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Different Bounds for an Original Double Integral Involving Concave Functions

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ABSTRACT

Establishing bounds for integrals involving concave functions is an important topic in mathematical analysis. In this article, we investigate a class of double integrals whose integrands are given by the ratio of the sum of two concave functions. This form is particularly interesting because of its connection with well-known inequalities of Hilbert integral type. We derive a series of upper and lower bounds using various valuable techniques, leading to elegant and new results. Where appropriate, multiple proofs of the same result are given. Numerical examples are also used to illustrate and support the findings.

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1. Introduction

1.1. Framework

The concept of concavity is fundamental to mathematical analysis. In particular, it plays a central role in the study of functional inequalities. In this article, we focus on concave functions because of their rich structural properties. More specifically, they allow for the establishment of sharp bounds and meaningful comparisons between function values and integrals. Formally, let $a, b \in \mathbb{R}$ with $a < b$, and let $f : [a, b] \mapsto \mathbb{R}$ be a (real-valued) function. The function f is said to be concave if and only if, for all $\lambda \in [0, 1]$ and all $x, y \in [a, b]$, the following inequality holds:

$$f[\lambda x + (1 - \lambda)y] \geq \lambda f(x) + (1 - \lambda)f(y).$$

This inequality expresses that the graph of f lies above the chord joining any two points on its graph. If the function f is twice differentiable on $[a, b]$, then concavity is equivalent to the second derivative being non-positive throughout the interval, i.e.,

$$f''(x) \leq 0 \quad \text{for all } x \in [a, b].$$

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Conversely, f is called convex if and only if, for all $\lambda \in [0,1]$ and all $x, y \in [a,b]$, the reverse inequality holds:

$$f[\lambda x + (1-\lambda)y] \leq \lambda f(x) + (1-\lambda) f(y).$$

If f is twice differentiable, then convexity is equivalent to

$$f''(x) \geq 0 \quad \text{for all } x \in [a,b].$$

These well-structured properties facilitate the derivation of inequalities that are both theoretically significant and practically useful. We refer to the foundational works in [8, 11, 12, 13, 2, 3, 15, 16, 21, 17, 10, 19].

A central problem in analysis involves deriving sharp upper and lower bounds for integrals involving concave (or convex) functions. One of the most renowned results in this context is the Hermite-Hadamard two-sided integral inequality. Formally, let $a, b \in \mathbb{R}$ with $a < b$, and let $f : [a,b] \mapsto \mathbb{R}$ be a concave function. Then we have

$$\frac{1}{2}[f(a) + f(b)] \leq \frac{1}{b-a}\int_a^b f(x)dx \leq f\left(\frac{a+b}{2}\right).$$

Conversely, if f is convex instead of concave, then the inequality is reversed, i.e.,

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{1}{2}[f(a) + f(b)].$$

Another famous integral inequality involving concave (or convex) functions include the Jensen inequality. Formally, let $a, b \in \mathbb{R}$ with $a < b$, let $f : \mathbb{R} \mapsto \mathbb{R}$ be a concave function, let $g : [a,b] \mapsto \mathbb{R}$ be a function, and let $p : [a,b] \mapsto [0,+\infty)$ satisfying $\int_a^b p(x)dx = 1$. Then we have

$$\int_a^b f[g(x)]p(x)dx \leq f\left[\int_a^b g(x)p(x)dx\right].$$

Conversely, if f is convex instead of concave, then the inequality is reversed, i.e.,

$$f\left[\int_a^b g(x)p(x)dx\right] \leq \int_a^b f[g(x)]p(x)dx.$$

These fundamental inequalities have many applications in areas such as optimization, probability theory, numerical integration and functional analysis. Over time, they have inspired numerous generalizations, refinements, and extensions. For further developments and references, see [19, 4, 6, 22, 25, 14, 20, 28, 7, 27, 1, 18, 26, 23, 24, 5].

1.2. Motivations

In this article, we consider an original integral inequality problem involving mainly concave functions. Formally, let $a, b \in \mathbb{R}$ with $a < b$, and let $f, g : [a,b] \mapsto [0,+\infty)$ be two concave functions. Then we want to find pertinent lower and upper bounds for the following double integral:

$$\int_a^b \int_a^b \frac{1}{f(x) + g(y)} dxdy. \tag{1.1}$$

This problem was inspired by the work of W.T. Sulaiman [25] (2008), who investigated a more general case by incorporating an exponent into the denominator of the form $[f(x)+g(y)]^{\iota}$ with $\iota > 0$, along with additional assumptions on f and g, i.e., $g(b) \leq f(a)+2g(a)$ and $f(b) \leq g(a)+2f(a)$. The results involved complex bounds, including those expressed via the beta function, the inclusion of three additional parameters, δ, γ and ϵ, which also interact with ι, and required technical arguments.

In contrast, we focus on the simpler case $\iota = 1$, which allows us to derive more accessible bounds using elementary tools. In this sense, we revisit the approach in [25] from a different perspective, highlighting simpler techniques applicable in this specific setting.

Moreover, the structure of the double integral resembles that of Hilbert-type integral inequalities, which are well known in functional analysis. See [9, 29, 30]. Our results may thus lead to new variants of such inequalities, developed through the prism of concave (or convex) analysis.

1.3. Contributions

We distinguish between the lower and upper bounds of the double integral in Equation (1.1). In both cases, several bounds are established using different techniques that exploit the concavity of the functions. As a representative example of the lower bounds, and as an introductory step, we highlight the following elegant inequality:

$$\int_a^b \int_a^b \frac{1}{f(x)+g(y)} dxdy \geq \frac{(b-a)^2}{f[(a+b)/2]+g[(a+b)/2]}.$$

A similar approach is taken for the upper bounds. In this case, we present the following sharp inequality:

$$\int_a^b \int_a^b \frac{1}{f(x)+g(y)} dxdy \leq \frac{(b-a)^2}{4}\sqrt{\left[\frac{1}{f(a)}+\frac{1}{f(b)}\right]\left[\frac{1}{g(a)}+\frac{1}{g(b)}\right]}.$$

For these two examples, the assumptions and results in [28] are significantly simplified and improved. Beyond these, we derive a variety of more refined and technical bounds, using minimal auxiliary results. Complete proofs are provided to ensure full mathematical rigor.

1.4. Article Organization

The rest of this article is organized as follows: In Section 2, we present preliminary results, including two lemmas of independent interest and a proposition that lays the foundation for our main analysis. Section 3 is devoted to deriving new lower bounds for the double integral in Equation (1.1), while Section 4 presents the corresponding upper bounds. Finally, Section 5 concludes the article with a summary and possible directions for future work.

2. Preliminary results

As a technical note, in all our statements and proofs, it will always be implicitly assumed that the denominator terms are not equal to 0, so that all the ratio terms are relevant in a mathematical sense.

2.1. Two Lemmas

We now present two lemmas which serve as intermediate results to our main results.

The lemma below relates the nature of a function and its ratio form to concavity and convexity.

Lemma 2.1. *Let $a, b \in \mathbb{R}$ with $a < b$, and let $f : [a, b] \mapsto [0, +\infty)$ be a twice differentiable concave function. Then the ratio function $1/f$ is convex.*

Proof. Using standard differentiation rules, we have

$$\left(\frac{1}{f(x)}\right)' = -\frac{f'(x)}{[f(x)]^2}, \quad \left(\frac{1}{f(x)}\right)'' = \left[\left(\frac{1}{f(x)}\right)'\right]' = \frac{2[f'(x)]^2 - f(x)f''(x)}{[f(x)]^3}.$$

We have $f(x) \geq 0$ for all $x \in [a, b]$ and, since f is twice differentiable and concave, we have $f''(x) \leq 0$ for all $x \in [a, b]$. This implies that

$$\left(\frac{1}{f(x)}\right)'' = \frac{2[f'(x)]^2 + f(x)[-f''(x)]}{[f(x)]^3} \geq 0,$$

which means that $1/f$ is convex. This ends the proof of Lemma 2.1. ∎

The reverse of this lemma is not always true: if f is convex, then $1/f$ is not necessarily concave. For example, consider $f(x) = e^x$, which is convex, but $1/f(x) = e^{-x}$ is obviously also convex. So Lemma 2.1 is not as trivial as it seems at first sight.

The lemma below shows a sharp upper bound on $1/(u + v)$ in terms of $1/u$ and $1/v$. Of interest is the introduction of two adjustable parameters which allow modulating $1/u$ and $1/v$.

Lemma 2.2. *For all $\alpha, \beta \geq 0$ with $\alpha + \beta > 0$ and $u, v > 0$, we have*

$$\frac{1}{u+v} \leq \frac{\alpha}{\left[\sqrt{\alpha} + \sqrt{\beta}\right]^2} \times \frac{1}{u} + \frac{\beta}{\left[\sqrt{\alpha} + \sqrt{\beta}\right]^2} \times \frac{1}{v}.$$

Proof. Considering the natural difference function and developing it using standard analytical tools, we get

$$\begin{aligned}
&\frac{\alpha}{\left[\sqrt{\alpha} + \sqrt{\beta}\right]^2} \times \frac{1}{u} + \frac{\beta}{\left[\sqrt{\alpha} + \sqrt{\beta}\right]^2} \times \frac{1}{v} - \frac{1}{u+v} \\
&= \frac{1}{uv(u+v)} \left\{ \frac{\alpha}{\left[\sqrt{\alpha} + \sqrt{\beta}\right]^2} v(u+v) + \frac{\beta}{\left[\sqrt{\alpha} + \sqrt{\beta}\right]^2} u(u+v) - uv \right\} \\
&= \frac{1}{uv(u+v)} \left[\frac{\alpha}{\left[\sqrt{\alpha} + \sqrt{\beta}\right]^2} v^2 + \frac{\beta}{\left[\sqrt{\alpha} + \sqrt{\beta}\right]^2} u^2 + uv \left\{ \frac{\alpha + \beta - \left[\sqrt{\alpha} + \sqrt{\beta}\right]^2}{\left[\sqrt{\alpha} + \sqrt{\beta}\right]^2} \right\} \right] \\
&= \frac{1}{uv(u+v)\left[\sqrt{\alpha} + \sqrt{\beta}\right]^2} \left[\alpha v^2 + \beta u^2 - 2\sqrt{\alpha}\sqrt{\beta} uv \right] \\
&= \frac{1}{uv(u+v)\left[\sqrt{\alpha} + \sqrt{\beta}\right]^2} \left[\sqrt{\alpha} v - \sqrt{\beta} u \right]^2 \geq 0.
\end{aligned}$$

This concludes the proof of Lemma 2.2. ∎

In particular, if we take $\alpha = 1$ and $\beta = 1$, we get the following elegant inequality:

$$\frac{1}{u+v} \leq \frac{\alpha}{\left[\sqrt{\alpha}+\sqrt{\beta}\right]^2} \times \frac{1}{u} + \frac{\beta}{\left[\sqrt{\alpha}+\sqrt{\beta}\right]^2} \times \frac{1}{v} = \frac{1}{4}\left(\frac{1}{u}+\frac{1}{v}\right).$$

This significantly improves the natural inequality $1/(u+v) \leq 1/u + 1/v$, gaining the factor constant $1/4$.

2.2. A Basic Proposition

The proposition below is part of our study because it departs from the concavity assumption; it provides valuable lower and upper bounds on the double integral of interest in the simple case where the functions are monotonic. Since such functions are more fundamental than concave (or convex) functions, this result must be seen as a benchmark for future work.

Proposition 2.3. *Let $a, b \in \mathbb{R}$ with $a < b$, and let $f, g : [a, b] \mapsto [0, +\infty)$ be two monotonic functions. Then the following two-sided inequality holds:*

$$\frac{(b-a)^2}{\min[f(a), f(b)] + \min[g(a), g(b)]} \geq \int_a^b \int_a^b \frac{1}{f(x)+g(y)} dxdy$$
$$\geq \frac{(b-a)^2}{\max[f(a), f(b)] + \max[g(a), g(b)]}.$$

Proof. Since f and g are monotonic functions, for all $x, y \in [a, b]$, we have $\max[f(a), f(b)] \geq f(x) \geq \min[f(a), f(b)]$ and $\max[g(a), g(b)] \geq g(y) \geq \min[g(a), g(b)]$, so that

$$\max[f(a), f(b)] + \max[g(a), g(b)] \geq f(x) + g(y) \geq \min[f(a), f(b)] + \min[g(a), g(b)].$$

We therefore have

$$\frac{(b-a)^2}{\min[f(a), f(b)] + \min[g(a), g(b)]} = \int_a^b \int_a^b \frac{1}{\min[f(a), f(b)] + \min[g(a), g(b)]} dxdy$$
$$\geq \int_a^b \int_a^b \frac{1}{f(x)+g(y)} dxdy \geq \int_a^b \int_a^b \frac{1}{\max[f(a), f(b)] + \max[g(a), g(b)]} dxdy$$
$$= \frac{(b-a)^2}{\max[f(a), f(b)] + \max[g(a), g(b)]}.$$

This can be reduced to

$$\frac{(b-a)^2}{\min[f(a), f(b)] + \min[g(a), g(b)]} \geq \int_a^b \int_a^b \frac{1}{f(x)+g(y)} dxdy$$
$$\geq \frac{(b-a)^2}{\max[f(a), f(b)] + \max[g(a), g(b)]}.$$

The proof of Proposition 2.3 ends. ∎

In particular, if $f = g$, then we have

$$\frac{(b-a)^2}{2\min[f(a), f(b)]} \geq \int_a^b \int_a^b \frac{1}{f(x)+f(y)} dxdy \geq \frac{(b-a)^2}{2\max[f(a), f(b)]}.$$

We are now able to give sharp lower and upper bounds on the double integral of interest under concavity assumptions, starting with the lower bounds.

3. Lower bounds

3.1. Main Theorem

The theorem below proposes a lower bound for our main double integral. The proof is based on a suitable decomposition of the integrand and Lemma 2.2.

Theorem 3.1. *Let $a, b \in \mathbb{R}$ with $a < b$, and let $f, g : [a, b] \mapsto [0, +\infty)$ be two concave functions. Then, for all $\alpha, \beta \geq 0$ with $\alpha + \beta > 0$, the following inequality holds:*

$$\int_a^b \int_a^b \frac{1}{f(x) + g(y)} dxdy \geq \frac{(b-a)^2 \left[\sqrt{\alpha} + \sqrt{\beta}\right]^2}{2(\alpha + \beta)\left\{f[(a+b)/2] + g[(a+b)/2]\right\}}.$$

Proof. We can write

$$\frac{(b-a)^2}{f[(a+b)/2] + g[(a+b)/2]} = \frac{1}{f[(a+b)/2] + g[(a+b)/2]} \int_a^b \int_a^b dxdy$$
$$= \int_a^b \int_a^b \frac{1}{f[(a+b)/2] + g[(a+b)/2]} dxdy$$
$$= \int_a^b \int_a^b \frac{1}{f[(a+b-x+x)/2] + g[(a+b-y+y)/2]} dxdy. \qquad (3.1)$$

Using the concave property of f and g, we have

$$f\left(\frac{a+b-x+x}{2}\right) + g\left(\frac{a+b-y+y}{2}\right)$$
$$\geq \frac{1}{2}[f(a+b-x) + f(x)] + \frac{1}{2}[g(a+b-y) + g(y)]$$
$$= \frac{1}{2}[f(a+b-x) + f(x) + g(a+b-y) + g(y)],$$

so that

$$\int_a^b \int_a^b \frac{1}{f[(a+b-x+x)/2] + g[(a+b-y+y)/2]} dxdy$$
$$\leq 2\int_a^b \int_a^b \frac{1}{f(a+b-x) + f(x) + g(a+b-y) + g(y)} dxdy$$
$$= 2\int_a^b \int_a^b \frac{1}{[f(a+b-x) + g(a+b-y)] + [f(x) + g(y)]} dxdy. \qquad (3.2)$$

Using Lemma 2.2, which introduces α and β, we get

$$\int_a^b \int_a^b \frac{1}{[f(a+b-x) + g(a+b-y)] + [f(x) + g(y)]} dxdy$$
$$\leq \frac{\alpha}{\left[\sqrt{\alpha} + \sqrt{\beta}\right]^2} \int_a^b \int_a^b \frac{1}{f(a+b-x) + g(a+b-y)} dxdy$$
$$+ \frac{\beta}{\left[\sqrt{\alpha} + \sqrt{\beta}\right]^2} \int_a^b \int_a^b \frac{1}{f(x) + g(y)} dxdy. \qquad (3.3)$$

Applying the change of variables $(s,t) = (a+b-x, a+b-y)$ in the first double integral, we obtain

$$\begin{aligned}
&\frac{\alpha}{\left[\sqrt{\alpha}+\sqrt{\beta}\right]^2}\int_a^b\int_a^b \frac{1}{f(a+b-x)+g(a+b-y)}dxdy \\
&+\frac{\beta}{\left[\sqrt{\alpha}+\sqrt{\beta}\right]^2}\int_a^b\int_a^b \frac{1}{f(x)+g(y)}dxdy \\
&=\frac{\alpha}{\left[\sqrt{\alpha}+\sqrt{\beta}\right]^2}\int_a^b\int_a^b \frac{1}{f(s)+g(t)}dsdt+\frac{\beta}{\left[\sqrt{\alpha}+\sqrt{\beta}\right]^2}\int_a^b\int_a^b \frac{1}{f(x)+g(y)}dxdy \\
&=\frac{\alpha+\beta}{\left[\sqrt{\alpha}+\sqrt{\beta}\right]^2}\int_a^b\int_a^b \frac{1}{f(x)+g(y)}dxdy. \qquad (3.4)
\end{aligned}$$

It follows from Equations (3.1), (3.2), (3.3) and (3.4) that

$$\frac{(b-a)^2}{f[(a+b)/2]+g[(a+b)/2]} \leq \frac{2(\alpha+\beta)}{\left[\sqrt{\alpha}+\sqrt{\beta}\right]^2}\int_a^b\int_a^b \frac{1}{f(x)+g(y)}dxdy,$$

so that

$$\int_a^b\int_a^b \frac{1}{f(x)+g(y)}dxdy \geq \frac{(b-a)^2\left[\sqrt{\alpha}+\sqrt{\beta}\right]^2}{2(\alpha+\beta)\left\{f[(a+b)/2]+g[(a+b)/2]\right\}}.$$

This concludes the proof of Theorem 3.1. ∎

Note that, for all $\alpha, \beta \geq 0$ such that $\alpha+\beta > 0$, we have $\left[\sqrt{\alpha}+\sqrt{\beta}\right]^2 - (\alpha+\beta) = \sqrt{\alpha}\sqrt{\beta} \geq 0$. This implies that

$$\begin{aligned}
&\int_a^b\int_a^b \frac{1}{f(x)+g(y)}dxdy \geq \frac{(b-a)^2\left[\sqrt{\alpha}+\sqrt{\beta}\right]^2}{2(\alpha+\beta)\left\{f[(a+b)/2]+g[(a+b)/2]\right\}} \\
&\geq \frac{(b-a)^2}{2}\times\frac{1}{f[(a+b)/2]+g[(a+b)/2]} \geq \frac{(b-a)^2}{4}\times\frac{1}{f[(a+b)/2]+g[(a+b)/2]}.
\end{aligned}$$

This last lower bound is the one obtained in [25, Theorem 3.1] with $\iota = 1$. In this case, therefore, we significantly improve it for all choices of α and β.

Obviously, the introduction of α and β adds a new degree of flexibility. For example, if we take $\alpha = 1$ and $\beta = 1$, then we have

$$\int_a^b\int_a^b \frac{1}{f(x)+g(y)}dxdy \geq \frac{(b-a)^2}{f[(a+b)/2]+g[(a+b)/2]}.$$

As another example, assuming $a, b > 0$, choosing $\alpha = a$ and $\beta = b$ leads to the following original inequality:

$$\int_a^b\int_a^b \frac{1}{f(x)+g(y)}dxdy \geq \frac{(b-a)^2\left[\sqrt{a}+\sqrt{b}\right]^2}{2(a+b)\left\{f[(a+b)/2]+g[(a+b)/2]\right\}}.$$

A more elegant choice will be proposed in Subsection 3.3.

3.2. Alternative Proof of Theorem 3.1

A new proof of a theorem is of multiple interest. It may reveal connections to other areas, or it may simplify understanding through alternative techniques. In this spirit, we now present an alternative proof of Theorem 3.1, but only for the special case $\alpha = 1$ and $\beta = 1$. It is mainly based on the Cauchy-Schwarz double integral inequality and the "concave version" of the Hermite-Hadamard two-sided integral inequality.

Alternative proof of Theorem 3.1 for the case $\alpha = 1$ and $\beta = 1$. An appropriate decomposition of the integrand and the Cauchy-Schwarz double integral inequality give

$$(b-a)^2 = \int_a^b \int_a^b dxdy = \int_a^b \int_a^b \frac{\sqrt{f(x)+g(y)}}{\sqrt{f(x)+g(y)}} dxdy$$
$$\leq \sqrt{\int_a^b \int_a^b \frac{1}{f(x)+g(y)} dxdy} \sqrt{\int_a^b \int_a^b [f(x)+g(y)] dxdy}$$
$$= \sqrt{\int_a^b \int_a^b \frac{1}{f(x)+g(y)} dxdy} \sqrt{(b-a)\int_a^b f(x)dx + (b-a)\int_a^b g(y)dy}. \tag{3.5}$$

Applying the right-hand side of the "concave version" of the Hermite-Hadamard two-sided integral inequality to the concave functions f and g, we get

$$(b-a)\int_a^b f(x)dx + (b-a)\int_a^b g(y)dy$$
$$= (b-a)^2 \left[\frac{1}{b-a}\int_a^b f(x)dx + \frac{1}{b-a}\int_a^b g(y)dy\right]$$
$$\leq (b-a)^2 \left[f\left(\frac{a+b}{2}\right) + g\left(\frac{a+b}{2}\right)\right]. \tag{3.6}$$

It follows from Equations (3.5) and (3.6) that

$$(b-a)^2 \leq \sqrt{\int_a^b \int_a^b \frac{1}{f(x)+g(y)} dxdy} \sqrt{(b-a)^2 \left[f\left(\frac{a+b}{2}\right) + g\left(\frac{a+b}{2}\right)\right]},$$

so that

$$(b-a)^2 \leq \int_a^b \int_a^b \frac{1}{f(x)+g(y)} dxdy \left[f\left(\frac{a+b}{2}\right) + g\left(\frac{a+b}{2}\right)\right],$$

which can be arranged as follows:

$$\int_a^b \int_a^b \frac{1}{f(x)+g(y)} dxdy \geq \frac{(b-a)^2}{f[(a+b)/2] + g[(a+b)/2]}.$$

This is the result of Theorem 3.1 in the case $\alpha = 1$ and $\beta = 1$, ending this alternative proof. □

It is interesting to see how the Cauchy-Schwarz double integral inequality and the Hermite-Hadamard two-sided integral inequality can be used in this context. However, setting $\alpha = 1$ and $\beta = 1$ loses an important interest of Theorem 3.1. We will emphasize this point in the next subsection.

3.3. An Elegant Corollary

Thanks to the dependence of α and β, Theorem 3.1 can be configured in different ways. One configuration, which leads to an elegant inequality, is considered in the corollary below.

Corollary 3.2. *Let $a, b \in \mathbb{R}$ with $a < b$, and let $f, g : [a, b] \mapsto [0, +\infty)$ be two concave functions. Then we have*

$$\int_a^b \int_a^b \frac{1}{f(x) + g(y)} dxdy \geq \frac{(b-a)^2}{2} \times \frac{f[(a+b)/2] + g[(a+b)/2]}{f^2[(a+b)/2] + g^2[(a+b)/2]}.$$

Proof. Applying Theorem 3.1 with $\alpha = f^2[(a+b)/2]$ and $\beta = g^2[(a+b)/2]$, we get

$$\int_a^b \int_a^b \frac{1}{f(x) + g(y)} dxdy \geq \frac{(b-a)^2 \left[\sqrt{\alpha} + \sqrt{\beta}\right]^2}{2(\alpha+\beta)\left\{f[(a+b)/2] + g[(a+b)/2]\right\}}$$

$$= \frac{(b-a)^2 \left[\sqrt{f^2[(a+b)/2]} + \sqrt{g^2[(a+b)/2]}\right]^2}{2[f^2[(a+b)/2] + g^2[(a+b)/2]]\left\{f[(a+b)/2] + g[(a+b)/2]\right\}}$$

$$= \frac{(b-a)^2}{2} \times \frac{f[(a+b)/2] + g[(a+b)/2]}{f^2[(a+b)/2] + g^2[(a+b)/2]}.$$

This concludes the proof of Corollary 3.2. ∎

To illustrate this result numerically, let us take $a = 0$, $b = 1$ and study two examples based on different choices for f and g.

Example 1. First, we consider $f(x) = 2 - e^{-x}$, $x \in [0, 1]$, and $g(y) = \log(1 + y)$, $y \in [0, 1]$, which are obviously non-negative and concave. Then we have

$$\int_a^b \int_a^b \frac{1}{f(x) + f(y)} dxdy = \int_0^1 \int_0^1 \frac{1}{2 - e^{-x} + \log(1+y)} dxdy \approx 0.5847$$

and

$$\frac{(b-a)^2}{2} \times \frac{f[(a+b)/2] + g[(a+b)/2]}{f^2[(a+b)/2] + g^2[(a+b)/2]} = \frac{1}{2} \times \frac{2 - e^{-1/2} + \log(3/2)}{(2 - e^{-1/2})^2 + \log^2(3/2)} \approx 0.42706.$$

We have $0.42706 < 0.5847$, which is consistent with Corollary 3.2.

Example 2. As another numerical example, let us consider $f(x) = 2 - e^{-x}$, $x \in [0, 1]$, and $g(y) = \sqrt{y/(1 + y)}$, $y \in [0, 1]$, which are obviously non-negative and concave. Then we have

$$\int_a^b \int_a^b \frac{1}{f(x) + f(y)} dxdy = \int_0^1 \int_0^1 \frac{1}{2 - e^{-x} + \sqrt{y/(1+y)}} dxdy \approx 0.535086$$

and

$$\frac{(b-a)^2}{2} \times \frac{f[(a+b)/2] + g[(a+b)/2]}{f^2[(a+b)/2] + g^2[(a+b)/2]} = \frac{1}{2} \times \frac{2 - e^{-1/2} + 1/\sqrt{3}}{(2 - e^{-1/2})^2 + 1/3} \approx 0.43313.$$

We obtain $0.43313 < 0.535086$, as expected.

In the next subsection, we will emphasize this corollary for the special case of $f = g$.

3.4. The Special Case $f = g$

Corollary 3.3. *Let $a, b \in \mathbb{R}$ with $a < b$, and let $f : [a, b] \mapsto [0, +\infty)$ be a concave function. Then we have*

$$\int_a^b \int_a^b \frac{1}{f(x) + f(y)} dxdy \geq \frac{(b-a)^2}{2} \times \frac{1}{f[(a+b)/2]}.$$

Proof. Applying Corollary 3.2 with $f = g$ yields directly

$$\int_a^b \int_a^b \frac{1}{f(x) + f(y)} dxdy \geq \frac{(b-a)^2}{2} \times \frac{f[(a+b)/2] + f[(a+b)/2]}{f^2[(a+b)/2] + f^2[(a+b)/2]}$$
$$= \frac{(b-a)^2}{2} \times \frac{1}{f[(a+b)/2]}.$$

This concludes the proof of Corollary 3.3. ∎

To illustrate this result numerically, let us take $a = 0$, $b = 1$ and study three examples based on different choices for f.

Example 1. First, we consider $f(x) = 2 - e^{-x}$, $x \in [0, 1]$, which is obviously non-negative and concave. Then we have

$$\int_a^b \int_a^b \frac{1}{f(x) + f(y)} dxdy = \int_0^1 \int_0^1 \frac{1}{4 - e^{-x} - e^{-y}} dxdy \approx 0.368878$$

and

$$\frac{(b-a)^2}{2} \times \frac{1}{f[(a+b)/2]} = \frac{1}{2} \times \frac{1}{2 - e^{-1/2}} \approx 0.35881.$$

We get $0.35881 < 0.368878$, illustrating the obtained lower bound. Furthermore, it is relatively sharp in this example.

Example 2. As another numerical example, let us consider $f(x) = \log(1 + x)$, $x \in [0, 1]$, which is obviously non-negative and concave. Then we have

$$\int_a^b \int_a^b \frac{1}{f(x) + f(y)} dxdy = \int_0^1 \int_0^1 \frac{1}{\log(1+x) + \log(1+y)} dxdy \approx 1.66504$$

and

$$\frac{(b-a)^2}{2} \times \frac{1}{f[(a+b)/2]} = \frac{1}{2} \times \frac{1}{\log(3/2)} \approx 1.233151.$$

We obtain $1.233151 < 1.66504$, as expected.

Example 3. As a last example, let us consider $f(x) = \sqrt{x/(1+x)}$, $x \in [0, 1]$, which is obviously non-negative and concave. Then we have

$$\int_a^b \int_a^b \frac{1}{f(x) + f(y)} dxdy = \int_0^1 \int_0^1 \frac{1}{\sqrt{x/(1+x)} + \sqrt{y/(1+y)}} dxdy \approx 0.992571$$

and

$$\frac{(b-a)^2}{2} \times \frac{1}{f[(a+b)/2]} = \frac{\sqrt{3}}{2} \approx 0.86602.$$

We find that $0.86602 < 0.992571$, which is consistent with the result of Corollary 3.3.

In the interest of full understanding, two alternative proofs of Corollary 3.3 are proposed in the subsection below.

3.5. Two Further Proofs of Corollary 3.3

Assuming that f is twice differentiable, two further proofs of Corollary 3.3 are now proposed, using different techniques. The first uses Lemma 2.1 and the left-hand side of the "convex version" of the Hermite-Hadamard two-sided integral inequality, while the second uses the "convex version" of the Jensen double integral inequality.

First alternative proof of Corollary 3.3. The concavity of f implies that

$$\frac{1}{2}f(x) + \frac{1}{2}f(y) \leq f\left(\frac{x+y}{2}\right),$$

so that

$$\frac{1}{f(x)+f(y)} \geq \frac{1}{2f[(x+y)/2]}.$$

We therefore have

$$\int_a^b \int_a^b \frac{1}{f(x)+f(y)} dxdy \geq \frac{1}{2} \int_a^b \int_a^b \frac{1}{f[(x+y)/2]} dxdy. \tag{3.7}$$

Since f is twice differentiable and concave, Lemma 2.1 ensures that $1/f$ is convex. In particular, this implies that $1/f[(x+y)/2]$ is also convex with respect to x or y. For the sake of clarity, let us now recall the left-hand side of the "convex version" of the Hermite-Hadamard two-sided integral inequality. For any convex function $h : [a, b] \mapsto [0, +\infty)$, this inequality states that

$$h\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b h(x)dx.$$

Applying it to the convex function $h(x) = 1/f[(x+y)/2]$, then to the convex function $h(x) = 1/f\{[(a+b)/2 + x]/2\}$, we get

$$\begin{aligned}
&\int_a^b \int_a^b \frac{1}{f[(x+y)/2]} dxdy = \frac{1}{2} \int_a^b \left[\int_a^b \frac{1}{f[(x+y)/2]} dx\right] dy \\
&\geq \int_a^b \left[(b-a)\frac{1}{f\{[(a+b)/2+y]/2\}}\right] dy = (b-a) \int_a^b \frac{1}{f\{[(a+b)/2+y]/2\}} dy \\
&\geq (b-a)^2 \frac{1}{f\{[(a+b)/2+(a+b)/2]/2\}} = (b-a)^2 \frac{1}{f[(a+b)/2]}.
\end{aligned} \tag{3.8}$$

It follows from Equations (3.7) and (3.8) that

$$\int_a^b \int_a^b \frac{1}{f(x)+g(y)} dxdy \geq \frac{(b-a)^2}{2} \times \frac{1}{f[(a+b)/2]}.$$

This concludes the first alternative proof of Corollary 3.3. □

Second alternative proof of Corollary 3.3. The concavity of f implies that

$$\frac{1}{2} f(x) + \frac{1}{2} f(y) \leq f\left(\frac{x+y}{2}\right),$$

so that

$$\frac{1}{f(x)+f(y)} \geq \frac{1}{2f[(x+y)/2]}.$$

We therefore have

$$\int_a^b \int_a^b \frac{1}{f(x)+f(y)} dxdy \geq \frac{1}{2} \int_a^b \int_a^b \frac{1}{f[(x+y)/2]} dxdy. \tag{3.9}$$

Since f is concave, Lemma 2.1 implies that $1/f$ is convex.

For the sake of clarity, let us now recall the "convex version" of the Jensen double integral inequality. For any convex function $h : \mathbb{R} \mapsto \mathbb{R}$, any function $g : [a,b]^2 \mapsto \mathbb{R}$, and any $p : [a,b]^2 \mapsto [0,+\infty)$ satisfying $\int_a^b \int_a^b p(x,y) dxdy = 1$, this inequality states that

$$h\left[\int_a^b \int_a^b g(x,y) p(x,y) dxdy\right] \leq \int_a^b \int_a^b h[g(x,y)] p(x,y) dxdy.$$

Applying it with $h = f$, we obtain

$$\int_a^b \int_a^b \frac{1}{f[(x+y)/2]} dxdy = (b-a)^2 \left[\frac{1}{(b-a)^2} \int_a^b \int_a^b \frac{1}{f[(x+y)/2]} dxdy\right]$$
$$\geq (b-a)^2 \frac{1}{\Upsilon}, \tag{3.10}$$

where

$$\Upsilon = f\left[\frac{1}{(b-a)^2} \int_a^b \int_a^b \frac{x+y}{2} dxdy\right].$$

We basically calculate

$$\int_a^b \int_a^b \frac{x+y}{2} dxdy = \frac{1}{2}(b-a) \int_a^b x dx + \frac{1}{2}(b-a) \int_a^b y dy = (b-a) \left[\frac{x^2}{2}\right]_{x=a}^{x=b}$$
$$= (b-a) \frac{b^2-a^2}{2} = \frac{(b-a)^2(a+b)}{2}.$$

We therefore have

$$\Upsilon = f\left[\frac{1}{(b-a)^2} \times \frac{(b-a)^2(a+b)}{2}\right] = f\left(\frac{a+b}{2}\right). \tag{3.11}$$

It follows from Equations (3.9), (3.10) and (3.11) that

$$\int_a^b \int_a^b \frac{1}{f(x)+g(y)} dxdy \geq \frac{(b-a)^2}{2} \times \frac{1}{f[(a+b)/2]}.$$

This ends the second alternative proof of Corollary 3.3. □

We have given three different proofs for a new and interesting integral inequality which combines two dimensions and concavity. It therefore deserves to be reported as such. This study is completed by the subsection below, which proposes different kinds of lower bounds.

3.6. Three Additional Propositions

The proposition below extends the scope of Theorem 3.1 by introducing an exponent parameter, in the spirit of [25], but with a much simpler and more efficient result.

Proposition 3.4. *Let $a, b \in \mathbb{R}$ with $a < b$, and let $f, g : [a, b] \mapsto [0, +\infty)$ be two concave functions. Then, for all $\alpha, \beta \geq 0$ with $\alpha + \beta > 0$ and $\theta \in [0, 1]$, the following inequality holds:*

$$\int_a^b \int_a^b \frac{1}{[f(x)+g(y)]^\theta} dxdy \geq \frac{(b-a)^2 \left[\sqrt{\alpha}+\sqrt{\beta}\right]^2}{2(\alpha+\beta)\left\{f^\theta[(a+b)/2]+g^\theta[(a+b)/2]\right\}}.$$

Proof. Since $\theta \in [0, 1]$, a well-known convex inequality ensures that, for all $\tau, \omega \geq 0$,

$$(\tau+\omega)^\theta \leq \tau^\theta + \omega^\theta.$$

Applying this with $\tau = f(x)$ and $\omega = g(x)$, we get

$$[f(x)+g(y)]^\theta \leq f^\theta(x) + g^\theta(x),$$

so that

$$\int_a^b \int_a^b \frac{1}{[f(x)+g(y)]^\theta} dxdy \geq \int_a^b \int_a^b \frac{1}{f^\theta(x)+g^\theta(y)} dxdy. \tag{3.12}$$

Since $\theta \in [0, 1]$, f^θ and g^θ are concave functions as compositions of an increasing concave power function with concave functions. Applying Theorem 3.1 with these functions, we obtain

$$\int_a^b \int_a^b \frac{1}{f^\theta(x)+g^\theta(y)} dxdy \geq \frac{(b-a)^2 \left[\sqrt{\alpha}+\sqrt{\beta}\right]^2}{2(\alpha+\beta)\left\{f^\theta[(a+b)/2]+g^\theta[(a+b)/2]\right\}}. \tag{3.13}$$

It follows from Equations (3.12) and (3.13) that

$$\int_a^b \int_a^b \frac{1}{[f(x)+g(y)]^\theta} dxdy \geq \frac{(b-a)^2 \left[\sqrt{\alpha}+\sqrt{\beta}\right]^2}{2(\alpha+\beta)\left\{f^\theta[(a+b)/2]+g^\theta[(a+b)/2]\right\}}.$$

This ends the proof of Proposition 3.4. ■

Obviously, if we take $\theta = 1$, Proposition 3.4 simplifies to Theorem 3.1. The other intermediate values lead to new integral inequalities.

The proposition below can be described as a variant of Theorem 3.1, which deals with a slight modification of the double integral.

Proposition 3.5. *Let $a, b \in \mathbb{R}$ with $a < b$, and let $f, g : [a, b] \mapsto [0, +\infty)$ be two concave functions. Then we have*

$$\int_a^b \int_a^b \frac{1}{f(a) + f(b) + g(a) + g(b) - f(x) - g(y)} dxdy \geq \frac{(b-a)^2 \left[\sqrt{\alpha} + \sqrt{\beta}\right]^2}{2(\alpha + \beta) \left\{f[(a+b)/2] + g[(a+b)/2]\right\}}.$$

Proof. Since f is concave, for all $x \in [a, b]$, we claim that

$$f(a + b - x) \geq f(a) + f(b) - f(x).$$

Let us present a concise proof of this claim. Since $x \in [a, b]$, we can express x as $x = \sigma a + (1-\sigma)b$ where $\sigma \in [0, 1]$. This, combined with the basic concave property, gives

$$\begin{aligned} f(a+b-x) &= f[a+b-\sigma a - (1-\sigma)b] = f[(1-\sigma)a + \sigma b] \geq (1-\sigma)f(a) + \sigma f(b) \\ &= f(a) + f(b) - \sigma f(a) - (1-\sigma)f(b) \geq f(a) + f(b) - f[\sigma a + (1-\sigma)b] \\ &= f(a) + f(b) - f(x). \end{aligned}$$

Similarly, since g is concave, for all $y \in [a, b]$, we have

$$g(a + b - y) \geq g(a) + g(b) - g(y).$$

We thus derive

$$\begin{aligned} f(a+b-x) + g(a+b-y) &\geq f(a) + f(b) - f(x) + g(a) + g(b) - g(y) \\ &= f(a) + f(b) + g(a) + g(b) - f(x) - g(y), \end{aligned}$$

so that

$$\int_a^b \int_a^b \frac{1}{f(a) + f(b) + g(a) + g(b) - f(x) - g(y)} dxdy \geq \int_a^b \int_a^b \frac{1}{f(a+b-x) + g(a+b-y)} dxdy. \tag{3.14}$$

Making the change of variables $(s, t) = (a + b - x, a + b - y)$, we get

$$\int_a^b \int_a^b \frac{1}{f(a+b-x) + g(a+b-y)} dxdy = \int_a^b \int_a^b \frac{1}{f(s) + g(t)} dsdt. \tag{3.15}$$

A direct application of Theorem 3.1 gives

$$\int_a^b \int_a^b \frac{1}{f(s) + g(t)} dsdt \geq \frac{(b-a)^2 \left[\sqrt{\alpha} + \sqrt{\beta}\right]^2}{2(\alpha + \beta) \left\{f[(a+b)/2] + g[(a+b)/2]\right\}}. \tag{3.16}$$

It follows from Equations (3.14), (3.15) and (3.16) that

$$\int_a^b \int_a^b \frac{1}{f(a) + f(b) + g(a) + g(b) - f(x) - g(y)} dxdy \geq \frac{(b-a)^2 \left[\sqrt{\alpha} + \sqrt{\beta}\right]^2}{2(\alpha + \beta) \left\{f[(a+b)/2] + g[(a+b)/2]\right\}}.$$

This concludes the proof of Proposition 3.5. ∎

In particular, combining Equations (3.14) and (3.15) gives

$$\int_a^b \int_a^b \frac{1}{f(a)+f(b)+g(a)+g(b)-f(x)-g(y)} dxdy \geq \int_a^b \int_a^b \frac{1}{f(x)+g(y)} dxdy,$$

which is not a trivial integral inequality in the general case.

The proposition below offers a new point of view. It determines an appropriate simple integral lower bound for the double integral of interest.

Proposition 3.6. *Let $a, b \in \mathbb{R}$ with $a < b$, and let $f : [a, b] \mapsto [0, +\infty)$ be a concave function. Then the following inequality holds:*

$$\int_0^{+\infty} \int_0^{+\infty} \frac{1}{f(x)+f(y)} dxdy \geq \int_a^b \frac{\min(b, 2x-a) - \max(a, 2x-b)}{f(x)} dx.$$

Proof. The concavity of f implies that, for all $x, y \in [a, b]$,

$$\frac{1}{2}[f(x)+f(y)] \leq f\left(\frac{x+y}{2}\right). \tag{3.17}$$

So we have

$$\begin{aligned} \int_a^b \int_a^b \frac{1}{f(x)+g(y)} dxdy &\geq \frac{1}{2} \int_a^b \int_a^b \frac{1}{f[(x+y)/2]} dxdy \\ &= \frac{1}{2} \int_a^b \left[\int_a^b \frac{1}{f[(x+y)/2]} dx \right] dy. \end{aligned} \tag{3.18}$$

Applying the change of variables $u = (x+y)/2$ with respect to x, we get

$$\int_a^b \left[\int_a^b \frac{1}{f[(x+y)/2]} dx \right] dy = 2 \int_a^b \left[\int_{(a+y)/2}^{(b+y)/2} \frac{1}{f(u)} du \right] dy. \tag{3.19}$$

Using the Fubini-Tonelli integral theorem, which allows the order of integration to be changed, with particular attention to the integration interval, we obtain

$$\begin{aligned} \int_a^b \left[\int_{(a+y)/2}^{(b+y)/2} \frac{1}{f(u)} du \right] dy &= \int_a^b \frac{1}{f(u)} \left[\int_{\max(a,2u-b)}^{\min(b,2u-a)} dy \right] du \\ &= \int_a^b \frac{\min(b, 2u-a) - \max(a, 2u-b)}{f(u)} du. \end{aligned} \tag{3.20}$$

It follows from Equations (3.17), (3.18), (3.19) and (3.20) that

$$\int_0^{+\infty} \int_0^{+\infty} \frac{1}{f(x)+f(y)} dxdy \geq \int_a^b \frac{\min(b, 2x-a) - \max(a, 2x-b)}{f(x)} dx.$$

This concludes the proof of Proposition 3.6. ∎

To illustrate this notable result numerically, let us take $a = 0$, $b = 1$ and study three examples based on different choices for f.

Example 1. First, we consider $f(x) = 2 - e^{-x}$, $x \in [0, 1]$, which is obviously non-negative and concave. Then we have

$$\int_a^b \int_a^b \frac{1}{f(x) + f(y)} dxdy = \int_0^1 \int_0^1 \frac{1}{4 - e^{-x} - e^{-y}} dxdy \approx 0.368878$$

and

$$\int_a^b \frac{\min(b, 2x - a) - \max(a, 2x - b)}{f(x)} dx = \int_0^1 \frac{\min(1, 2x) - \max(0, 2x - 1)}{2 - e^{-x}} dx$$
$$\approx 0.365379.$$

We get $0.365379 < 0.368878$, which illustrates the lower bound obtained. The sharpness is particularly remarkable.

Example 2. As another numerical example, let us consider $f(x) = \log(1 + x)$, $x \in [0, 1]$, which is obviously non-negative and concave. Then we have

$$\int_a^b \int_a^b \frac{1}{f(x) + f(y)} dxdy = \int_0^1 \int_0^1 \frac{1}{\log(1 + x) + \log(1 + y)} dxdy \approx 1.66504$$

and

$$\int_a^b \frac{\min(b, 2x - a) - \max(a, 2x - b)}{f(x)} dx = \int_0^1 \frac{\min(1, 2x) - \max(0, 2x - 1)}{\log(1 + x)} dx$$
$$\approx 1.61985$$

We obtain $1.61985 < 1.66504$, as expected.

Example 3. As a last example, let us consider $f(x) = \sqrt{x/(1 + x)}$, $x \in [0, 1]$, which is obviously non-negative and concave. Then we have

$$\int_a^b \int_a^b \frac{1}{f(x) + f(y)} dxdy = \int_0^1 \int_0^1 \frac{1}{\sqrt{x/(1 + x)} + \sqrt{y/(1 + y)}} dxdy \approx 0.992571$$

and

$$\int_a^b \frac{\min(b, 2x - a) - \max(a, 2x - b)}{f(x)} dx = \int_0^1 \frac{\min(1, 2x) - \max(0, 2x - 1)}{\sqrt{x/(1 + x)}} dx$$
$$\approx 0.935104.$$

We find that $0.935104 < 0.992571$. This is consistent with the result of Proposition 3.6.

The next section is devoted to the upper bounds on the main double integral of interest.

4. Upper Bounds

4.1. Two Simple Propositions

The proposition below suggests a simple upper bound using a basic majorization approach and the "convex version" of the Hermite-Hadamard two-sided integral inequality.

Proposition 4.1. *Let $a, b \in \mathbb{R}$ with $a < b$, and let $f, g : [a, b] \mapsto [0, +\infty)$ be two twice differentiable concave functions. Then we have*

$$\int_a^b \int_a^b \frac{1}{f(x)+g(y)} dxdy \leq \frac{(b-a)^2}{2} \min\left[\frac{1}{f(a)} + \frac{1}{f(b)}, \frac{1}{g(a)} + \frac{1}{g(b)}\right].$$

Proof. Since f and g are non-negative, we have, for all $x, y \in [a, b]$,

$$\frac{1}{f(x)+g(y)} \leq \frac{1}{f(x)}, \quad \frac{1}{f(x)+g(y)} \leq \frac{1}{g(y)},$$

so that

$$\int_a^b \int_a^b \frac{1}{f(x)+g(y)} dxdy \leq \int_a^b \int_a^b \frac{1}{f(x)} dxdy = (b-a) \int_a^b \frac{1}{f(x)} dx \tag{4.1}$$

and

$$\int_a^b \int_a^b \frac{1}{f(x)+g(y)} dxdy \leq \int_a^b \int_a^b \frac{1}{g(y)} dxdy = (b-a) \int_a^b \frac{1}{g(y)} dy. \tag{4.2}$$

Since f and g are twice differentiable and concave, Lemma 2.1 ensures that $1/f$ and $1/g$ are convex. The right-hand side of the "convex version" of the Hermite-Hadamard two-sided integral inequality applied to $1/f$ and $1/g$ gives

$$\begin{aligned}\int_a^b \frac{1}{f(x)} dx &= (b-a)\left[\frac{1}{b-a}\int_a^b \frac{1}{f(x)} dx\right] \leq (b-a) \times \frac{1}{2}\left[\frac{1}{f(a)} + \frac{1}{f(b)}\right] \\ &= \frac{b-a}{2}\left[\frac{1}{f(a)} + \frac{1}{f(b)}\right]\end{aligned} \tag{4.3}$$

and, similarly,

$$\begin{aligned}\int_a^b \frac{1}{g(y)} dy &= (b-a)\left[\frac{1}{b-a}\int_a^b \frac{1}{g(y)} dy\right] \leq (b-a) \times \frac{1}{2}\left[\frac{1}{g(a)} + \frac{1}{g(b)}\right] \\ &= \frac{b-a}{2}\left[\frac{1}{g(a)} + \frac{1}{g(b)}\right].\end{aligned} \tag{4.4}$$

It follows from Equations (4.1), (4.2), (4.3) and (4.4) that

$$\int_a^b \int_a^b \frac{1}{f(x)+g(y)} dxdy \leq \frac{(b-a)^2}{2} \min\left[\frac{1}{f(a)} + \frac{1}{f(b)}, \frac{1}{g(a)} + \frac{1}{g(b)}\right].$$

This achieves the proof of Proposition 4.1. ∎

The obtained upper bound is much more simpler and manageable than that in [25, Theorem 3.1] for the case $\iota = 1$.

The proposition below provides an alternative based on the Cauchy-Schwarz (simple) integral inequality, Lemma 2.1 and the "convex version" of the Hermite-Hadamard two-sided integral inequality.

Proposition 4.2. *Let $a, b \in \mathbb{R}$ with $a < b$, and let $f, g : [a, b] \mapsto [0, +\infty)$ be two twice differentiable concave functions. Then we have*

$$\int_a^b \int_a^b \frac{1}{f(x)+g(y)} dxdy \leq \frac{(b-a)^2}{4} \sqrt{\left[\frac{1}{f(a)} + \frac{1}{f(b)}\right]\left[\frac{1}{g(a)} + \frac{1}{g(b)}\right]}.$$

Proof. For all $\tau, \omega \geq 0$, the basic inequality $[\sqrt{\tau} - \sqrt{\omega}]^2 \geq 0$ implies that

$$\tau + \omega \geq 2\sqrt{\tau}\sqrt{\omega}.$$

Applying it with $\tau = f(x)$ and $\omega = g(y)$ yields

$$f(x) + g(y) \geq 2\sqrt{f(x)}\sqrt{g(y)},$$

so that

$$\int_a^b \int_a^b \frac{1}{f(x)+g(y)} dxdy \leq \frac{1}{2} \int_a^b \int_a^b \frac{1}{\sqrt{f(x)}\sqrt{g(y)}} dxdy$$
$$= \frac{1}{2}\left[\int_a^b \frac{1}{\sqrt{f(x)}} dx\right]\left[\int_a^b \frac{1}{\sqrt{g(y)}} dy\right]. \tag{4.5}$$

On the other hand, the Cauchy-Schwarz integral inequality gives

$$\int_a^b \frac{1}{\sqrt{f(x)}} dx \leq \sqrt{\int_a^b \frac{1}{f(x)} dx}\sqrt{\int_a^b dx} \leq \sqrt{\int_a^b \frac{1}{f(x)} dx}\sqrt{b-a}$$

and

$$\int_a^b \frac{1}{\sqrt{g(y)}} dy \leq \sqrt{\int_a^b \frac{1}{g(y)} dy}\sqrt{\int_a^b dy} \leq \sqrt{\int_a^b \frac{1}{g(y)} dy}\sqrt{b-a},$$

so that

$$\left[\int_a^b \frac{1}{\sqrt{f(x)}} dx\right]\left[\int_a^b \frac{1}{\sqrt{g(y)}} dy\right] \leq \sqrt{\int_a^b \frac{1}{f(x)} dx}\sqrt{b-a} \times \sqrt{\int_a^b \frac{1}{g(y)} dy}\sqrt{b-a}$$
$$= (b-a)\sqrt{\left[\int_a^b \frac{1}{f(x)} dx\right]\left[\int_a^b \frac{1}{g(y)} dy\right]}. \tag{4.6}$$

Since f and g are twice differentiable and concave, Lemma 2.1 ensures that $1/f$ and $1/g$ are convex. The right-hand side of the "convex version" of the Hermite-Hadamard two-sided integral inequality applied to $1/f$ and $1/g$ gives

$$\left[\int_a^b \frac{1}{f(x)} dx\right]\left[\int_a^b \frac{1}{g(y)} dy\right] = (b-a)^2 \left[\frac{1}{b-a}\int_a^b \frac{1}{f(x)} dx\right]\left[\frac{1}{b-a}\int_a^b \frac{1}{g(y)} dy\right]$$
$$\leq (b-a)^2 \times \frac{1}{2}\left[\frac{1}{f(a)} + \frac{1}{f(b)}\right] \times \frac{1}{2}\left[\frac{1}{g(a)} + \frac{1}{g(b)}\right]$$

$$= \frac{(b-a)^2}{4}\left[\frac{1}{f(a)}+\frac{1}{f(b)}\right]\left[\frac{1}{g(a)}+\frac{1}{g(b)}\right]. \tag{4.7}$$

It follows from Equations (4.5), (4.6) and (4.7) that

$$\int_a^b\int_a^b \frac{1}{f(x)+g(y)}dxdy \leq \frac{b-a}{2}\sqrt{\frac{(b-a)^2}{4}\left[\frac{1}{f(a)}+\frac{1}{f(b)}\right]\left[\frac{1}{g(a)}+\frac{1}{g(b)}\right]}$$
$$= \frac{(b-a)^2}{4}\sqrt{\left[\frac{1}{f(a)}+\frac{1}{f(b)}\right]\left[\frac{1}{g(a)}+\frac{1}{g(b)}\right]}.$$

This concludes the proof of Proposition 4.2. ∎

The upper bounds obtained in Propositions 4.1 and 4.2 are difficult to compare. For this reason, they must be viewed as complementary.

To illustrate this result numerically, let us take $a = 0$, $b = 1$ and study two examples based on different choices for f and g.

Example 1. First, we consider $f(x) = 2 - e^{-x}$, $x \in [0,1]$, and $g(y) = \log(2+y)$, $y \in [0,1]$, which are obviously non-negative and concave. Then we have

$$\int_a^b\int_a^b \frac{1}{f(x)+f(y)}dxdy = \int_0^1\int_0^1 \frac{1}{2-e^{-x}+\log(2+y)}dxdy \approx 0.443191$$

and

$$\frac{(b-a)^2}{4}\sqrt{\left[\frac{1}{f(a)}+\frac{1}{f(b)}\right]\left[\frac{1}{g(a)}+\frac{1}{g(b)}\right]} = \frac{1}{4}\sqrt{\left[1+\frac{1}{2-e^{-1}}\right]\left[\frac{1}{\log(2)}+\frac{1}{\log(3)}\right]}$$
$$\approx 0.48699.$$

We obtain $0.443191 < 0.48699$, which is consistent with Proposition 4.2.

Example 2. As another numerical example, let us consider $f(x) = 2 - e^{-x}$, $x \in [0,1]$, and $g(y) = 1 + \sqrt{y/(1+y)}$, $y \in [0,1]$, which are obviously non-negative and concave. Then we have

$$\int_a^b\int_a^b \frac{1}{f(x)+f(y)}dxdy = \int_0^1\int_0^1 \frac{1}{2-e^{-x}+1+\sqrt{y/(1+y)}}dxdy \approx 0.34715$$

and

$$\frac{(b-a)^2}{4}\sqrt{\left[\frac{1}{f(a)}+\frac{1}{f(b)}\right]\left[\frac{1}{g(a)}+\frac{1}{g(b)}\right]} = \frac{1}{4}\sqrt{\left[1+\frac{1}{2-e^{-1}}\right]\left[1+\frac{1}{1+1/\sqrt{2}}\right]}$$
$$\approx 0.399796.$$

We get $0.34715 < 0.399796$, as expected.

4.2. A Technical Theorem

The theorem below gives an upper bound on the double integral of interest, which is closely related to the concavity of the functions involved. No intermediate results are used.

Theorem 4.3. *Let $a, b \in \mathbb{R}$ with $b > a$, and $f, g : [a, b] \mapsto [0, +\infty)$ be two concave functions. Then the following inequality holds:*

$$\int_a^b \int_a^b \frac{1}{f(x) + g(y)} dxdy \leq$$
$$\frac{1}{f(b) - f(a)} \left\{ \left[\frac{g(a) + f(b)}{g(b) - g(a)} \right] \log \left[\frac{f(b) + g(b)}{g(a) + f(b)} \right] + \log \left[\frac{f(b) + g(b)}{f(a) + g(b)} \right] \right.$$
$$\left. + \left[\frac{g(a) + f(a)}{g(b) - g(a)} \right] \log \left[\frac{g(a) + f(a)}{f(a) + g(b)} \right] \right\}.$$

Proof. Using the change of variables $(x, y) = ((1 - w)a + wb, (1 - z)a + zb)$, we get

$$\int_a^b \int_a^b \frac{1}{f(x) + g(y)} dxdy = (b - a)^2 \int_0^1 \int_0^1 \frac{1}{f[(1 - w)a + wb] + g[(1 - z)a + zb]} dwdz. \tag{4.8}$$

Using the basic concave property of f and g, we obtain

$$f[(1 - w)a + wb] \geq (1 - w)f(a) + wf(b) = f(a) + w[f(b) - f(a)]$$

and

$$g[(1 - z)a + zb] \geq (1 - z)g(a) + zg(b) = g(a) + z[g(b) - g(a)].$$

So we have

$$\int_0^1 \int_0^1 \frac{1}{f[(1 - w)a + wb] + g[(1 - z)a + zb]} dwdz$$
$$\leq \int_0^1 \int_0^1 \frac{1}{f(a) + w[f(b) - f(a)] + g(a) + z[g(b) - g(a)]} dwdz$$
$$= \int_0^1 \left\{ \int_0^1 \frac{1}{f(a) + g(a) + w[f(b) - f(a)] + z[g(b) - g(a)]} dw \right\} dz. \tag{4.9}$$

Using various logarithmic primitives with respect to w, then with respect to z, we get

$$= \int_0^1 \left\{ \int_0^1 \frac{1}{f(a) + g(a) + w[f(b) - f(a)] + z[g(b) - g(a)]} dw \right\} dz$$
$$\int_0^1 \left[\frac{1}{f(b) - f(a)} \log \{ f(a) + g(a) + w[f(b) - f(a)] + z[g(b) - g(a)] \} \right]_{w=0}^{w=1} dz$$
$$= \frac{1}{f(b) - f(a)} \left\{ \int_0^1 \log \{ g(a) + f(b) + z[g(b) - g(a)] \} dz \right.$$
$$\left. - \int_0^1 \log \{ g(a) + f(a) + z[g(b) - g(a)] \} dz \right\}$$

$$= \frac{1}{f(b)-f(a)}\left\{\left[\left[\frac{g(a)+f(b)}{g(b)-g(a)}+z\right]\log\{g(a)+f(b)+z[g(b)-g(a)]\}-z\right]_{z=0}^{z=1}\right.$$
$$\left.-\left[\left[\frac{g(a)+f(a)}{g(b)-g(a)}+z\right]\log\{g(a)+f(a)+z[g(b)-g(a)]\}-z\right]_{z=0}^{z=1}\right\}$$
$$= \frac{1}{f(b)-f(a)}\left\{\left[\frac{g(a)+f(b)}{g(b)-g(a)}+1\right]\log[f(b)+g(b)]-1\right.$$
$$-\left[\frac{g(a)+f(b)}{g(b)-g(a)}\right]\log[g(a)+f(b)]-\left[\frac{g(a)+f(a)}{g(b)-g(a)}+1\right]\log[f(a)+g(b)]+1$$
$$\left.+\left[\frac{g(a)+f(a)}{g(b)-g(a)}\right]\log[g(a)+f(a)]\right\}$$
$$= \frac{1}{f(b)-f(a)}\left\{\left[\frac{g(a)+f(b)}{g(b)-g(a)}\right]\log\left[\frac{f(b)+g(b)}{g(a)+f(b)}\right]+\log\left[\frac{f(b)+g(b)}{f(a)+g(b)}\right]\right.$$
$$\left.+\left[\frac{g(a)+f(a)}{g(b)-g(a)}\right]\log\left[\frac{g(a)+f(a)}{f(a)+g(b)}\right]\right\}. \tag{4.10}$$

It follows from Equations (4.8), (4.9) and (4.10) that

$$\int_a^b\int_a^b \frac{1}{f(x)+g(y)}dxdy \leq$$
$$\frac{1}{f(b)-f(a)}\left\{\left[\frac{g(a)+f(b)}{g(b)-g(a)}\right]\log\left[\frac{f(b)+g(b)}{g(a)+f(b)}\right]+\log\left[\frac{f(b)+g(b)}{f(a)+g(b)}\right]\right.$$
$$\left.+\left[\frac{g(a)+f(a)}{g(b)-g(a)}\right]\log\left[\frac{g(a)+f(a)}{f(a)+g(b)}\right]\right\}.$$

This concludes the proof of Theorem 4.3. ∎

If f is convex instead of concave, the final inequality is reversed. This can be proved as in the current proof by reversing all the inequalities line by line. The upper bound thus becomes a lower bound in this case.

To illustrate this result numerically, let us take $a=0$, $b=1$ and study one example based on different choices for f and g. We consider $f(x)=2-e^{-x}$, $x\in[0,1]$, and $g(y)=\log(2+y)$, $y\in[0,1]$, which are obviously non-negative and concave. Then we have

$$\int_a^b\int_a^b \frac{1}{f(x)+f(y)}dxdy = \int_0^1\int_0^1 \frac{1}{2-e^{-x}+\log(2+y)}dxdy \approx 0.443191$$

and

$$\frac{1}{f(b)-f(a)}\left\{\left[\frac{g(a)+f(b)}{g(b)-g(a)}\right]\log\left[\frac{f(b)+g(b)}{g(a)+f(b)}\right]+\log\left[\frac{f(b)+g(b)}{f(a)+g(b)}\right]\right.$$
$$\left.+\left[\frac{g(a)+f(a)}{g(b)-g(a)}\right]\log\left[\frac{g(a)+f(a)}{f(a)+g(b)}\right]\right\}$$
$$= \frac{1}{1-e^{-1}}\left\{\left[\frac{\log(2)+2-e^{-1}}{\log(3/2)}\right]\log\left[\frac{2-e^{-1}+\log(3)}{\log(2)+2-e^{-1}}\right]+\log\left[\frac{2-e^{-1}+\log(3)}{1+\log(3)}\right]\right.$$

$$+ \left[\frac{\log(2)+1}{\log(3/2)}\right] \log\left[\frac{1+\log(2)}{1+\log(3)}\right]\Bigg\}$$
$$= 1.5819767068 \times (0.92178463 + 0.2632937041 - 0.89649) \approx 0.45654.$$

We get $0.443191 < 0.45654$, which is consistent with Theorem 4.3. This upper bound is very sharp, also better than that obtained with Theorem 4.2, with the same setting, i.e., 0.48699.

5. Conclusion

In this article, we investigated bounds for a class of double integrals involving concave functions. We completely revisit, and significantly improve and extend the scope of a special case in [25, Theorem 3.1]. In a sense, our results complement classical Hilbert-type integral inequalities. Several upper and lower bounds have been established using different techniques, highlighting the richness of the problem. Whenever possible, multiple proofs have been given to provide deeper understanding. Numerical examples support and illustrate the theoretical results.

Future work could explore extensions to more general classes of functions, such as the s-concave functions. It would also be interesting to study analogous problems in higher dimensions. We can, of course, think of finding bounds for the multiple integral

$$\int_a^b \dots \int_a^b \frac{1}{\sum_{i=1}^n f_i(x_i)} dx_1 \dots dx_n,$$

where $n \in \mathbb{N}\backslash\{0\}$ and $f_1, \dots f_n : [a,b] \mapsto [0,+\infty)$ are n functions assumed to be concave. Based on our results, some conjectures can be made. Another promising direction is the application of these bounds to operator theory or integral equations. Finally, connections with probabilistic and geometric inequalities deserve further investigation.

Competing Interests

The authors declare that they have no competing interests.

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