $\lambda > 0$;

(iii) $\omega_{\lambda+\mu}(\xi, \zeta) \leq \omega_\lambda(\xi, \gamma) + \omega_\mu(\gamma, \zeta)$ for all $\lambda, \mu > 0$.

According to Chistyakov [3], suppose we only have condition (i') in place of condition (i) $\omega_\lambda(\xi, \xi) = 0$ for all $\lambda > 0$, then ω is said to be a (metric) pseudomodular on X. The main property of a (pseudo) modular ω on a set X is the following: given $\xi, \zeta \in X$, the function $0 < \lambda \to \omega_\lambda(\xi, \zeta) \in [0, \infty]$ is non-increasing on $(0, \infty)$. In fact, if $0 < \mu < \lambda$, then (iii), (i') and (ii) imply $\omega_\lambda(\xi, \zeta) \leq \omega_{\lambda-\mu}(\xi, \xi) + \omega_\mu(\xi, \zeta) = \omega_\mu(\xi, \zeta)$. Thus, at every point where $\lambda > 0$, the right limit $\omega_{\lambda+0}(\xi, \zeta) := \lim_{\epsilon\to 0^+} \omega_{\lambda+\epsilon}(\xi, \zeta)$ and the left limit $\omega_{\lambda-0}(\xi, \zeta) : \lim_{\epsilon\to 0^+} \omega_{\lambda-\epsilon}(\xi, \zeta)$ exist in $[0, \infty]$ and the two inequalities that follow are true: $\omega_{\lambda+0}(\xi, \zeta) \leq \omega_\lambda(\xi, \zeta) \leq \omega_{\lambda-0}(\xi, \zeta)$.

If we replaced pseudomodular with modular, the same claim remains true. The metric space X_ω is modular if ω is metric modular in X. By the property of modular and metric spaces, the following are defined.

Definition 2.3. [16, Definition 2.4] Consider the modular metric space X_ω:

(i) Convergence of a sequence $(\xi_j)_{j\in\mathbb{N}}$ in X_ω is defined as $j \to \infty$ for all $\lambda > 0$ if $\omega_\lambda(\xi_j, \xi) \to 0$;

(ii) The Cauchy sequence $(\xi_j)_{j\in\mathbb{N}}$ *in* X_ω is defined as $\omega_\lambda(\xi_k, \xi_j) \to 0$, as $k, j \to \infty$ for all $\lambda > 0$;

(iii) If a convergent sequence of A always has a limit that belongs to A, then A is a closed subset of X_ω;

(iv) To be considered complete, a subset A of X_ω must have a convergent Cauchy sequence in A with a limit in A.

Let a function $\alpha : X \times X \to [0, 1)$. A mapping $\Lambda : X \to X$ is α-orbital admissible ([22]) if $\alpha(\xi, \Lambda\xi)$ is greater than or equal to 1 implies $\alpha(\Lambda\xi, \Lambda^2\xi) \geq 1$, for all $\xi \in X$.

α-orbital admissible mapping Λ is called triangular α-orbital admissible ([22]) whenever $\alpha(\xi, \zeta) \geq 1$ and $\alpha(\zeta, \Lambda\zeta) \geq 1$ implies $\alpha(\xi, \Lambda\zeta) \geq 1$, for every $\xi, \zeta \in X$.

Lemma 2.4. *[9, Theorem 1.2] Suppose that for a triangular α-orbital admissible mapping $\Lambda : X \to X$, $\exists\, \xi_0 \in X$ s.t $\alpha(\xi_0, \Lambda\xi_0) \geq 1$. Then $\alpha(\xi_j, \xi_k) \geq 1$, $\forall\, j, k \in \mathbb{N}$, where the sequence $\{\xi_j\}$ is defined by $\xi_{j+1} = \Lambda\xi_j$, $j \in \mathbb{N}$.*

In order to formulate and prove theorems in this article, we consider the following notations:

$$\delta_{j_0}(\omega_\lambda, \Omega, \xi_0) = \sup\{\omega_\lambda(\Omega^j\xi_0, \Omega^k\xi_0) : j, k \in \mathbb{N}, j, k \geq j_0\},$$

where $j_0 \in \mathbb{N}$, and

$$\delta(\omega_\lambda, \Omega, \xi_0) = \sup\{\omega_\lambda(\Omega^j\xi_0, \Omega^k\xi_0) : j, k \in \mathbb{N}\}.$$

Let us consider the orbit of an element ξ_0 by an operator $\Omega : X_\omega \to X_\omega$ using the symbol $O_\Omega(\xi_0) = \{\Omega^j\xi_0 : j \in \mathbb{N}\}$.

3. Main Results

Motivated by the notions of a Reich-Istratescu-type contraction introduced in [10], in this section we establish the conditions for the existence and uniqueness of a fixed point for convex contraction type mapping in modular metric spaces.

Definition 3.1. Let $(X_\omega, \omega_\lambda)$ be a modular metric space and $\alpha : X_\omega \times X_\omega \longrightarrow [0, \infty)$ be a function. A mapping $\Omega : X_\omega \longrightarrow X_\omega$ is a hybrid-interpolative Reich-Istrăţescu-type contraction in the case there exist $\eta_i \in [0, 1)$, for $i = 1, 2, 3, 4, 5$ and $\delta \geq 0$, $\kappa \in (0, 1)$ such that,

$$\alpha(\xi, \zeta)\Omega_\lambda(\Omega^2\xi, \Omega^2\zeta) \leq \kappa J^\lambda(\xi, \zeta) \tag{3.1}$$

where

$$J^\lambda(\xi,\zeta) = \begin{cases} \kappa[\eta_1\omega_\lambda(\Omega\xi,\Omega\zeta)^\lambda + \eta_2\omega_\lambda(\Omega\xi,\Omega^2\xi)^\lambda + \eta_3\omega_\lambda(\zeta,\Omega\zeta)^\lambda \\ +\eta_4\omega_\lambda(\Omega\zeta,\Omega^2\zeta)^\lambda + \eta_5\omega_\lambda(\Omega\zeta,\Omega^2\xi)^\lambda + \delta\omega_\lambda(\zeta,\Omega\xi)^\lambda]^{\frac{1}{\lambda}} \\ \text{if } \lambda > 0, \textit{with} \sum_{i=1}^{5}\eta_i + \delta \leq 1, \xi,\zeta \in X_\omega, \xi \neq \zeta, \\ \\ \kappa[\omega_\lambda(\Omega\xi,\Omega\zeta)^{\eta_1} \cdot \omega_\lambda(\Omega\xi,\Omega^2\xi)^{\eta_2} \cdot \omega_\lambda(\zeta,\Omega\zeta)^{\eta_3} \\ \cdot\omega_\lambda(\Omega\zeta,\Omega^2\zeta)^{\eta_4} \cdot \omega_\lambda(\Omega\zeta,\Omega^2\xi)^{\eta_5} \cdot \omega_\lambda(\zeta,\Omega\xi)^{\delta}] \\ \text{if } \lambda = 0, \textit{with} \sum_{i=1}^{5}\eta_i + \delta = 1, \xi,\zeta \in X_\omega \setminus F_\Omega(X_\omega), \end{cases}$$

Theorem 3.2. *Let $(X_\omega, \omega_\lambda)$ be a complete modular metric space, $\Omega : X_\omega \longrightarrow X_\omega$ be an operator. Suppose that the following conditions are satisfied:*

(i) Ω is a hybrid-interpolative Reich-Istrăţescu-type contraction;

(ii) Ω is α-admissible;

(iii) there exists $\xi_0 \in X_\omega$ such that $\alpha(\xi_0, \Omega\xi_0) \geq 1$,

for all distinct $\xi, \zeta \in O_\Omega(\xi_0)$. Then the mapping Ω has an approximate fixed point property.

Proof. Let $\xi_0 \in X_\omega$ be arbitrary point. Then, by Property (iii), $\alpha(\xi_0, \Omega\xi_0) \geq 1$. Again, for $\xi_1 \in X_\omega$, we have $\alpha(\xi_1, \Omega\xi_1) \geq 1$. In general, $\alpha(\xi_j, \Omega\xi_j) \geq 1$, for all $j \in \mathbb{N}$. Starting from this point $\xi_0 \in X_\omega$, we define the sequence $\{\xi_n\}$ in X_ω as follows:

$$\xi_1 = \Omega\xi_0,$$

$$\xi_2 = \Omega\xi_1 = \Omega^2\xi_0, \dots,$$
$$\xi_n = \Omega\xi_{n-1} = \Omega^n\xi_0.$$

If there is some $n \in \mathbb{N}$ satisfying that $\xi_n = \xi_{n+1}$, then ξ_n is a fixed point of T, and the proof finishes here. On the contrary case, suppose that $\xi_n \neq \xi_{n+1}$ for all $n \in \mathbb{N}$. Now, using Property (i), we have two cases:

CASE 1: When $\lambda > 0$, we have

$$\begin{aligned}
\omega_\lambda(\xi_2, \xi_3) &= \omega_\lambda(\Omega^2\xi_0, \Omega^2\xi_1) \leq \alpha(\xi_0, \xi_1)\omega_\lambda(\Omega^2\xi_0, \Omega^2\xi_1) \\
&\leq \kappa[\eta_1\omega_\lambda(\Omega\xi_0, \Omega\xi_1)^\lambda + \eta_2\omega_\lambda(\Omega\xi_0, \Omega^2\xi_0)^\lambda + \eta_3\omega_\lambda(\xi_1, \Omega\xi_1)^\lambda \\
&\quad + \eta_4\omega_\lambda(\Omega\xi_1, \Omega^2\xi_1)^\lambda + \eta_5\omega_\lambda(\Omega\xi_1, \Omega^2\xi_0)^\lambda + \delta\omega_\lambda(\xi_1, \Omega\xi_0)^\lambda]^{\frac{1}{\lambda}} \\
&= \kappa[\eta_1\omega_\lambda(\xi_1, \xi_2)^\lambda + \eta_2\omega_\lambda(\xi_1, \xi_2)^\lambda + \eta_3\omega_\lambda(\xi_1, \xi_2)^\lambda \\
&\quad + \eta_4\omega_\lambda(\xi_1, \xi_2)^\lambda + \eta_5\omega_\lambda(\xi_1, \xi_2)^\lambda + \delta\omega_\lambda(\xi_2, \xi_3)^\lambda]^{\frac{1}{\lambda}} \\
&= \kappa[(\eta_1 + \eta_2 + \eta_3 + \eta_4 + \eta_5)\omega_\lambda(\xi_1, \xi_2)^\lambda + \delta\omega_\lambda(\xi_2, \xi_3)^\lambda]^{\frac{1}{\lambda}} \\
&\leq \kappa[(1 - \delta)\omega_\lambda(\xi_1, \xi_2)^\lambda + \delta\omega_\lambda(\xi_2, \xi_3)^\lambda]^{\frac{1}{\lambda}}.
\end{aligned} \tag{3.2}$$

Using powers of λ, inequality (3.2), gives

$$\omega_\lambda(\xi_2, \xi_3)^\lambda \leq \kappa^\lambda(1 - \delta)\omega_\lambda(\xi_1, \xi_2)^\lambda + \kappa^\lambda\delta\omega_\lambda(\xi_2, \xi_3)^\lambda. \tag{3.3}$$

Therefore (3.3) yields

$$(1 - \kappa^\lambda\delta)\omega_\lambda(\xi_2, \xi_3)^\lambda \leq \kappa^\lambda(1 - \delta)\omega_\lambda(\xi_1, \xi_2)^\lambda. \tag{3.4}$$

Thus, from (3.4) we get

$$\begin{aligned}
\omega_\lambda(\xi_2, \xi_3)^\lambda &\leq \frac{\kappa^\lambda(1 - \delta)}{(1 - \kappa^\lambda\delta)}\omega_\lambda(\xi_1, \xi_2)^\lambda \\
&\leq \left(\frac{\kappa^\lambda - \kappa^\lambda\delta}{1 - \kappa^\lambda\delta}\right)\omega_\lambda(\xi_1, \xi_2)^\lambda.
\end{aligned} \tag{3.5}$$

Hence (3.5) becomes

$$\begin{aligned}
\omega_\lambda(\xi_2, \xi_3) &\leq \left(\frac{\kappa^\lambda - \kappa^\lambda\delta}{1 - \kappa^\lambda\delta}\right)^{\frac{1}{\lambda}}\omega_\lambda(\xi_1, \xi_2) \\
&\leq c^{\frac{1}{\lambda}}\omega_\lambda(\xi_1, \xi_2), \text{where } c = \frac{\kappa^\lambda - \kappa^\lambda\delta}{1 - \kappa^\lambda\delta}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\omega_\lambda(\xi_3, \xi_4) &= \omega_\lambda(\Omega^2\xi_1, \Omega^2\xi_2) \leq \alpha(\xi_1, \xi_2)\omega_\lambda(\Omega^2\xi_1, \Omega^2\xi_2) \\
&\leq \kappa[\eta_1\omega_\lambda(\Omega\xi_1, \Omega\xi_2)^\lambda + \eta_2\omega_\lambda(\Omega\xi_1, \Omega^2\xi_1)^\lambda + \eta_3\omega_\lambda(\xi_2, \Omega\xi_2)^\lambda \\
&\quad + \eta_4\omega_\lambda(\Omega\xi_2, \Omega^2\xi_2)^\lambda + \eta_5\omega_\lambda(\Omega\xi_2, \Omega^2\xi_1)^\lambda + \delta\omega_\lambda(\xi_2, \Omega\xi_1)^\lambda]^{\frac{1}{\lambda}} \\
&= \kappa[\eta_1\omega_\lambda(\xi_2, \xi_3)^\lambda + \eta_2\omega_\lambda(\xi_2, \xi_3)^\lambda + \eta_3\omega_\lambda(\xi_2, \xi_3)^\lambda \\
&\quad + \eta_4\omega_\lambda(\xi_2, \xi_3)^\lambda + \eta_5\omega_\lambda(\xi_2, \xi_3)^\lambda + \delta\omega_\lambda(\xi_3, \xi_4)^\lambda]^{\frac{1}{\lambda}}
\end{aligned}$$

$$\begin{aligned} &= \kappa[(\eta_1+\eta_2+\eta_3+\eta_4+\eta_5)\omega_\lambda(\xi_2,\xi_3)^\lambda + \delta\omega_\lambda(\xi_3,\xi_4)^\lambda]^{\frac{1}{\lambda}} \\ &\leq \kappa[(1-\delta)\omega_\lambda(\xi_2,\xi_3)^\lambda + \delta\omega_\lambda(\xi_3,\xi_4)^\lambda]^{\frac{1}{\lambda}}. \end{aligned} \tag{3.6}$$

Using powers of λ inequality (3.6), gives

$$\omega_\lambda(\xi_3,\xi_4)^\lambda \leq \kappa^\lambda(1-\delta)\omega_\lambda(\xi_2,\xi_3)^\lambda + \kappa^\lambda\delta\omega_\lambda(\xi_3,\xi_4)^\lambda. \tag{3.7}$$

Therefore (3.7) yield

$$(1-\kappa^\lambda\delta)\omega_\lambda(\xi_3,\xi_4)^\lambda \leq \kappa^\lambda(1-\delta)\omega_\lambda(\xi_2,\xi_3)^\lambda. \tag{3.8}$$

Thus, from (3.8) we get

$$\begin{aligned} \omega_\lambda(\xi_3,\xi_4)^\lambda &\leq \frac{\kappa^\lambda(1-\delta)}{(1-\kappa^\lambda\delta)}\omega_\lambda(\xi_2,\xi_3)^\lambda \\ &\leq \left(\frac{\kappa^\lambda-\kappa^\lambda\delta}{1-\kappa^\lambda\delta}\right)\omega_\lambda(\xi_2,\xi_3)^\lambda. \end{aligned} \tag{3.9}$$

Hence (3.9) becomes

$$\begin{aligned} \omega_\lambda(\xi_3,\xi_4) &\leq \left(\frac{\kappa^\lambda-\kappa^\lambda\delta}{1-\kappa^\lambda\delta}\right)^{\frac{1}{\lambda}}\omega_\lambda(\xi_2,\xi_3) \\ &\leq c^{\frac{1}{\lambda}}\omega_\lambda(\xi_2,\xi_3) \\ &\leq c^{\frac{1}{\lambda}}\left[c^{\frac{1}{\lambda}}\omega_\lambda(\xi_1,\xi_2)\right] \\ &\leq c^{(\frac{1}{\lambda})^2}\omega_\lambda(\xi_1,\xi_2). \end{aligned} \tag{3.10}$$

Inductively,

$$\omega_\lambda(\xi_j,\xi_{j+1}) \leq c^{(\frac{1}{\lambda})^{j-1}}\omega_\lambda(\xi_1,\xi_2), \text{where } j\geq 2. \tag{3.11}$$

Clearly, $c=\frac{\kappa^\lambda-\kappa^\lambda\delta}{1-\kappa^\lambda\delta}<1$. Hence $c\in(0,1)$. So, by taking limit as $j\to\infty$ in (3.11) gives $\lim_{j\to\infty}\omega_\lambda(\xi_j,\xi_{j+1})=0$. According to this definition, there exists $j_\epsilon\in\mathbb{N}$ such that for every $\epsilon>0$,

$$\omega_\lambda(\Omega^j\xi_0,\Omega^{j+1}\xi_0)=\omega_\lambda(\Omega^j\xi_0,\Omega\Omega^j\xi_0)<\epsilon, \text{ for all } j\geq j_\epsilon.$$

Let $\xi=\Omega^j\xi_0$, so for every $\epsilon>0$, we see that there exists $\xi\in O_\Omega(\xi_0)$ such that $\omega_\lambda(\xi,\Omega\xi)<\epsilon$.

CASE 2: When $\lambda=0$, we have

$$\begin{aligned} \omega_\lambda(\xi_2,\xi_3) &= \omega_\lambda(\Omega^2\xi_0,\Omega^2\xi_1) \leq \alpha(\xi_0,\xi_1)\omega_\lambda(\Omega^2\xi_0,\Omega^2\xi_1) \\ &\leq \kappa[\omega_\lambda(\Omega\xi_0,\Omega\xi_1)^{\eta_1}\cdot\omega_\lambda(\Omega\xi_0,\Omega^2\xi_0)^{\eta_2}\cdot\omega_\lambda(\xi_1,\Omega\xi_1)^{\eta_3} \\ &\quad \cdot\omega_\lambda(\Omega\xi_1,\Omega^2\xi_1)^{\eta_4}\cdot\omega_\lambda(\Omega\xi_1,\Omega^2\xi_0)^{\eta_5}\cdot\omega_\lambda(\xi_1,\Omega\xi_0)^{\delta}] \\ &= \kappa[\omega_\lambda(\xi_1,\xi_2)^{\eta_1}\cdot\omega_\lambda(\xi_1,\xi_2)^{\eta_2}\cdot\omega_\lambda(\xi_1,\xi_2)^{\eta_3} \\ &\quad \cdot\omega_\lambda(\xi_1,\xi_2)^{\eta_4}\cdot\omega_\lambda(\xi_1,\xi_2)^{\eta_5}\cdot\omega_\lambda(\xi_2,\xi_3)^{\delta}] \\ &= \kappa[\omega_\lambda(\xi_1,\xi_2)^{(\eta_1+\eta_2+\eta_3+\eta_4+\eta_5)}\cdot\omega_\lambda(\xi_2,\xi_3)^{\delta}] \\ &\leq \kappa[\omega_\lambda(\xi_1,\xi_2)^{(1-\delta)}\cdot\omega_\lambda(\xi_2,\xi_3)^{\delta}], \end{aligned} \tag{3.12}$$

thus inequality (3.12), gives

$$\omega_\lambda(\xi_2,\xi_3)^{(1-\delta)} \leq \kappa\omega_\lambda(\xi_1,\xi_2)^{(1-\delta)}. \tag{3.13}$$

Therefore (3.13) yields

$$\omega_\lambda(\xi_2,\xi_3) \leq \kappa^{\frac{1}{(1-\delta)}}\omega_\lambda(\xi_1,\xi_2). \tag{3.14}$$

Similarly,

$$\begin{aligned}
\omega_\lambda(\xi_3,\xi_4) = \omega_\lambda(\Omega^2\xi_1,\Omega^2\xi_2) &\leq \alpha(\xi_1,\xi_2)\omega_\lambda(\Omega^2\xi_1,\Omega^2\xi_2)\\
&\leq \kappa[\omega_\lambda(\Omega\xi_1,\Omega\xi_2)^{\eta_1}\cdot\omega_\lambda(\Omega\xi_1,\Omega^2\xi_1)^{\eta_2}\cdot\omega_\lambda(\xi_2,\Omega\xi_2)^{\eta_3}\\
&\quad\cdot\omega_\lambda(\Omega\xi_2,\Omega^2\xi_2)^{\eta_4}\cdot\omega_\lambda(\Omega\xi_2,\Omega^2\xi_1)^{\eta_5}\cdot\omega_\lambda(\xi_2,\Omega\xi_1)^{\delta}]\\
&= \kappa[\omega_\lambda(\xi_2,\xi_3)^{\eta_1}\cdot\omega_\lambda(\xi_2,\xi_3)^{\eta_2}\cdot\omega_\lambda(\xi_2,\xi_3)^{\eta_3}\\
&\quad\cdot\omega_\lambda(\xi_2,\xi_3)^{\eta_4}\cdot\omega_\lambda(\xi_2,\xi_3)^{\eta_5}\cdot\omega_\lambda(\xi_3,\xi_4)^{\delta}]\\
&= \kappa[\omega_\lambda(\xi_2,\xi_3)^{(\eta_1+\eta_2+\eta_3+\eta_4+\eta_5)}\cdot\omega_\lambda(\xi_3,\xi_4)^{\delta}]\\
&\leq \kappa[\omega_\lambda(\xi_2,\xi_3)^{(1-\delta)}\cdot\omega_\lambda(\xi_3,\xi_4)^{\delta}].
\end{aligned} \tag{3.15}$$

Thus inequality (3.15), gives

$$\omega_\lambda(\xi_3,\xi_4)^{(1-\delta)} \leq k\omega_\lambda(\xi_2,\xi_3)^{(1-\delta)}. \tag{3.16}$$

Therefore (3.16) yields

$$\begin{aligned}
\omega_\lambda(\xi_3,\xi_4) &\leq \kappa^{\frac{1}{(1-\delta)}}\omega_\lambda(\xi_2,\xi_3)\\
&\leq \kappa^{\frac{1}{(1-\delta)}}\left[\kappa^{\frac{1}{(1-\delta)}}\omega_\lambda(\xi_1,\xi_2)\right]\\
&= \kappa^{\left(\frac{1}{1-\delta}\right)^2}\omega_\lambda(\xi_1,\xi_2).
\end{aligned}$$

Inductively,

$$\omega_\lambda(\xi_j,\xi_{j+1}) \leq \kappa^{\left(\frac{1}{1-\delta}\right)^{j-1}}\omega_\lambda(\xi_1,\xi_2), \text{where } j\geq 2. \tag{3.17}$$

So, by taking limit as $j\to\infty$ in (3.17) gives $\lim_{j\to\infty}\omega_\lambda(\xi_j,\xi_{j+1}) = 0$. According to this definition, there exists $j_\epsilon \in \mathbb{N}$ such that for every $\epsilon > 0$,

$$\omega_\lambda(\Omega^j\xi_0,\Omega^{j+1}\xi_0) = \omega_\lambda(\Omega^j\xi_0,\Omega\Omega^j\xi_0) < \epsilon, \text{for all } j\geq j_\epsilon.$$

Let $\xi = \Omega^j\xi_0$, so for every $\epsilon > 0$, we see that there exists $\xi \in O_\Omega(\xi_0)$ such that $\omega_\lambda(\xi,\Omega\xi) < \epsilon$. ∎

Theorem 3.3. *Suppose $(X_\omega,\omega_\lambda)$ is a complete modular metric space, $\Omega : X_\omega \longrightarrow X_\omega$ is a self-map. Suppose that the following conditions are satisfied:*

(i) Ω is a triangular α-admissible;

(ii) there is $\xi_0 \in X_\omega$ such that $\alpha(\xi_0,\Omega\xi_0)\geq 1$;

(iii) Ω is hybrid-interpolative Reich-Istrățescu-type contraction;

(iv) Ω^2 *is continuous.*

After that, the sequence $\{\Omega^j \xi_0\}$ *converges to a point* $\xi \in X_\omega$. *Additionally,* ξ *is a distinct fixed point of* Ω.

Proof. Let $\xi_0 \in X_\omega$ be arbitrary point. Then as in the previous result, the sequence $\{\xi_j\}$ converges to the point $\xi_0 \in X_\omega$. Since Ω is triangular α-admissible, then by Lemma 2.4, it follows that $\alpha(\xi_j, \xi_k) \geq 1$, for all $j, k \in \mathbb{N}$, $k > j$. Also, from the previous result, for each $\epsilon, \lambda > 0$, there exists $j_\epsilon \in \mathbb{N}$ such that $\omega_\lambda(\xi_{j+2}, \xi_{j+3}) < \epsilon$, for all $j \geq j_\epsilon$. Consequently, for $\frac{\lambda}{k-j} > 0$, there exists $j_{\frac{\lambda}{k-j}}$ such that $\omega_{\frac{\lambda}{k-j}}(\xi_{j+2}, \xi_{j+3}) < \frac{\epsilon}{k-j}$, $\forall\, j \geq j_{\frac{\lambda}{k-j}}$. Now,

$$\begin{aligned}
\omega_\lambda(\xi_{j+2}, \xi_{k+2}) &= \omega_{\frac{\lambda}{k-j}}(\xi_{j+2}, \xi_{j+3}) + \omega_{\frac{\lambda}{k-j}}(\xi_{j+3}, \xi_{j+4}) + \ldots + \omega_{\frac{\lambda}{k-j}}(\xi_{k+1}, \xi_{k+2}) \\
&= \frac{\lambda}{k-j} + \frac{\lambda}{k-j} + \ldots + \frac{\lambda}{k-j} \\
&= \epsilon, \text{for all } j, k \geq j_{\frac{\lambda}{k-j}}.
\end{aligned}$$

This implies that the sequence $\{\Omega^j \xi_0\}$ is Cauchy. There is a point $\xi \in X$ such that $\xi_j \to \xi$ as $j \to \infty$ according to the completeness of $(X_\omega, \omega_\lambda)$. Also, since Ω^2 is continuous, it is clear that

$$\omega_\lambda(\xi, \Omega^2 \xi) = \lim_{j \to \infty} \omega_\lambda(\xi_{j+2}, \Omega^2 \xi) = \lim_{j \to \infty} \omega_\lambda(\Omega^2 \xi_j, \Omega^2 \xi) = 0, \tag{3.18}$$

for all $\lambda > 0$. Thus, $\Omega^2 \xi = \xi$. Hence, ξ is a fixed point of Ω^2.
Consider ξ and ζ as two fixed points of Ω for uniqueness. Using Property (iii), we have two cases:
CASE 1: When $\lambda > 0$ we see that

$$\begin{aligned}
\omega_\lambda(\xi, \zeta) &= \omega_\lambda(\Omega^2 \xi, \Omega^2 \zeta) \leq \alpha(\xi, \zeta) \omega_\lambda(\Omega^2 \xi, \Omega^2 \zeta) \\
&\leq \kappa[\eta_1 \omega_\lambda(\Omega\xi, \Omega\zeta)^\lambda + \eta_2 \omega_\lambda(\Omega\xi, \Omega^2 \xi)^\lambda \\
&\quad + \eta_3 \omega_\lambda(\zeta, \Omega\zeta)^\lambda + \eta_4 \omega_\lambda(\Omega\zeta, \Omega^2 \zeta)^\lambda \\
&\quad + \eta_5 \omega_\lambda(\Omega\zeta, \Omega^2 \xi)^\lambda + \delta \omega_\lambda(\zeta, \Omega\xi)^\lambda]^{\frac{1}{\lambda}} \\
&= \kappa[\eta_1 \omega_\lambda(\xi, \zeta)^\lambda + \eta_2 \omega_\lambda(\xi, \xi)^\lambda + \eta_3 \omega_\lambda(\zeta, \zeta)^\lambda \\
&\quad + \eta_4 \omega_\lambda(\zeta, \zeta)^\lambda + \eta_5 \omega_\lambda(\zeta, \xi)^\lambda + \delta \omega_\lambda(\zeta, \xi)^\lambda]^{\frac{1}{\lambda}}.
\end{aligned} \tag{3.19}$$

Using powers of λ inequality (3.19), gives

$$\begin{aligned}
\omega_\lambda(\xi, \zeta)^\lambda &\leq \kappa^\lambda[\eta_1 \omega_\lambda(\xi, \zeta)^\lambda + \eta_2 \omega_\lambda(\xi, \xi)^\lambda + \eta_3 \omega_\lambda(\zeta, \zeta)^\lambda \\
&\quad + \eta_4 \omega_\lambda(\zeta, \zeta)^\lambda + \eta_5 \omega_\lambda(\zeta, \xi)^\lambda + \delta \omega_\lambda(\zeta, \xi)^\lambda] \\
&= \kappa^\lambda[\eta_1 \omega_\lambda(\xi, \zeta)^\lambda + \eta_5 \omega_\lambda(\xi, \zeta)^\lambda + \delta \omega_\lambda(\xi, \zeta)^\lambda] \\
&= \kappa^\lambda(\eta_1 + \eta_5 + \delta) \omega_\lambda(\xi, \zeta)^\lambda.
\end{aligned} \tag{3.20}$$

Thus, inequality (3.20) becomes

$$(1 - \kappa^\lambda(\eta_1 + \eta_5 + \delta)) \omega_\lambda(\xi, \zeta) \leq 0, \tag{3.21}$$

so, inequality (3.21) yields $\omega_\lambda(\xi,\zeta) \leq 0$. Hence $\omega_\lambda(\xi,\zeta) = 0$, for all $\lambda > 0$ and this implies that $\xi = \zeta$. Hence, ξ is the unique fixed point of Ω and the proof is complete.
CASE 2: When $\lambda > 0$, we see that

$$\begin{aligned}
\omega_\lambda(\xi,\zeta) &= \omega_\lambda(\Omega^2\xi,\Omega^2\zeta) \leq \alpha(\xi,\zeta)\omega_\lambda(\Omega^2\xi,\Omega^2\zeta) \\
&\leq \kappa[\omega_\lambda(\Omega\xi,\Omega\zeta)^{\eta_1} \cdot \omega_\lambda(\Omega\xi,\Omega^2\xi)^{\eta_2} \\
&\quad \cdot \omega_\lambda(\zeta,\Omega\zeta)^{\eta_3} \cdot \omega_\lambda(\Omega\zeta,\Omega^2\zeta)^{\eta_4} \\
&\quad \cdot \omega_\lambda(\Omega\zeta,\Omega^2\xi)^{\eta_5} \cdot \omega_\lambda(\zeta,\Omega\xi)^{\delta}] \\
&= \kappa[\omega_\lambda(\xi,\zeta)^{\eta_1} \cdot \omega_\lambda(\xi,\xi)^{\eta_2} \cdot \omega_\lambda(\zeta,\zeta)^{\eta_3} \\
&\quad \cdot \omega_\lambda(\zeta,\zeta)^{\eta_4} \cdot \omega_\lambda(\zeta,\xi)^{\eta_5} \cdot \omega_\lambda(\zeta,\xi)^{\delta}] \\
&\leq \kappa[\omega_\lambda(\xi,\zeta)^{\eta_1} \cdot \omega_\lambda(\xi,\zeta)^{\eta_5} \cdot \omega_\lambda(\xi,\zeta)^{\delta}] \\
&\leq \kappa[\omega_\lambda(\xi,\zeta)^{\eta_1+\eta_5+\delta}].
\end{aligned} \tag{3.22}$$

Thus, inequality (3.22), becomes

$$\omega_\lambda(\xi,\zeta)^{(1-(\eta_1+\eta_5+\delta))} \leq \kappa, \tag{3.23}$$

hence, inequality (3.23) gives

$$\omega_\lambda(\xi,\zeta) \leq \kappa^{(1-(\eta_1+\eta_5+\delta))}, \tag{3.24}$$

letting $\kappa \to \infty$ in inequality (3.24), yields $\omega_\lambda(\xi,\zeta) \leq 0$, hence $\omega_\lambda(\xi,\zeta) = 0$, for all $\lambda > 0$ and this implies that $\xi = \zeta$. Hence, ξ is the unique fixed point of Ω and the proof is complete. ∎

Corollary 3.4. *Let $(X_\omega,\omega_\lambda)$ be a complete modular metric space, $\Omega : X_\omega \longrightarrow X_\omega$ be an operator. Suppose that the following conditions are satisfied:*

(i) Ω is a triangular α-admissible;

(ii) there is $\xi_0 \in X_\omega$ such that $\alpha(\xi_0,\Omega\xi_0) \geq 1$;

(iii) there are $\eta_1,\eta_2,\eta_3,\eta_5 \in [0,1)$, with $\eta_1+\eta_2+\eta_3+\eta_5 \leq 1$, $\kappa \in (0,1)$ such that

$$\begin{aligned}
\alpha(\xi,\zeta)\omega_\lambda(\Omega^2\xi,\Omega^2\zeta) \leq \kappa[&\eta_1\omega_\lambda(\Omega\xi,\Omega\zeta) + \eta_2\omega_\lambda(\Omega\xi,\Omega^2\xi) \\
&+ \eta_3\omega_\lambda(\zeta,\Omega\zeta) + \eta_5\omega_\lambda(\Omega\zeta,\Omega^2\xi)],
\end{aligned}$$

for all $\xi,\zeta \in O_\Omega(\xi_0)$, $\lambda > 0$;

(iv) Ω^2 is continuous.

After that, the sequence $\{\Omega^j\xi_0\}$ converges to a point $\xi \in X_\omega$. Additionally, ξ is a distinct fixed point of Ω.

Proof. Setting $\eta_4 = \delta = 0$ in Theorem 3.3, the result follows immediately. ∎

Corollary 3.5. *Let $(X_\omega,\omega_\lambda)$ be a complete modular metric space, $\Omega : X_\omega \longrightarrow X_\omega$ be an operator. Suppose that the following conditions are satisfied:*

(i) Ω is a triangular α-admissible;

(ii) *there is* $\xi_0 \in X_\omega$ *such that* $\alpha(\xi_0, \Omega\xi_0) \geq 1$;

(iii) $\eta_1, \eta_2, \eta_4, \delta \in [0, 1)$, *with* $\eta_1 + \eta_2 + \eta_4 + \delta \leq 1$, *such that*

$$\alpha(\xi, \zeta)\omega_\lambda(\Omega^2\xi, \Omega^2\zeta) \leq \kappa[\eta_1\omega_\lambda(\Omega\xi, \Omega\zeta) + \eta_2\omega_\lambda(\Omega\xi, \Omega^2\xi) + \eta_4\omega_\lambda(\Omega\zeta, \Omega^2\zeta) + \eta_6\omega_\lambda(\zeta, \Omega\xi)],$$

for all $\xi, \zeta \in O_\Omega(\xi_0)$, $\lambda > 0$;

(iv) Ω^2 *is continuous.*

After that, the sequence $\{\Omega^j\xi_0\}$ *converges to a point* $\xi \in X_\omega$. *Additionally,* ξ *is a distinct fixed point of* Ω.

Corollary 3.6. *Let* $(X_\omega, \omega_\lambda)$ *be a complete modular metric space,* $\Omega : X_\omega \longrightarrow X_\omega$ *be an operator. Suppose that the following conditions are satisfied:*

(i) Ω *is a triangular* α*-admissible;*

(ii) *there is* $\xi_0 \in X_\omega$ *such that* $\alpha(\xi_0, \Omega\xi_0) \geq 1$;

(iii) *there are* $\eta_1, \eta_2, \eta_3, \eta_5 \in [0, 1)$, *with* $\eta_1 + \eta_2 + \eta_3 + \eta_5 \leq 1$, $\kappa \in (0, 1)$ *such that*

$$\alpha(\xi, \zeta)\omega_\lambda(\Omega^2\xi, \Omega^2\zeta) \leq \kappa[\omega_\lambda(\Omega\xi, \Omega\zeta)^{\eta_1} \cdot \omega_\lambda(\Omega\xi, \Omega^2\xi)^{\eta_2} \cdot \omega_\lambda(\zeta, \Omega\zeta)^{\eta_3} \cdot \omega_\lambda(\Omega\zeta, \Omega^2\xi)^{\eta_5}],$$

for all $\xi, \zeta \in O_\Omega(\xi_0)$, $\lambda = 0$;

(iv) Ω *is continuous.*

After that, the sequence $\{\Omega^j\xi_0\}$ *converges to a point* $\xi \in X_\omega$. *Additionally,* ξ *is a distinct fixed point of* Ω.

Proof. Setting $\eta_4 = \delta = 0$ in Theorem 3.3, the result follows immediately. ∎

Corollary 3.7. *Let* $(X_\omega, \omega_\lambda)$ *be a complete modular metric space,* $\Omega : X_\omega \longrightarrow X_\omega$ *be an operator. Suppose that the following conditions are satisfied:*

(i) Ω *is a triangular* α*-admissible;*

(ii) *there is* $\xi_0 \in X_\omega$ *such that* $\alpha(\xi_0, \Omega\xi_0) \geq 1$;

(iii) $\eta_1, \eta_2, \eta_4, \delta \in [0, 1)$, *with* $\eta_1 + \eta_2 + \eta_4 + \delta \leq 1$, $\kappa \in (0, 1)$ *such that*

$$\alpha(\xi, \zeta)\omega_\lambda(\Omega^2\xi, \Omega^2\zeta) \leq \kappa[\omega_\lambda(\Omega\xi, \Omega\zeta)^{\eta_1} \cdot \omega_\lambda(\Omega\xi, \Omega^2\xi)^{\eta_2} \cdot \omega_\lambda(\Omega\zeta, \Omega^2\zeta)^{\eta_4} \cdot \omega_\lambda(\zeta, \Omega\xi)^{\delta}],$$

for all $\xi, \zeta \in O_\Omega(\xi_0)$, $\lambda = 0$;

(iv) Ω *is continuous.*

After that, the sequence $\{\Omega^j\xi_0\}$ *converges to a point* $\xi \in X_\omega$. *Additionally,* ξ *is a distinct fixed point of* Ω.

Other important corollaries can be obtained by considering particular cases of modular metric spaces and by varying the constant $\eta_i \in [0,1)$, for $i = 1,2,3,4,5$ and $\delta \geq 0$. An example is constructed in this section to bolster the assumptions of Theorem 3.3.

Example 3.8. Let $X_\omega = [0,1]$ endowed with the metric $\omega_\lambda : X_\omega \times X_\omega \to [0,\infty]$ defined by

$$\omega_\lambda(\xi,\zeta) = \begin{cases} \frac{|\xi-\zeta|}{\lambda} & \text{if } \xi = \zeta, \\ 0 & \textit{otherwise}. \end{cases}$$

For all $\xi,\zeta \in X_\omega$ with $\xi \leq \zeta$ and $\lambda > 0$. Clearly, $(X_\omega, \omega_\lambda)$ is a complete modular metric space. Choose

$$\alpha(\xi,\zeta) = \begin{cases} 1 & \text{if } \xi,\zeta \in [0,\infty], \\ \frac{1}{2} & \textit{otherwise}. \end{cases}$$

Define $\Omega : X_\omega \to X_\omega$ as $\Omega\xi = \frac{\xi}{3}$ for all $\xi \in X_\omega$ and take $\eta_1 = \frac{1}{6}$, $\eta_2 = \frac{1}{4}$, $\eta_3 = \frac{1}{8}$, $\eta_4 = \frac{1}{16}$, $\eta_5 = \delta = \frac{1}{32}$. Then $\eta_1 + \eta_2 + \eta_3 + \eta_4 + \eta_5 + \delta \leq 1$. Furthermore,

$$\begin{aligned}
\omega_\lambda(\Omega^2\xi,\Omega^2\zeta) &= \frac{|\Omega^2\xi - \Omega^2\zeta|}{\lambda} \\
&= \frac{|\xi-\zeta|}{9\lambda} \\
&= \frac{|\xi-\zeta|}{18\lambda} + \frac{|\xi-\zeta|}{18\lambda} \\
&= \frac{|\xi-\zeta|}{18\lambda} + \frac{|\xi-\zeta|}{36\lambda} + \frac{|\xi-\zeta|}{36\lambda} \\
&= \frac{|\xi-\zeta|}{18\lambda} + \frac{|\xi-\zeta|}{36\lambda} + \frac{|\xi-\zeta|}{72\lambda} + \frac{|\xi-\zeta|}{72\lambda} \\
&= \frac{|\xi-\zeta|}{18\lambda} + \frac{|\xi-\zeta|}{36\lambda} + \frac{|\xi-\zeta|}{72\lambda} + \frac{|\xi-\zeta|}{144\lambda} + \frac{|\xi-\zeta|}{144\lambda} \\
&= \frac{|\xi-\zeta|}{18\lambda} + \frac{|\xi-\zeta|}{36\lambda} + \frac{|\xi-\zeta|}{72\lambda} + \frac{|\xi-\zeta|}{144\lambda} + \frac{|\xi-\zeta|}{288\lambda} + \frac{|\xi-\zeta|}{288\lambda} \\
&= \frac{|\frac{\xi}{3}-\frac{\zeta}{3}|}{6\lambda} + \frac{|\frac{\xi}{9}-\frac{\zeta}{9}|}{4\lambda} + \frac{|\frac{\xi}{9}-\frac{\zeta}{9}|}{8\lambda} + \frac{|\frac{\xi}{9}-\frac{\zeta}{9}|}{16\lambda} + \frac{|\frac{\xi}{9}-\frac{\zeta}{9}|}{32\lambda} + \frac{|\frac{\xi}{9}-\frac{\zeta}{9}|}{32\lambda} \\
&\leq \frac{|\frac{\xi}{3}-\frac{\zeta}{3}|}{6\lambda} + \frac{|\frac{\xi}{3}-\frac{\zeta}{9}|}{4\lambda} + \frac{|\xi-\frac{\zeta}{3}|}{8\lambda} + \frac{|\frac{\xi}{3}-\frac{\zeta}{9}|}{16\lambda} + \frac{|\frac{\xi}{9}-\frac{\zeta}{3}|}{32\lambda} + \frac{|\frac{\xi}{3}-\zeta|}{32\lambda} \\
&\leq \frac{|\frac{\xi}{3}-\frac{\zeta}{3}|}{6\lambda} + \frac{|\frac{\xi}{3}-\frac{\zeta}{9}+\frac{\zeta}{9}-\frac{\xi}{9}|}{4\lambda} + \frac{|\xi-\frac{\zeta}{3}|}{8\lambda} + \frac{|\frac{\xi}{3}-\frac{\zeta}{9}|}{16\lambda} + \frac{|\frac{\xi}{9}-\frac{\zeta}{3}|}{32\lambda} + \frac{|\frac{\xi}{3}-\zeta|}{32\lambda} \\
&\leq \frac{|\frac{\xi}{3}-\frac{\zeta}{3}|}{6\lambda} + \frac{|\frac{\xi}{3}-\frac{\xi}{9}|+|\frac{\zeta}{9}-\frac{\zeta}{9}|}{4\lambda} + \frac{|\xi-\frac{\zeta}{3}|}{8\lambda} + \frac{|\frac{\xi}{3}-\frac{\zeta}{9}|}{16\lambda} + \frac{|\frac{\xi}{9}-\frac{\zeta}{3}|}{32\lambda} + \frac{|\frac{\xi}{3}-\zeta|}{32\lambda} \\
&\leq \frac{|\frac{\xi}{3}-\frac{\zeta}{3}|}{6\lambda} + \frac{|\frac{\xi}{3}-\frac{\xi}{9}|+|\frac{\zeta}{9}-\frac{\zeta}{9}|}{4\lambda} + \frac{|\zeta-\frac{\zeta}{3}|}{8\lambda} + \frac{|\frac{\zeta}{3}-\frac{\zeta}{9}|}{16\lambda} + \frac{|\frac{\xi}{9}-\frac{\zeta}{3}|}{32\lambda} + \frac{|\frac{\xi}{3}-\zeta|}{32\lambda} \\
&\leq \frac{|\frac{\xi}{3}-\frac{\zeta}{3}|}{6\lambda} + \frac{|\frac{\xi}{3}-\frac{\xi}{9}|}{4\lambda} + \frac{|\zeta-\frac{\zeta}{3}|}{8\lambda} + \frac{|\frac{\zeta}{3}-\frac{\zeta}{9}|}{16\lambda} + \frac{|\frac{\xi}{9}-\frac{\zeta}{3}|}{32\lambda} + \frac{|\frac{\xi}{3}-\zeta|}{32\lambda} \\
&\leq \frac{1}{6}\omega_\lambda(\Omega\xi,\Omega\zeta) + \frac{1}{4}\omega_\lambda(\Omega\xi,\Omega^2\xi) + \frac{1}{8}\omega_\lambda(\zeta,\Omega\zeta)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{16}\omega_\lambda(\Omega\zeta,\Omega^2\zeta)+\frac{1}{32}\omega_\lambda(\Omega\zeta,\Omega^2\xi)+\frac{1}{32}\omega_\lambda(\zeta,\Omega\xi)\\
\le\ &\eta_1\omega_\lambda(\Omega\xi,\Omega\zeta)+\eta_2\omega_\lambda(\Omega\xi,\Omega^2\xi)+\eta_3\omega_\lambda(\zeta,\Omega\zeta)\\
&+\eta_4\omega_\lambda(\Omega\zeta,\Omega^2\zeta)+\eta_5\omega_\lambda(\Omega\zeta,\Omega^2\xi)+\delta\omega_\lambda(\zeta,\Omega\xi).
\end{aligned}
$$

This indicates that Ω meets every requirement of Theorem 3.2. Because of this, Ω has a unique fixed point $\xi = 0$.

4. Conclusion

In connection with modular metric spaces, this study has established types of contractions and related Theorems of fixed points. Theorem 3.2 dealt with the requirements for the existence of an approximate fixed point, while the existence and uniqueness of the fixed point for the specified operator were established by Theorem 3.3. The hypothesis of the primary finding is supported by Example 3.8. Moreover, several relevant findings in the literature have been found to be improved by the notion in these studies.

Competing Interests

The authors declare that they have no competing interests.

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