Nonlinear Convex Analysis and Optimization
Vol. 4, No. 2, 2025, pp. 43–58
https://bangmodjmcs.com/index.php/ncao

NONLINEAR CONVEX ANALYSIS AND OPTIMIZATION
An International Journal on Numerical, Computation and Applications

Topological Space in Finite Geometry with Variables in Integer Modulo d

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ABSTRACT

In this paper, a special class of finite geometry is studied. Lines in the geometry are discussed. The existence of a partially ordered relation leads to the emergence of subgeometries via partial order, which by extension leads to the presence of topological spaces between geometry and its subgeometries, with subgeometries embedded in a topological space.

Article History
Received 16 Apr 2025
Revised 29 May 2025
Accepted 2 May 2025

Keywords:
Lines; Non-near-linear finite geometry; Ring of integer modulo d; Topology; Topological space;

MSC
47H05; 47J25

1. Introduction

Let Z_d denotes the set of integers modulo d. The Cartesian product $Z_d \times Z_d$ gives a finite geometry $G(d)$. The geometry is defined as the pair (L_d, P_d) where L_d denotes the set of lines in the geometry and P_d denotes the set of points in the line. Currently, a lot of works focus on finite geometry, where, for instance, two lines in a finite geometry intersect mostly at a point. The findings in [3] negated this common axiom by restricting the correctness of the axiom to lines in near-linear geometry. It was affirmed in [13] that in non-near-linear finite geometry, two lines of the geometry intersect at at least two points.

Over the years, researchers in topology have always been challenged with many questions focusing on the application of topology outside classrooms and the relevance of the area to human endeavour. For instance, the authors in [1, 9, 15] have asked some pertinent questions about the areas of application or relevance of topology to humanity outside teaching. In what can be described as an answer to those questions, Mayila et al. [10] developed a mathematical representation of a decision space and a topology on a notion using some properties of topological operators. This further underscores the importance of topology and

Please cite this article as: S.O. Oladejo and A.M. Awwal, Topological Space in Finite Geometry with Variables in Integer Modulo d , Nonlinear Convex Anal. & Optim., Vol. 4, No. 2, 2025, 43–58. https://doi.org/10.58715/ncao.2025.4.4

the need for more studies to explore other areas of real-life applications. For more details, interested readers may refer to the following references [2, 4, 6, 8, 18].

In this work, we investigate the structure of topology on finite geometry. We examine how topological structure is established on non-near-linear finite geometry. We demonstrate how the finite geometry under consideration satisfies the conditions of topology. Finally, we demonstrate how the divisor functions play an essential role in establishing the topology on the geometry under discussion. This work is divided as follows. Section 1 covers the introduction. Section 2 focuses on the terminologies used in this work. It is titled Preliminaries. Lines in $\mathbb{Z}_p \times \mathbb{Z}_p$ are discussed in Section 3. Lines in $\mathbb{Z}_d \times \mathbb{Z}_d$ and $\prod \mathbb{Z}_{dl} \times \mathbb{Z}_{dl}$ are discussed in Section 4. Topological space in finite geometry is discussed in Section 5 of this work. Section 6 showcases illustrations of this work using numerical examples. Section 7 focuses on the conclusion.

2. Preliminaries

Definition 2.1.

1. The ring of integers modulo d is denoted by Z_d where $d \in Z^+$, and Z^+ represents set of positive integers. In this work, $G_d = Z_d^2$, so we use them interchangeably.
2. $|Z^*|$ is the cardinality of Z^*. It represents the number of invertible elements in Z_d.
3. $|Z^*| = \phi(d)$. $\phi(d)$ is called Euler Phi function. It is defined as
$$\phi(d) = \prod_{j=1}^{\ell}(p_j - 1), \ \ p_j \text{ is prime for all } j.$$
4. The divisors function is denoted by the symbol $\sigma(d)$, where $\sigma(d) = \sum_{i=1}^{k}(d_i)$
5. $\psi(d)$ is called Dedekind psi function where;
$$\psi(d) = \prod_{j=1}^{\ell}(p_j + 1), \ \ p_j = \text{prime}.$$
Here in this work, d is expressed as the product of two distinct primes.
6. Let $g\colon (X, \tau) \to (Y, \theta)$ be a continuous function. Another function is denoted by $\beta\colon G_{d_1} \to G_{d_2}$.
7. For $d_1|d_2|...|d_k$; $G_{d_1} \cong G_{d_2} \subset G_{d_k}$
8. Greatest Common Divisor is denoted by GCD
9. Least Common Multiple is represented by LCM

3. Lines in $\mathbf{Z}_p \times \mathbf{Z}_p$

In this section, lines in near linear finite geometry are discussed. This concept forms the theoretical background through which this work is built on. The concepts are defined as follows:

Definition 3.1. A space $S(P, L)$ is a system of points P and line L such that every line L is a subset of P.

Definition 3.2. A near linear space is an incident structure $I(P, L)$ of points P and lines L such that the following axioms are satisfied:

1. Any line has at-least two points.
2. Two lines meet in at most one point.

A near-linear space is defined as follows:

$$G_d = (L_d, P_d).$$

Here, P_d represents points on the line L_d.

L_d denotes lines with points P_d, where

$$L_d = \{(\alpha a, \alpha b) | a, b \epsilon Z_d, \alpha \in Z_d\}. \tag{3.1}$$

Lemma 3.3. *Let d be a prime. Two distinct lines of a near-linear finite geometry G_d intersects at one point.*

Proof. Let $G_d = Z_d^2$. $Z_d^2 = Z_d \times Z_d$ represents lines with points in G_d. For any prime d, the intersection of any pair of arbitrary lines yields a point. ∎

Definition 3.4. A partial ordered relation R in a set Z_d is a relation $R \subseteq \mathbf{Z}_d \times \mathbf{Z}_d$ which satisfies the following conditions;

1. Reflexivity; that is, for $a \in \mathbf{Z}_d$, $(a, a) \in \mathbf{Z}_d \times \mathbf{Z}_d$.
2. Antisymmetric: that is, $\mathbf{a}, \mathbf{b} \in \mathbf{Z}_d$, if $a \leqslant b$ and $b \leqslant a$, then $a = b$.
3. Transitivity: that is, $\mathbf{a}, \mathbf{b}, c \in \mathbf{Z}_d$, if $a \leqslant b$ and $b \leqslant c$, then $a \leqslant c$.

4. Lines in $\mathbf{Z}_d \times \mathbf{Z}_d$ and $\prod_{i=1}^{k} \mathbf{Z}_{d_i} \times \mathbf{Z}_{d_i}$

This subsection focuses on lines in non-near-linear finite geometry, where two lines in a phase-space $Z_d \times Z_d$ meet in at least two points. Equation (4) denotes a line through the origin (0,0).

Our investigation in this work focuses on lines on any arbitrary point in the geometry, as mentioned in [19] is named a shifted in this work. It is defined as follows:

$$L_d = \{\alpha a + \vartheta, \alpha b + s | a, b, \vartheta, s \in Z_d, \}, \alpha \in Z_d. \tag{4.1}$$

Mathematically, it is defined as the pair (P_d, L_d) in $G_d = Z_d^2$, where P_d represents points in a line and L_d represents lines in G_d with $P_d = \{(e, f) | e, f \in Z_d\}$. From some results obtained in the previous work of [13, 14, 17], we confirmed the following propositions.

1. If $b \in Z_d^*$ then $L(\alpha, \beta) = L(b\alpha, b\beta)$. Now, Z_d^* represents the set of invertible elements in Z_d. Also, if $b \in Z_d - Z_d^*$ then $L(\alpha, \beta)\, mod(d) \subset L(b\alpha, b\beta)$ Hence $L(b\alpha, b\beta) \prec L(\alpha, \beta)$, where $\prec$ represents partial ordering. We confirm that $L(\alpha, \beta)$ is a maximal line in G_d if $GCD(\alpha, \beta) \in Z_d^*$ and $L(\alpha, \beta)$ is a subline in G_d if $GCD(\alpha, \beta) \in Z_d - Z_d^*$

2. A finite geometry G_d shown in equation (4.1) can also be defined as $L(s\alpha, s\beta) = \{(s\xi\alpha, s\xi\beta)|\xi \in Z_d\}\xi \in Z_{\xi d}$. Furthermore, the line $L(\xi\alpha, \xi\beta)$ in $G_{\xi d}$ is a subline of $L(\alpha, \beta) = \{(s'\alpha, s'\beta)|s' = 0, ..., \xi d - 1\}$.

3. For $q|d$, if two maximal lines have q points in common, then the q points give a subline $L(\alpha, \beta)$ where $\alpha, \beta \in \frac{d}{q}Z_q$.

If we consider the sub-geometry G_q, the subline $L(\alpha, \beta)$ in G_d is a maximal line in G_q. The existence of maximal lines $\psi(q)$ is confirmed in the subgeometry G_q of finite geometry G_d. A set of lines in non-near linear finite geometry G_d together with its set of subgeometries forms a topological space via partial ordering with its subgeometries embedded in the space.

5. Topological Space in $Z_d \times Z_d$

In this subsection, $Z_d \times Z_d$ is used extensively to form a finite geometry where all the lines are derived. For $d \neq$ prime, $Z_d \times Z_d$ forms a non-near-linear finite geometry. In this section, we present how a phase-space finite geometry forms a topological space with its subset as topology.

Definition 5.1. A pair (X, τ) where X denotes a nonempty set X and τ denotes a collection of subsets is called a topological space if it meets the following conditions:

1. The empty set, φ, and the whole set, X, are elements of τ, that is, $\varphi, X \in \tau$.

2. The arbitrary union of members of τ is also an element of τ.

3. The finite intersection of members of τ is also an element of τ.

A set X together with the topology τ, that is (X, τ) is a topological space [5, 7, 11, 12, 16, 20].

Here, a non-near-linear geometry with variables in Z_d where d is a non-prime integer, forms a topological space as follows: Let X represents the ring of integer modulo d, *that* is Z_d, while the geometric combination $G_d = Z_d^2 = Z_d \times Z_d$ is taken as the collection of all subsets of Z_d. $(Z_d, Z_d \times Z_d)$ is a topological space. This phenomenon is shown to exist both when the geometric lines are taken through any arbitrary points in the geometry as defined in equation (4.1). For $q|d$, any subset of $\{Z_d \times Z_d\}$ is an open set, and the elements of $Z_d \times Z_d$ form open sets. A finite intersection of subsets of the collection $Z_d \times Z_d$ yields an open set. The set X and φ of $Z_d \times Z_d$ are both open and closed, where φ denotes an empty set. The set G_d together with the collection of its subsets $\tau_{G(d)}$ form a topological space with $Z_d \times Z_d$ as its topology.

Suppose $x, y \in Z_d$, we define $M_{x,y} = \{x + ym|m \in Z_d\}$. A non-empty subset $S \subseteq Z_d$ is open if it is a union of sets of the form $M_{x,y}$. The collection of subsets obtained from $M_{x,y}$ forms a topology. Furthermore, if we take a finite intersection of any finite subsets of the whole collection, it gives an open set.

Let X be the ring of integers modulo d and $G_d = Z_d^2 = Z_d \times Z_d$ be a collection of all subsets of Z_d. Then (G_d, τ_{G_d}) forms a topological space with the subset of G_d as topology.

Definition 5.2. Let X, Y be topological spaces and $\alpha\colon X \to Y$ be an arbitrary function. Then α is a continuous function if the inverse image of every open set in Y is open in X, i.e. $\forall U \subseteq Y$ open, $\alpha^{-1}(U) \subseteq X$ is open.

Definition 5.3. A function $\alpha\colon X \to Y$ between topological spaces is a homeomorphism because of an existence of one-to-one correspondence between continuous functions α and α^{-1}.

The topological space of finite geometry is analogous to the expression of an integer as products of its prime. An existence of a non-trivial subgeometry in a finite geometry is linked to expression of non-prime integer as products of its prime. A finite intersection of lines in the geometry is related to finding the highest common factor (H.C.F) of two or more integers while the union of two or more lines of the finite geometry is related to finding the least common multiplier (L.C.M) of two or more integers. The result of each is a member of the topological space in this context.

Proposition 5.4. *Let d be a prime. The pair (G_d, τ_{G_d}) forms an indiscrete topological space.*

Proof. Let $G_d = Z_d \times Z_d$, where $d = prime$, the following conditions hold:

1. Since d is a prime then Z_d is a field of integer modulo d. All the non-zero elements in this regard are invertible. Hence $L(\alpha x, \alpha y) \cong L(\beta x, \beta y), \alpha, \beta, x, y \in Z_d$.

   $L(\alpha x, \alpha y) \in \tau_{G_d}$

2. Intersection of $L_1(\beta x, \beta y)$ and $L_2(\beta x, \beta y)$ gives $L_k(x, y) \in \tau_{G_d}$ where $L_k(x, y)$ is a line with at most one point.

3. $L_1(\beta x, \beta y) \cup L_2(\beta x, \beta y) \cup ... \cup L_k(\beta x, \beta y) = G_d \in \tau_{G_d}$

Hence, form an indiscrete topological space. ∎

Proposition 5.5. *Let d be a prime. (G_d, τ_{G_d}) forms a discrete topological space.*

Proof. Let $G_d = Z_d \times Z_d$, where $d \neq prime$, the following conditions hold:

1. Since d is a non-prime, Z_d is a ring of integer modulo d. In this case, not all non-zero elements are invertible. Hence $L(\alpha x, \alpha y) \subseteq L(\beta x, \beta y), \alpha, \beta, x, y \in Z_d$, and $L(\alpha x, \alpha y), L(\beta x, \beta y) \in \tau_{G_d}$.

2. Intersection of $L_1(\beta x, \beta y)$ and $L_2(\beta x, \beta y)$ gives $L_k(x, y) \in \tau_{G_d}$ where $L_k(x, y)$ is a line with at least one point.

3. $L_1(\beta x, \beta y) \cup L_2(\beta x, \beta y) \cup ... \cup L_k(\beta x, \beta y) = G_d \in \tau_{G_d}$.

∎

6. Examples

Topology and topological space in finite Geometry $(G_d, \tau_{d|d_i})$ is verified as follows:

1. Collection of subset $\tau_{d|d_i}$ is a topology since G_{d_i} (for $i = 1$) and $G_d, d \neq prime$ are members of the topology as G_1 is a line with one point and G_d has lines with $d_1, d_2, ..., d_k$ points for $d|d_i$.

2. $G_1 \cup G_2 \cup ... \cup G_k \in \tau_{G_d}$

3. $G_1 \cap G_2 \cap ... \cap G_k \in \tau_{G_d}$ Where $d_1, d_2, ...d_k$ are non-trivial divisors of d

Furthermore, it was observed that for $d = 10$, G_{10} forms a discrete topology with the set of integers modulo 10 with its subgeometries as topology. This is discussed thus: in G_{10}, there are lines with 10, 5,2, and 1 points. A union of lines with 10 points is a topology in the space. An intersection of any two arbitrary lines with the same number of points gives another line whose representation is in the topological space. Each member of the space is an open set.

In addition, L(1,2) about the origin in the geometry has the following set of points, $\{(0,0)(1,2)(2,4)(3,6)(4,8)(5,0)(6,2)(7,4)(8,6)(9,8)\}$. Point (3,6) for an instance, has every other point in the set as its neighbourhood. The union of all neighbourhoods form a complete set of neighbourhood at point $(1,2) \in G_{10}$.

Consider $L(1,2) \in G_5 \subset G_{10}$, this confirms a homomorphism between G_5 and G_{10} and lines in $G_5 \cong 2G_5 \subset G_{10}$. Hence, $\beta\colon G_{d_1} \to G_{d_2} \to G_{d_3} \to \cdots \to G_{d_k} \to$ hence demonstrates continuity.

The existence of $\phi(d)$ lines is bijective to each other in the geometry.

The following results were generated by equation (3.1)

$$L(0,1) \cong L(0,3) \cong L(0,7) \cong L(0,9)$$
$$L(1,0) \cong L(3,0) \cong L(7,0) \cong L(9,0)$$
$$L(1,1) \cong L(3,3) \cong L(7,7) \cong L(9,9)$$
$$L(1,2) \cong L(3,6) \cong L(7,4) \cong L(9,8)$$
$$L(1,3) \cong L(3,9) \cong L(7,1) \cong L(9,7)$$
$$L(1,4) \cong L(3,8) \cong L(7,2) \cong L(9,6)$$
$$L(1,5) \cong L(3,5) \cong L(7,5) \cong L(9,5)$$
$$L(1,6) \cong L(3,8) \cong L(7,2) \cong L(9,4)$$
$$L(1,7) \cong L(3,1) \cong L(7,9) \cong L(9,3)$$
$$L(1,8) \cong L(3,4) \cong L(7,6) \cong L(9,2)$$
$$L(1,9) \cong L(3,7) \cong L(7,3) \cong L(9,1)$$
$$L(2,1) \cong L(6,3) \cong L(4,7) \cong L(8,9)$$
$$L(2,3) \cong L(6,9) \cong L(4,1) \cong L(8,7)$$
$$L(2,5) \cong L(6,5) \cong L(4,5) \cong L(8,5)$$
$$L(2,7) \cong L(6,1) \cong L(4,9) \cong L(8,3)$$
$$L(0,2) \cong L(0,4) \cong L(0,6) \cong L(0,8)$$
$$L(2,9) \cong L(4,3) \cong L(6,7) \cong L(8,8)$$
$$L(5,1) \cong L(5,3) \cong L(5,7) \cong L(5,9)$$
$$L(5,2) \cong L(5,4) \cong L(5,6) \cong L(5,8)$$
$$L(2,0) \cong L(4,0) \cong L(6,0) \cong L(8,0)$$
$$L(2,2) \cong L(4,4) \cong L(6,6) \cong L(8,8)$$
$$L(2,4) \cong L(4,8) \cong L(6,2) \cong L(8,6)$$
$$L(2,6) \cong L(4,2) \cong L(6,8) \cong L(8,4)$$
$$L(2,8) \cong L(4,6) \cong L(6,4) \cong L(8,2)$$

$L(0,5)$, $L(5,0)$, and $L(5,5)$ are lines with two points as expressed thus $L(0,5) = \{(0,0),(0,5)\}$, $L(5,0) = \{(0,0),(5,0)\}$, and $L(5,5) = \{(0,0),(5,5)\}$, respectively. We confirm the Definition 5.1 using the following as a check.

Condition 1: $\varphi, X \in \tau$, here $X = Z_{10}$, $\tau = Z_{10} \times Z_{10}$. Clearly, the empty set φ is an element of τ_{G_d}. That is $\varphi = L(0,0) \in \tau_{G_d}$.

Again, the whole set X is an element of τ_{G_d}. Hence, condition 1 is satisfied.

Condition 2: The union of subsets of $Z_{10} \times Z_{10}$ is also an element of $Z_{10} \times Z_{10}$.

Clearly the union of any pair of subsets of $Z_{10} \times Z_{10}$ with the same number of points is a member of $Z_{10} \times Z_{10}$. For instance, we consider the following for illustration.

1. $L(1,1) \cup L(2,5) \cup L(8,2) = \{L(1,1), L(2,5), L(8,2)\} \in \tau_{G_d}$
2. $L(3,4) \cup L(5,6) = \{L(3,4), L(5,6)\} \in \tau_{G_d}$

3. $L(0,5) \cup L(5,0) \cup L(5,5) = \{L(0,5), L(5,0), L(5,5)\} \in \tau_{G_d}$

Hence Condition 2 is satisfied.

Condition 3 Finite intersection of elements of $Z_{10} \times Z_{10}$ is again an element of $Z_{10} \times Z_{10}$
Clearly the finite intersection of elements of $Z_{10} \times Z_{10}$ is an element of $Z_{10} \times Z_{10}$

1. $L(1,1) \cap L(2,5) = \{(0,0)\} \in \tau_{G_d}$
2. $L(3,4) \cap L(5,6) = \{(0,0),(5,0)\} \in \tau_{G_d}$

Hence condition 3 is also satisfied.

Thus, we conclude that at point (0,0), the geometric combination $Z_{10} \times Z_{10}$ is a topology, and the combination $(Z_{10}, Z_{10} \times Z_{10})$ forms a topological space.

For a shifted origin, say a line through the point (2,3), we check for topological space using the axioms of topology and topological space.

Taking an arbitrary point (2,3) as a reference point, we obtain the following results. When origin is shifted to (a,b)=(2,3), the line defined by as follows;

$$
\begin{aligned}
L(0,0) &= \{(2,3),(2,3),(2,3),(2,3),(2,3),(2,3),(2,3),(2,3),(2,3)\}\\
L(0,1) &= \{(2,3),(2,4),(2,5),(2,6),(2,7),(2,8),(2,9),(2,0),(2,1),(2,2)\}\\
L(0,2) &= \{(2,3),(2,5),(2,7),(2,9),(2,1)\}\\
L(0,3) &= \{(2,3),(2,6),(2,9),(2,2),(2,7),(2,8),(2,1),(2,4),(2,7),(2,0)\}\\
L(0,4) &= \{(2,3),(2,7),(2,1),(2,5),(2,9)\}\\
L(0,5) &= \{(2,3),(2,8)\}\\
L(1,0) &= \{(2,3),(3,3),(4,3),(5,3),(6,3),(7,3),(8,3),(9,3),(0,3),(1,3)\}\\
L(1,2) &= \{(2,3),(3,5),(4,7),(5,9),(6,1),(7,3),(8,5),(9,7),(0,9),(1,1)\}\\
L(1,3) &= \{(2,3),(3,6),(4,9),(5,2),(6,5),(7,8),(8,1),(9,4),(0,7),(1,0)\}\\
L(1,4) &= \{(2,3),(3,7),(4,1),(5,5),(6,9),(7,3),(8,7),(9,1),(0,5),(1,9)\}\\
L(1,5) &= \{(2,3),(3,8),(4,3),(5,8),(6,3),(7,8),(8,3),(9,8),(0,3),(1,8)\}\\
L(1,6) &= \{(2,3),(3,9),(4,5),(5,1),(6,7),(7,3),(8,9),(9,5),(0,1),(1,7)\}\\
L(1,7) &= \{(2,3),(3,0),(4,7),(5,4),(6,1),(7,8),(8,5),(9,2),(0,9),(1,6)\}\\
L(1,8) &= \{(2,3),(3,1),(4,9),(5,7),(6,5),(7,3),(8,1),(9,9),(0,7),(1,5)\}\\
L(1,9) &= \{(2,3),(3,2),(4,1),(5,0),(6,9),(7,8),(8,7),(9,6),(0,5),(1,4)\}\\
L(2,0) &= \{(2,3),(4,3),(6,3),(8,3),(0,3)\}\\
L(2,1) &= \{(2,8),(4,4),(6,5),(8,6),(0,7),(2,8),(4,9),(6,0),(8,1),(0,2)\}\\
L(2,2) &= \{(2,3),(4,5),(6,7),(8,9),(0,1)\}\\
L(2,3) &= \{(2,3),(4,6),(6,9),(8,2),(0,5),(2,8),(4,1),(6,4),(8,7),(0,0)\}\\
L(2,4) &= \{(2,3),(4,7),(6,1),(8,5),(0,9)\}
\end{aligned}
$$

$L(2,5) = \{(2,3),(4,8),(6,3),(8,8),(0,3),(2,8),(4,3),(6,8),(8,3),(0,8)\}$
$L(2,6) = \{(2,3),(4,9),(6,5),(8,1),(0,7)\}$
$L(2,7) = \{(2,3),(4,0),(6,7),(8,4),(0,1),(2,8),(4,5),(6,2),(8,9),(0,5)\}$
$L(2,8) = \{(2,3),(4,1),(6,9),(8,7),(0,5)\}$
$L(2,9) = \{(2,3),(4,2),(6,1),(8,0),(0,9),(2,8),(4,7),(6,6),(8,5),(0,4)\}$
$L(3,0) = \{(2,3),(5,3),(8,3),(1,3),(4,3),(7,3),(4,3),(3,3),(6,3),(9,3)\}$
$L(3,1) = \{(2,3),(5,4),(8,5),(1,6),(4,7),(7,8),(0,9),(3,0),(6,1),(9,2)\}$
$L(3,2) = \{(2,3),(5,5),(8,7),(1,9),(4,1),(7,3),(0,5),(3,7),(6,9),(9,1)\}$
$L(3,3) = \{(2,3),(5,6),(8,9),(1,2),(4,5),(7,8),(0,1),(3,4),(6,7),(9,0)\}$
$L(3,4) = \{(2,3),(5,7),(8,1),(1,5),(4,9),(7,3),(0,7),(3,1),(6,5),(9,9)\}$
$L(3,5) = \{(2,3),(5,8),(8,3),(1,8),(4,3),(7,8),(0,3),(3,8),(6,3),(9,8)\}$
$L(3,6) = \{(2,3),(5,9),(8,5),(1,1),(4,7),(7,3),(0,9),(3,5),(6,1),(9,7)\}$
$L(3,7) = \{(2,3),(5,0),(8,7),(1,4),(4,1),(7,8),(0,5),(3,2),(6,9),(9,6)\}$
$L(3,8) = \{(2,3),(5,1),(8,9),(1,7),(4,5),(7,3),(0,1),(3,9),(5,7),(9,5)\}$
$L(3,9) = \{(2,3),(5,2),(8,1),(1,0),(4,9),(7,8),(0,7),(3,6),(6,5),(9,4)\}$
$L(4,0) = \{(2,3),(6,3),(0,3),(4,3),(9,3)\}$
$L(4,1) = \{(2,3),(6,4),(0,5),(4,6),(9,7),(2,8),(6,9),(0,0),(4,1),(8,2)\}$
$L(4,2) = \{(2,3),(6,5),(0,7),(4,9),(9,1)\}$
$L(4,3) = \{(2,3),(6,6),(0,9),(4,2),(9,5),(2,8),(6,1),(0,4),(4,7),(8,0)\}$
$L(4,4) = \{(2,3),(6,7),(0,1),(4,5),(9,9)\}$
$L(4,5) = \{(2,3),(6,8),(0,3),(4,8),(9,3),(2,8),(6,3),(0,8),(4,3),(8,8)\}$
$L(4,6) = \{(2,3),(6,9),(0,5),(4,1),(9,7)\}$
$L(4,7) = \{(2,3),(6,0),(0,7),(4,4),(9,1),(2,8),(6,5),(0,2),(4,9),(8,6)\}$
$L(4,8) = \{(2,3),(6,1),(0,9),(4,7),(9,5)\}$
$L(4,9) = \{(2,3),(6,2),(0,1),(4,0),(9,9),(2,8),(6,7),(0,6),(4,5),(8,4)\}$
$L(5,0) = \{(2,3),(7,3)\}$
$L(5,1) = \{(2,3),(7,4),(2,5),(7,b),(2,7),(7,8),(2,9),(7,0),(2,1),(7,2)\}$
$L(5,2) = \{(2,3),(7,5),(2,7),(7,9),(2,1),(7,3),(2,5),(7,7),(2,9),(7,1)\}$
$L(5,3) = \{(2,3),(7,6),(2,9),(7,2),(2,5),(7,8),(2,1),(7,4),(2,7),(7,0)\}$
$L(5,4) = \{(2,3),(7,7),(2,1),(7,5),(2,0),(7,3),(2,7),(7,1),(2,5),(7,9)\}$
$L(5,5) = \{(2,3),(7,8)\}$
$L(5,6) = \{(2,3),(7,9),(2,5),(7,1),(2,7),(7,3),(2,9),(7,5),(2,1),(7,7)\}$
$L(5,7) = \{(2,3),(7,0),(2,7),(7,4),(2,1),(7,8),(2,5),(7,2),(2,9),(7,6)\}$
$L(5,8) = \{(2,3),(7,1),(2,9),(7,7),(2,5),(7,3),(2,1),(7,9),(2,7),(7,5)\}$
$L(5,9) = \{(2,3),(7,2),(2,1),(7,0),(2,9),(7,8),(2,7),(7,6),(2,5),(7,4)\}$
$L(6,0) = \{(2,3),(8,3),(4,3),(0,3),(6,3)\}$

$L(6,1) = \{(2,3),(8,4),(4,5),(4,6),(6,7),(2,8),(8,9),(4,0),(0,1),(6,2)\}$
$L(6,2) = \{(2,3),(8,5),(4,7),(0,9),(6,1)\}$
$L(6,3) = \{(2,3),(8,6),(4,0),(0,2),(6,5),(2,8),(8,1),(4,4),(0,7),(6,1)\}$
$L(6,4) = \{(2,3),(8,7),(4,1),(0,5),(6,9)\}$
$L(6,5) = \{(2,3),(8,8),(4,3),(0,8),(6,3),(2,8),(8,3),(4,8),(0,3),(6,8)\}$
$L(6,6) = \{(2,3),(8,9),(4,5),(0,1),(6,7)\}$
$L(6,7) = \{(2,3),(8,0),(4,7),(0,4),(6,1),(2,8),(8,5),(4,2),(0,9),(6,6)\}$
$L(6,8) = \{(2,3),(8,1),(4,9),(0,7),(6,5)\}$
$L(6,9) = \{(2,3),(8,2),(4,1),(0,0),(6,9),(2,8),(8,7),(4,6),(0,5),(6,4)\}$
$L(7,0) = \{(2,3),(9,3),(6,3),(3,3),(0,3),(7,3),(8,3),(1,3),(8,3),(5,3)\}$
$L(7,1) = \{(2,3),(9,4),(6,5),(3,6),(0,7),(7,8),(8,9),(1,0),(8,1),(0,2)\}$
$L(7,2) = \{(2,3),(0,5),(6,7),(3,1),(0,1),(7,3),(8,5),(1,7),(8,9),(5,1)\}$
$L(7,3) = \{(2,3),(9,6),(6,9),(3,2),(0,5),(7,8),(8,1),(1,4),(8,7),(5,0)\}$
$L(7,4) = \{(2,3),(9,7),(6,1),(3,5),(0,9),(7,3),(8,7),(1,1),(8,5),(5,9)\}$
$L(7,5) = \{(2,3),(9,8),(6,3),(3,8),(0,3),(7,8),(8,3),(1,8),(8,3),(5,8)\}$
$L(7,6) = \{(2,3),(9,9),(6,5),(3,1),(0,7),(7,3),(8,9),(1,5),(8,1),(5,7)\}$
$L(7,7) = \{(2,3),(9,0),(6,7),(3,4),(0,1),(7,8),(8,5),(1,2),(8,9),(5,6)\}$
$L(7,8) = \{(2,3),(9,1),(6,9),(3,7),(0,5),(7,3),(8,1),(1,9),(8,7),(5,5)\}$
$L(7,9) = \{(2,3),(9,2),(0,1),(3,0),(0,9),(7,8),(8,7),(1,6),(8,5),(5,4)\}$
$L(8,0) = \{(2,3),(0,3),(8,3),(6,3),(4,3)\}$
$L(8,1) = \{(2,3),(0,4),(8,5),(0,6),(4,7),(2,8),(0,9),(8,0),(5,1),(4,2)\}$
$L(8,2) = \{(2,3),(0,5),(8,7),(6,9),(4,1)\}$
$L(8,3) = \{(2,3),(0,6),(8,9),(G,2),(4,5),(2,8),(0,1),(4,0),(0,1),(4,0)\}$
$L(8,4) = \{(2,3),(0,7),(8,1),(6,5),(4,9)\}$
$L(8,5) = \{(2,3),(0,8),(8,3),(6,8),(4,3),(2,8),(8,1),(4,0),(0,1),(4,8)\}$
$L(8,6) = \{(2,3),(0,9),(8,5),(6,1),(4,7)\}$
$L(8,7) = \{(2,3),(0,0),(8,7),(6,4),(4,1),(2,8),(8,1),(4,0),(0,1),(4,6)\}$
$L(8,8) = \{(2,3),(0,1),(8,9),(6,7),(4,5)\}$
$L(8,9) = \{(2,3),(0,2),(8,1),(6,0),(4,9),(2,8),(8,1),(4,0),(0,1),(4,4)\}$
$L(9,0) = \{(2,3),(1,3),(0,3),(9,3),(8,3),(7,3),(6,3),(5,3),(4,3),(3,3)\}$
$L(9,1) = \{(2,3),(1,4),(0,5),(9,6),(8,7),(7,8),(6,9),(5,0),(4,1),(3,2)\}$
$L(9,2) = \{(2,3),(1,5),(0,7),(9,9),(8,1),(7,3),(6,5),(5,7),(4,9),(3,1)\}$
$L(9,3) = \{(2,3),(1,6),(0,9),(9,2),(8,5),(5,4),(4,7),(3,0),(6,1),(7,8)\}$
$L(9,4) = \{(2,3),(1,7),(0,1),(9,5),(8,9),(7,3),(6,7),(5,1),(4,5),(3,9)\}$
$L(9,5) = \{(2,3),(1,8),(0,3),(9,5),(8,3),(7,8),(6,3),(5,8),(4,3),(3,8)\}$

$$L(9,6) = \{(2,3),(1,9),(0,5),(9,1),(8,7),(7,5),(6,5),(5,5),(4,1),(3,7)\}$$
$$L(9,7) = \{(2,3),(1,0),(0,7),(9,4),(8,1),(7,5),(6,2),(5,9),(4,9),(3,6)\}$$
$$L(9,8) = \{(2,3),(1,1),(0,9),(9,7),(8,5),(7,3),(6,1),(5,9),(4,7),(3,5)\}$$
$$L(9,9) = \{(2,3),(1,2),(0,1),(9,0),(8,9),(7,8),(6,7),(5,6),(4,5),(3,4)\}$$

Result Showing Intersecting Lines and Their Points of Intersection
Lines with 10 Points

$$L(0,1) = L(0,3) = L(0,7) = L(0,9) = \{(2,3),(3,3),(4,3),(5,3),(6,3),(7,3),(8,3),(9,3),(0,3),(1,3)\}$$
$$L(1,0) = L(3,0) = L(7,0) = L(9,0) = \{(2,3),(3,3),(4,3),(5,3),(6,3),(7,3),(8,3),(9,3),(0,3),(1,3)\}$$
$$L(1,1) = L(3,3) = L(7,7) = L(9,9) = \{(2,3),(3,4),(4,5),(5,6),(6,7),(7,8),(8,9),(9,0),(0,1),(1,2)\}$$
$$L(1,2) = L(3,6) = L(7,4) = L(9,8) = \{(2,3),(3,5),(4,7),(5,9),(6,1),(7,3),(8,5),(9,7),(0,9),(1,1)\}$$
$$L(1,3) = L(3,9) = L(7,1) = L(9,7) = \{(2,3),(3,6),(4,9),(5,2),(6,5),(7,8),(8,1),(9,4),(0,7),(1,0)\}$$
$$L(1,4) = L(3,8) = L(7,2) = L(9,6) = \{(2,3),(3,7),(4,1),(5,5),(6,9),(7,3),(8,7),(9,1),(0,5),(1,9)\}$$
$$L(1,5) = L(3,5) = L(7,5) = L(9,5) = \{(2,3),(3,8),(4,3),(5,8),(6,3),(7,8),(8,3),(9,8),(0,3),(1,8)\}$$
$$L(1,6) = L(3,8) = L(7,2) = L(9,4) = \{(2,3),(3,9),(4,5),(5,1),(6,7),(7,3),(8,0),(9,5),(0,1),(1,7)\}$$
$$L(1,7) = L(3,1) = L(7,9) = L(9,3) = \{(2,3),(3,0),(4,7),(5,4),(6,1),(7,8),(8,5),(9,2),(0,9),(1,6)\}$$
$$L(1,8) = L(3,4) = L(7,6) = L(9,2) = \{(2,3),(3,1),(4,0),(5,7),(6,5),(7,3),(8,1),(9,0),(0,7),(1,5)\}$$
$$L(1,9) = L(3,7) = L(7,3) = L(9,1) = \{(2,3),(3,2),(4,1),(5,0),(6,9),(7,8),(8,7),(9,6),(0,5),(1,4)\}$$
$$L(2,1) = L(6,3) = L(4,7) = L(8,9) = \{(2,3),(4,4),(6,5),(8,6),(0,7),(2,8),(4,9),(6,0),(8,1),(0,2)\}$$
$$L(2,3) = L(6,9) = L(4,1) = L(8,7) = \{(2,3),(4,6),(6,9),(8,2),(0,5),(2,8),(4,1),(6,4),(8,7),(0,0)\}$$
$$L(2,5) = L(6,5) = L(4,5) = L(8,5) = \{(2,3),(4,8),(6,3),(8,8),(0,3),(2,8),(4,3),(6,8),(8,3),(0,8)\}$$
$$L(2,7) = L(6,1) = L(4,9) = L(8,3) = \{(2,3),(4,7),(6,7),(8,0),(0,1),(2,8),(4,7),(9,6),(8,1),(8,4)\}$$
$$L(2,9) = L(6,7) = L(4,3) = L(8,1) = \{(2,3),(4,2),(6,1),(8,1),(0,1),(2,8),(4,7),(6,6),(8,5),(0,4)\}$$
$$L(5,1) = L(5,3) = L(5,7) = L(5,9) = \{(2,3),(7,4),(2,5),(7,6),(2,7),(7,8),(2,9),(7,4),(7,1),(7,1)\}$$
$$L(5,2) = L(5,6) = L(5,4) = L(5,8) = \{(2,3),(7,5),(2,7),(7,4),(2,5),(7,3),(2,5),(7,7),(2,9),(7,1)\}$$

2. Lines with 5 Points

$$L(0,2) = L(0,4) = L(0,6) = L(0,8) = \{(2,3),(2,5),(2,7),(2,9),(2,1)\}$$
$$L(2,0) = L(4,0) = L(6,0) = L(8,0) = \{(2,3),(4,3),(6,3),(8,3),(0,3)\}$$
$$L(2,2) = L(4,4) = L(6,6) = L(8,8) = \{(2,3),(4,5),(6,7),(8,9),(0,1)\}$$
$$L(2,4) = L(4,8) = L(6,2) = L(8,6) = \{(2,3),(4,7),(6,1),(8,5),(0,9)\}$$
$$L(2,6) = L(4,2) = L(6,8) = L(8,4) = \{(2,3),(4,9),(6,5),(8,1),(0,7)\}$$
$$L(2,8) = L(4,6) = L(6,4) = L(8,2) = \{(2,3),(4,1),(6,9),(8,7),(0,5)\}$$

3. Lines with 2 Points

$$L(0,5) = \{(2,3),(2,8)\},$$
$$L(5,0) = \{(2,3),(7,3)\},$$
$$L(5,5) = \{(2,3),(7,8)\}$$

Condition1: $\varphi, X \in \tau$, here $X = Z_{10}$, $\tau = Z_{10} \times Z_{10}$.
Clearly the empty set φ is an element of the topology. That is $\varphi = L(2,3) \in \tau$
Again, the whole set X is an element of τ

Hence condition 1 is satisfied.

Condition 2 The union of subsets of $Z_{10} \times Z_{10}$ is also an element of $Z_{10} \times Z_{10}$.

Clearly the union of any subsets of $Z_{10} \times Z_{10}$ is a member element of $Z_{10} \times Z_{10}$. That is;

1. $L(1,1) \cup L(2,5) \cup L(8,2) = \{L(1,1), L(2,5), L(8,2)\} \in Z_{10} \times Z_{10}$
2. $L(3,4) \cup L(5,6) = \{L(3,4), L(5,6)\} \in Z_{10} \times Z_{10}$
3. $L(0,5) \cup L(5,0) \cup L(5,5) = \{L(0,5), L(5,0), L(5,5)\} \in Z_{10} \times Z_{10}$

Hence condition 2 is satisfied

Condition 3 Furthermore, finite intersection of elements of $Z_{10} \times Z_{10}$ is an element of $Z_{10} \times Z_{10}$. Below are simple illustrations.

1. $L(1,1) \cap L(2,5) = \{(2,3)\} \in Z_{10} \times Z_{10}$
2. $L(3,4) \cap L(5,6) = \{(2,3),(7,3)\} \in Z_{10} \times Z_{10}$

Hence condition 3 is satisfied.

Thus, we conclude that the geometric combination $Z_{10} \times Z_{10}$ is a topology. Furthermore, $(Z_{10}, Z_{10} \times Z_{10})$ with shifted origin forms a topological space.

6.1. PICTORIAL ILLUSTRATIONS

Figure 1 is the graph of $G_5 = Z_5 \times Z_5$. It comprises the following ψ(d) lines. L(0,1),L(1,0),L(1,1),L(1,2), L(1,3), and L(1,4). The ψ(d) lines intersect at the point (0,0).

$$\begin{aligned}&L(0,1) \cup L(1,0) \cup L(1,1) \cup L(1,2) \cup L(1,3) \cup L(1,4)\\&= (L(0,1), L(1,0), L(1,1), L(1,2), L(1,3), L(1,4)) \in G_5\end{aligned}$$

$L(0,1) \cap L(1,0) \cap L(1,1) \cap L(1,2) \cap L(1,3) \cap L(1,4) = L(0,0)$

The union of the lines in G_5 is a member of the space. The intersection of the lines in G_5 gives a line with point (0,0), which is a member of the topological space. Hence, G_5 forms an indiscrete topological space with $G_5 = \{L(0,1), L(1,0), L(1,1), L(1,2), L(1,3), L(1,4)\}$ and $G_1 = \{L(0,0)\}$ as topology.

Figure 2 is the graph of $G_6 = Z_6 \times Z_6$.
It comprises the following:
$\psi(6) = 12$ lines with 6 points each. That is, L(0,1),L(1,0),L(1,1),L(1,2), L(1,3), L(1,4),L(1,5), L(2,1), L(2,3),L(2,5),L(3,1) and L(3,2).
$\psi(3)$ lines with 3 points.
$\psi(2)$ lines with 2 points.
$\psi(1)$ line with 1 point (0,0).
The $\psi(6)$ lines intersect at the point (0,0).

1.

$$\begin{aligned}&L(0,1) \cup L(1,0) \cup L(1,1) \cup L(1,2) \cup L(1,3) \cup L(1,4) \cup L(1,5) \cup L(2,1) \cup L(2,3)\\&\cup L(2,5) \cup L(3,1) \cup L(3,2) = \{L(0,1), L(1,0), L(1,1), L(1,2), L(1,3), L(1,4),\\&L(1,5), L(2,1), L(2,3), L(2,5), L(3,1), L(3,2)\} \in G_6\end{aligned}$$

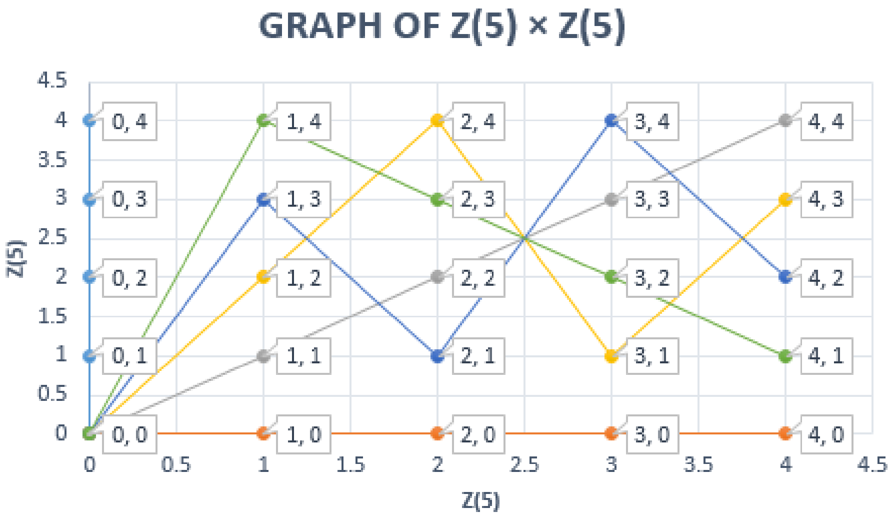

Fig. 1. The Graph of $G_5 = Z_5 \times Z_5$

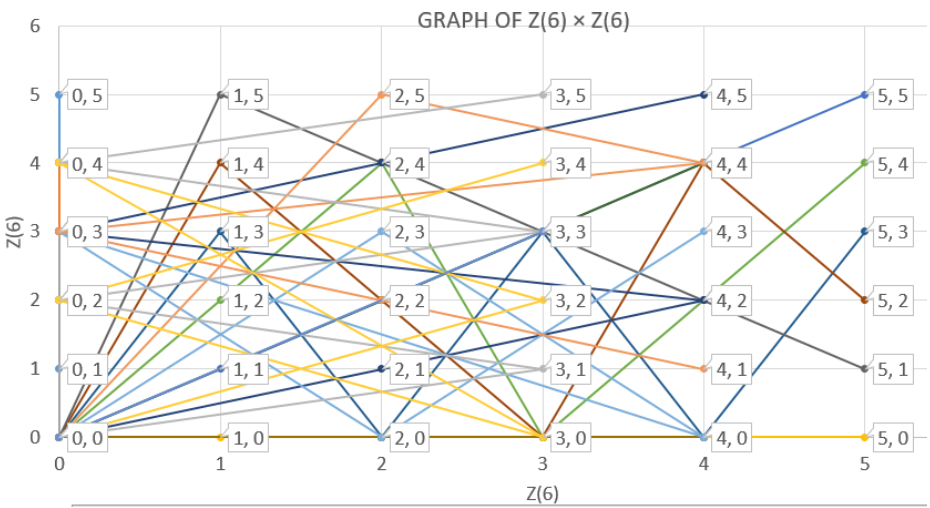

Fig. 2. The Graph of $G_6 = Z_6 \times Z_6$

2. $L(1,2) \cap L(1,5) = \{L(0,0), L(2,4), L(4,2)\} \in G_6$
3. $G_6 \subseteq G_6$
4. $G_1 = \{(0,0)\} \in G_6$

The union of the lines in G_6, that is, the set containing lines with 6,3, and 2 points are members of the space. The intersection of any pair of arbitrary lines with the same number of points in the geometry G_6 gives a line $L_k(x,y)$, which is a member of the topological space.

Hence, G_6 forms a discrete topological space with its subgeometries G_1, G_2, and G_3 as topology.

6.2. REMARKS:

The ideas presented in this work could be applicable in different areas. For example, it is used in generating the social security number of a population and is also used to generate the International Standard Book Number (ISBN).

Moreover, suppose that there are two individuals, Alice and Bob, who want to exchange a message. Alice has a digitized message to send to Bob via a noisy channel. Such a message is coded before sending, and Bob receives a coded message.

A code in the context of this work is a finite geometry G_d over Z_d which can be a set of messages stored in a medium such as a magnetic disk over a period of time. Alice is communicating to herself by storing information in the disk in the same location over a period of time.

In this work, we let $M(x,y)$ be the message Alice sent. Bob, who is far away received $M(\alpha x, \alpha y)$. To confirm whether the message has been tampered with or not, he checks for a bijection between the content of what he received and what was communicated initially by Alice. That is, $M(x,y) \cong M(\alpha x, \alpha y)$.

Between the storage time and retrieval time, there are possibilities that events may have happened to the content of the disk. This incident could be a virus, worms, Trojan, etc, among other forms of attack which has the potential to tamper with the content of the disk.

In order to protect the content of the disk or the message sent by Alice to Bob over a noisy channel, error detection or content validation code safeguards the integrity of the message. It can be established that if the noise is tolerable, then $M(x,y) \cong \alpha M(x,y)$, otherwise the message receive is corrupt. As the reason for encoding is to safeguard the integrity of the data, this is carried out by adding redundancy to a message, which in the context of this work is represented by α.

7. Conclusion

This work investigated a topological space that exists in non-near-linear finite geometry. It was discovered that two or more lines, members of topology in geometry, meet at a minimum of one point, and the point is a member of the topology. A union of lines in the geometry is a member of the topology. The existence of non-near linear finite geometry, which originated from non-prime dimensional finite geometry, leads to a discrete topology, while near-linear finite geometry leads to an indiscrete topology. More importantly, an introduction and usage of the divisor function creates a partial ordered relation between non-near-linear finite geometry and its subgeometries, which by extension leads to a topological space.

Competing Interests

The authors declare that they have no competing interests.

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