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Extending the Scope of the Fractional Hardy Integral Inequality

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ABSTRACT

In this article, we establish two theorems that generalize the fractional Hardy integral inequality by incorporating several intermediate functions and parameters. The assumptions made are tractable. Some of them can be related to the notions of sub-additivity and sub-multiplicativity. The proofs are self-contained, without intermediate results, and all details are given. We thus extend the applicability of a classical integral inequality and offer new potential applications in mathematical analysis.

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1. Introduction

Integral inequalities are one of the most useful tools in mathematics. They include fundamental results, such as the Cauchy-Schwarz integral inequality, the Grönwall integral inequality, the Hardy integral inequality, the Hilbert integral inequality, the Hölder integral inequality, the Jensen integral inequality, the Minkowski integral inequality, the Steffensen integral inequality and the Young integral inequality. They are designed to give tractable bounds on integrals that cannot be determined in a standard way, with the aim of simplifying and solving challenging mathematical problems. Integral inequalities also help to analyze relationships between functions, which are particularly important in real analysis, functional analysis including operator theory, optimization theory and probability theory. Their applications extend to the study of differential equations and variational problems. A comprehensive treatment of classical integral inequalities, along with their theory and applications, can be found in [17, 4, 32, 3, 34].

Integral inequalities of the Hardy type are diverse in form and scope. An overview of the subject can be found in [18, 25, 16, 17]. In this article, we emphasize a special Hardy integral inequality called the fractional Hardy integral inequality. A formal statement is given below. Let $p \in [1, +\infty)$, $\lambda \in (0, +\infty)\backslash\{1\}$, i.e., $(0, +\infty)$ excluding the point 1, and $f : (0, +\infty) \mapsto \mathbb{R}$

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(be a function) such that

$$\int_0^{+\infty} \frac{|f(x)|^p}{x^\lambda} dx < +\infty.$$

Then there exists a constant $A \in (0, +\infty)$ satisfying

$$\int_0^{+\infty} \frac{|f(x)|^p}{x^\lambda} dx \leq A \int_0^{+\infty} \int_0^{+\infty} \frac{|f(x) - f(y)|^p}{|x - y|^{1+\lambda}} dxdy.$$

For the technical details, including discussions of the expression of A, see [18]. Over the years, many improvements, extensions and refinements have been made to this inequality. Notable developments can be found over several decades, including [7, 31] from 1990 to 2000, [12, 14, 2, 13] from 2001 to 2010, [29, 20, 28, 11, 33, 23, 9, 24, 5, 35] from 2011 to 2020, and [30, 26, 1, 27, 8] from 2021 to 2025. In particular, [8] revisits the methodology introduced in [24], with special emphasis on [24, Lemma 2], leading to several significant contributions. Among them is a detailed investigation of the fractional Hardy integral inequality for the full range $p \in (0, +\infty)$, including the challenging case $p \in (0, 1)$, as shown in [8, Proposition 2.1]. Furthermore, [8] provides an exact upper bound for a fractional-type integral, expressed as follows:

$$\int_0^{+\infty} \frac{|f(x)|^p}{x} dx.$$

This corresponds to the left-hand side of the fractional Hardy integral inequality evaluated at $\lambda = 1$, a case that is excluded from the original formulation. For details, see [8, Proposition 3.1]. More precisely, let $p \in (0, +\infty)$ and $f : (0, +\infty) \mapsto \mathbb{R}$ such that

$$\int_0^{+\infty} \frac{|f(x)|^p}{x} dx < +\infty.$$

Then the following holds:

$$\int_0^{+\infty} \frac{|f(x)|^p}{x} dx \leq B_{p,\sigma,\alpha} \int_0^{+\infty} \int_0^{+\infty} \frac{|f(x) - f(y)|^p}{|x^\sigma - y|^{1+1/\sigma}} dydx,$$

where

$$B_{p,\sigma,\alpha} = \max(2^{p-1}, 1) \left[\sigma - \max(2^{p-1}, 1)\right]^{-1} \alpha^{-1} \sigma \left\{\max\left[|1 - \alpha|, |1 - 2\alpha|\right]\right\}^{1+1/\sigma},$$

$\sigma > \max(2^{p-1}, 1)$ and $\alpha \in (0, +\infty)$. As this case remains relatively underexplored, the result gives new ideas for applications in mathematical analysis. Moreover, the issue of restricting the domain of integration to a finite interval is addressed in [8, Proposition 4.1].

In this article, we generalize the framework of [8] by establishing two theorems. These theorems extend the scope of the fractional Hardy integral inequality in important ways. Specifically, the first theorem generalizes [8, Propositions 2.1 and 3.1] by incorporating multiple functions and additional parameters. Formally, it states that

$$\text{"} \int_0^{+\infty} |f(x)|^p g(x) h(x) dx \leq C \int_0^{+\infty} \int_0^{+\infty} \frac{|f(x) - f(y)|^p}{|h(x) - y|^\gamma} dxdy, \text{"}$$

where g and h are intermediate functions with complementary roles and C is a constant. They are subject to manageable assumptions involving monotonicity and inequality constraints, governed by several adjustable parameters. Notably, the special case $g(x)h(x) = 1/x^{\lambda}$ with $\lambda \in (0, +\infty)\backslash\{1\}$ recovers the classical fractional Hardy integral inequality, as well as [8, Proposition 2.1], while the case $g(x)h(x) = 1/x$ corresponds to [8, Propositions 3.1]. The second theorem introduces an even greater degree of generality by relaxing the dependence on p, and going beyond the traditional weighted L^p norm. Under similar tractable assumptions, it states that

$$\text{"}\int_0^{+\infty} \varphi\left[f(x)\right] g(x)h(x)dx \leq D \int_0^{+\infty}\int_0^{+\infty} \frac{\varphi[f(x)-f(y)]}{\psi[h(x)-y]}dydx,\text{"}$$

where φ and ψ are specific functions and D is a constant. Some of the assumptions made on these functions are related to classical properties such as sub-additivity and sub-multiplicativity. Note that the choice of $\varphi(x) = x^p$ and $\psi(x) = x^{\gamma}$ reduces the second theorem to the newly established first. The proofs are presented in a completely self-contained manner, with all details given and no reliance on auxiliary lemmas. Through these new contributions, combined with an original methodological approach, we significantly extend the theoretical framework and potential applications of fractional Hardy-type integral inequalities.

The remainder of the article is structured as follows: Section 2 presents the statements of the two main theorems. Section 3 is devoted to their proofs. Finally, Section 4 offers concluding remarks and potential directions for future research.

2. Two Theorems

The first generalized fractional Hardy integral inequality is presented in the theorem below. We emphasize the assumptions made on the new intermediate functions, g and h.

Theorem 2.1. *Let $p \in (0, +\infty)$, $f : (0, +\infty) \mapsto \mathbb{R}$ and $g, h : (0, +\infty) \mapsto (0, +\infty)$. We assume that*

- *h is increasing with $\lim_{x\to 0} h(x) = 0$ and $\lim_{x\to+\infty} h(x) = +\infty$,*
- *we have*

$$\int_0^{+\infty} |f(x)|^p g(x)h(x)dx < +\infty,$$

- *there exist two constants, $\theta \in (0, +\infty)$ and $\kappa \in (0, [\max(2^{p-1}, 1)]^{-1})$, such that*

$$\sup_{y\in(0,+\infty)} \left[\frac{1}{g(y)h(y)} \int_{h^{-1}[y/(2\theta)]}^{h^{-1}(y/\theta)} g(x)dx\right] \leq \theta\kappa, \tag{2.1}$$

- *there exist two constants, $\gamma \in (0, +\infty)$ and $\zeta \in (0, +\infty)$, such that*

$$\sup_{x\in(0,+\infty)} [h(x)^{\gamma} g(x)] \leq \zeta. \tag{2.2}$$

Then the following holds:

$$\int_0^{+\infty} |f(x)|^p g(x)h(x)dx \leq C_{p,\theta,\kappa,\gamma,\zeta} \int_0^{+\infty}\int_0^{+\infty} \frac{|f(x)-f(y)|^p}{|h(x)-y|^\gamma} dxdy,$$

where

$$C_{p,\theta,\kappa,\gamma,\zeta} = \left[1-\kappa \max(2^{p-1},1)\right]^{-1} \theta^{-1} \max(2^{p-1},1) \left\{\max\left[|1-\theta|,|1-2\theta|\right]\right\}^\gamma \zeta. \quad (2.3)$$

The proof is based on a thorough extension of that of [8, Propositions 2.1 and 3.1]. The details are given in Section 3.

In this theorem, choosing $g(x) = 1/x^{1+\lambda}$ with $\lambda \in (0,+\infty)\backslash\{1\}$ and $h(x) = x$, we find that $\kappa = (2^\lambda - 1)\theta^{\lambda-1}/\lambda$, $\gamma = 1+\lambda$ and $\zeta = 1$, and the main result reduces to

$$\int_0^{+\infty} \frac{|f(x)|^p}{x^\lambda} dx \leq C_{p,\theta,\kappa,\gamma,\zeta} \int_0^{+\infty}\int_0^{+\infty} \frac{|f(x)-f(y)|^p}{|x-y|^{1+\lambda}} dxdy,$$

with the corresponding edited expression of $C_{p,\theta,\kappa,\gamma,\zeta}$, as given in Equation (2.3). We thus obtain the standard fractional Hardy integral inequality extended to the case $p \in (0,+\infty)$, as shown in [8, Proposition 2.1].

As another example related to the literature, choosing $g(x) = 1/x^{1+\sigma}$ with $\sigma > \max(2^{p-1},1)$ and $h(x) = x^\sigma$, we find that $\kappa = 1/\sigma$, $\gamma = 1+1/\sigma$ and $\zeta = 1$, and the main result reduces to

$$\int_0^{+\infty} \frac{|f(x)|^p}{x} dx \leq C_{p,\theta,\kappa,\gamma,\zeta} \int_0^{+\infty}\int_0^{+\infty} \frac{|f(x)-f(y)|^p}{|x^\sigma-y|^{1+1/\sigma}} dydx,$$

with the corresponding edited expression of $C_{p,\theta,\kappa,\gamma,\zeta}$. We thus obtain the special fractional Hardy integral inequality as presented in [8, Proposition 3.1].

Continuing in the spirit of generalization, the theorem below can be viewed as an extended version of Theorem 2.1, incorporating additional intermediate functions and parameters. Particular attention is given to the assumptions made on the new functions g, h, ψ and φ, some of which will be examined in more detail after the statement of the theorem.

Theorem 2.2. *Let $f : (0,+\infty) \mapsto \mathbb{R}$, $g, h, \psi : (0,+\infty) \mapsto (0,+\infty)$ and $\varphi : \mathbb{R} \mapsto (0,+\infty)$. We assume that*

- *h is increasing with $\lim_{x\to 0} h(x) = 0$ and $\lim_{x\to+\infty} h(x) = +\infty$,*
- *for any $(x,y) \in \mathbb{R}^2$, there exists a constant $\alpha > 0$ such that*

$$\varphi(x+y) \leq \alpha[\varphi(x)+\varphi(y)], \quad (2.4)$$

- *ψ is continuous, monotonic and, for any $(x,y) \in (0,+\infty)^2$, there exists a constant $\beta > 0$ such that*

$$\psi(xy) \leq \beta\psi(x)\psi(y), \quad (2.5)$$

- *we have*

$$\int_0^{+\infty} \varphi\left[f(x)\right] g(x)h(x)dx < +\infty,$$

- *there exist two constants,* $\theta \in (0,+\infty)$ *and* $\kappa \in (0,\alpha^{-1})$*, such that*

$$\sup_{y\in(0,+\infty)}\left[\frac{1}{g(y)h(y)}\int_{h^{-1}[y/(2\theta)]}^{h^{-1}(y/\theta)} g(x)dx\right] \leq \theta\kappa, \tag{2.6}$$

- *there exists a constant* $\zeta \in (0,+\infty)$ *such that*

$$\sup_{x\in(0,+\infty)}\{\psi\,[h(x)]\,g(x)\} \leq \zeta. \tag{2.7}$$

Then the following holds:

$$\int_0^{+\infty} \varphi\,[f(x)]\,g(x)h(x)dx \leq D_{\alpha,\beta,\theta,\kappa,\zeta}\int_0^{+\infty}\int_0^{+\infty}\frac{\varphi[f(x)-f(y)]}{\psi[h(x)-y]}dydx,$$

where

$$D_{\alpha,\beta,\theta,\kappa,\zeta} = (1-\kappa\alpha)^{-1}\,\alpha\theta^{-1}\beta\max\left[\psi(1-\theta),\psi(1-2\theta)\right]\zeta. \tag{2.8}$$

The proof builds upon and extends the arguments used in Theorem 2.1, with full details provided in Section 3. Notably, choosing $\varphi(x) = x^p$ and $\psi(x) = x^\gamma$, the theorem reduces to Theorem 2.1 as a special case.

Note that the classes of functions characterized by the assumptions in Equations (2.4) and (2.5) are very large, beyond the classical power function.

In particular, the class of functions defined in Equation (2.4) with $\alpha = 1$ is that of the sub-additive functions. Examples of such functions include $\varphi(x) = x^\tau$ with $\tau \in [0,1]$, $\varphi(x) = \arctan(x)$, $\varphi(x) = \log(1+x)$ and $\varphi(x) = x/(\omega + x)$ with $\omega > 0$. Further details on this class can be found in [6, 10].

The class of functions defined in Equation (2.5) with $\beta = 1$ is the well-known class of sub-multiplicative functions. Representative examples include $\psi(x) = x^\iota$ with $\iota \in \mathbb{R}$, $\psi(x) = \log(\delta + x)$ with $\delta \geq e$ and $\psi(x) = 1/\tanh(\eta x)$ with $\eta > 0$. Additional information on this class can be found in [21, 15, 22].

The diversity of these classes of functions highlights the flexibility and generality of Theorem 2.2, and thus extends the range of potential applications of fractional Hardy-type integral inequalities.

3. Proofs

This section contains the detailed proofs of Theorems 2.1 and 2.2, one after another.

3.1. Proof of Theorem 2.1

Proof. A classic power-convexity inequality gives

$$|f(x)|^p = |f(y) + [f(x)-f(y)]|^p \leq \max(2^{p-1},1)|f(y)|^p + \max(2^{p-1},1)|f(x)-f(y)|^p.$$

Multiplying both sides by $\theta^{-1}g(x) \geq 0$, we obtain

$$\theta^{-1}|f(x)|^p g(x) \leq \max(2^{p-1},1)\theta^{-1}|f(y)|^p g(x) + \max(2^{p-1},1)\theta^{-1}|f(x)-f(y)|^p g(x).$$

Integrating both sides with respect to the variable $y \in (\theta h(x), 2\theta h(x))$, we get

$$\theta^{-1} \int_{\theta h(x)}^{2\theta h(x)} |f(x)|^p g(x) dy \le \max(2^{p-1}, 1)\theta^{-1} \int_{\theta h(x)}^{2\theta h(x)} |f(y)|^p g(x) dy + \max(2^{p-1}, 1)\theta^{-1} \int_{\theta h(x)}^{2\theta h(x)} |f(x) - f(y)|^p g(x) dy.$$

For the left-hand side term, we have

$$\theta^{-1} \int_{\theta h(x)}^{2\theta h(x)} |f(x)|^p g(x) dy = \theta^{-1} |f(x)|^p g(x) \int_{\theta h(x)}^{2\theta h(x)} dy = |f(x)|^p g(x) h(x).$$

Using this and integrating both sides with respect to the variable $x \in (0, +\infty)$, we find that

$$\int_0^{+\infty} |f(x)|^p g(x) h(x) dx \le \max(2^{p-1}, 1)\Phi + \max(2^{p-1}, 1)\Psi, \tag{3.1}$$

where

$$\Phi = \theta^{-1} \int_0^{+\infty} \int_{\theta h(x)}^{2\theta h(x)} |f(y)|^p g(x) dy dx$$

and

$$\Psi = \theta^{-1} \int_0^{+\infty} \int_{\theta h(x)}^{2\theta h(x)} |f(x) - f(y)|^p g(x) dy dx.$$

We now want to majorize Φ and Ψ appropriately.

For Φ, using the Fubini-Tonelli integral theorem, justified by the non-negativity of the integrand, which ensures the validity of interchanging the order of integration, and the fact that h is increasing with $\lim_{x\to 0} h(x) = 0$ and $\lim_{x\to+\infty} h(x) = +\infty$, we obtain

$$\Phi = \theta^{-1} \int_0^{+\infty} \int_{h^{-1}[y/(2\theta)]}^{h^{-1}(y/\theta)} |f(y)|^p g(x) dx dy = \theta^{-1} \int_0^{+\infty} |f(y)|^p \left[\int_{h^{-1}[y/(2\theta)]}^{h^{-1}(y/\theta)} g(x) dx \right] dy.$$

The assumption in Equation (2.1) says that there exist $\theta \in (0, +\infty)$ and $\kappa \in (0, [\max(2^{p-1}, 1)]^{-1})$ such that, for any $y \in (0, +\infty)$,

$$\int_{h^{-1}[y/(2\theta)]}^{h^{-1}(y/\theta)} g(x) dx \le \theta\kappa g(y) h(y).$$

This implies that

$$\Phi \le \kappa \int_0^{+\infty} |f(y)|^p g(y) h(y) dy. \tag{3.2}$$

For Ψ, considering the constant γ in Equation (2.2) and applying a fractional-type decomposition of the integrand, we have

$$\Psi = \theta^{-1}\int_0^{+\infty}\int_{\theta h(x)}^{2\theta h(x)} |h(x)-y|^{\gamma} g(x) \times \frac{|f(x)-f(y)|^p}{|h(x)-y|^{\gamma}} dydx$$

$$\leq \theta^{-1}\left\{\sup_{x\in(0,+\infty)}\sup_{y\in(\theta h(x),2\theta h(x))}[|h(x)-y|^{\gamma}g(x)]\right\}\int_0^{+\infty}\int_{\theta h(x)}^{2\theta h(x)}\frac{|f(x)-f(y)|^p}{|h(x)-y|^{\gamma}} dydx$$

$$\leq \theta^{-1}\left\{\sup_{x\in(0,+\infty)}\sup_{y\in(\theta h(x),2\theta h(x))}[|h(x)-y|^{\gamma}g(x)]\right\}\int_0^{+\infty}\int_0^{+\infty}\frac{|f(x)-f(y)|^p}{|h(x)-y|^{\gamma}} dydx.$$

By Equation (2.2), we have $\sup_{x\in(0,+\infty)}[h(x)^{\gamma}g(x)] \leq \zeta$. Using this, we get

$$\begin{aligned}
&\sup_{x\in(0,+\infty)}\sup_{y\in(\theta h(x),2\theta h(x))}[|h(x)-y|^{\gamma}g(x)] \\
&= \sup_{x\in(0,+\infty)}\max\left[|h(x)-\theta h(x)|^{\gamma}g(x), |h(x)-2\theta h(x)|^{\gamma}g(x)\right] \\
&= \{\max[|1-\theta|,|1-2\theta|]\}^{\gamma}\left\{\sup_{x\in(0,+\infty)}[h(x)^{\gamma}g(x)]\right\} \\
&\leq \{\max[|1-\theta|,|1-2\theta|]\}^{\gamma}\zeta.
\end{aligned}$$

We therefore obtain

$$\Psi \leq \theta^{-1}\{\max[|1-\theta|,|1-2\theta|]\}^{\gamma}\zeta\int_0^{+\infty}\int_0^{+\infty}\frac{|f(x)-f(y)|^p}{|h(x)-y|^{\gamma}} dydx. \tag{3.3}$$

Combining Equations (3.1), (3.2) and (3.3), we find that

$$\int_0^{+\infty}|f(x)|^p g(x)h(x)dx \leq \kappa\max(2^{p-1},1)\int_0^{+\infty}|f(x)|^p g(x)h(x)dx$$
$$+\theta^{-1}\max(2^{p-1},1)\{\max[|1-\theta|,|1-2\theta|]\}^{\gamma}\zeta\int_0^{+\infty}\int_0^{+\infty}\frac{|f(x)-f(y)|^p}{|h(x)-y|^{\gamma}} dydx.$$

This gives

$$\left[1-\kappa\max(2^{p-1},1)\right]\int_0^{+\infty}|f(x)|^p g(x)h(x)dx$$
$$\leq \theta^{-1}\max(2^{p-1},1)\{\max[|1-\theta|,|1-2\theta|]\}^{\gamma}\zeta\int_0^{+\infty}\int_0^{+\infty}\frac{|f(x)-f(y)|^p}{|h(x)-y|^{\gamma}} dydx.$$

Since $\kappa \in (0, [\max(2^{p-1},1)]^{-1})$, we can isolate the main integral of interest, as follows:

$$\begin{aligned}
&\int_0^{+\infty}|f(x)|^p g(x)h(x)dx \\
&\leq \left[1-\kappa\max(2^{p-1},1)\right]^{-1}\theta^{-1}\max(2^{p-1},1)\{\max[|1-\theta|,|1-2\theta|]\}^{\gamma}\zeta\times \\
&\int_0^{+\infty}\int_0^{+\infty}\frac{|f(x)-f(y)|^p}{|h(x)-y|^{\gamma}} dydx \\
&= C_{p,\theta,\kappa,\gamma,\zeta}\int_0^{+\infty}\int_0^{+\infty}\frac{|f(x)-f(y)|^p}{|h(x)-y|^{\gamma}} dxdy,
\end{aligned}$$

where $C_{p,\theta,\kappa,\gamma,\zeta}$ is indicated in Equation (2.3). The proof of Theorem 2.1 ends. ∎

3.2. Proof of Theorem 2.2

Proof. The proof follows the lines of that of Theorem 2.1, with the consideration of the new intermediate functions and parameters. Applying the inequality in Equation (2.4), we have

$$\varphi[f(x)] = \varphi[f(y) + [f(x) - f(y)]] \leq \alpha\varphi[f(y)] + \alpha\varphi[f(x) - f(y)].$$

Multiplying both sides by $\theta^{-1}g(x) \geq 0$, we have

$$\theta^{-1}\varphi[f(x)]g(x) \leq \alpha\theta^{-1}\varphi[f(y)]g(x) + \alpha\theta^{-1}\varphi[f(x) - f(y)]g(x).$$

Integrating both sides with respect to the variable $y \in (\theta h(x), 2\theta h(x))$, we get

$$\theta^{-1}\int_{\theta h(x)}^{2\theta h(x)} \varphi[f(x)]g(x)dy \leq \alpha\theta^{-1}\int_{\theta h(x)}^{2\theta h(x)} \varphi[f(y)]g(x)dy + \alpha\theta^{-1}\int_{\theta h(x)}^{2\theta h(x)} \varphi[f(x) - f(y)]g(x)dy.$$

For the left-hand side term, we have

$$\theta^{-1}\int_{\theta h(x)}^{2\theta h(x)} \varphi[f(x)]g(x)dy = \theta^{-1}\varphi[f(x)]g(x)\int_{\theta h(x)}^{2\theta h(x)} dy = \varphi[f(x)]g(x)h(x).$$

Using this and integrating both sides with respect to the variable $x \in (0, +\infty)$, we obtain

$$\int_0^{+\infty} \varphi[f(x)]g(x)h(x)dx \leq \alpha\Upsilon + \alpha\Omega, \tag{3.4}$$

where

$$\Upsilon = \theta^{-1}\int_0^{+\infty}\int_{\theta h(x)}^{2\theta h(x)} \varphi[f(y)]g(x)dydx$$

and

$$\Omega = \theta^{-1}\int_0^{+\infty}\int_{\theta h(x)}^{2\theta h(x)} \varphi[f(x) - f(y)]g(x)dydx.$$

We now want to majorize Υ and Ω appropriately.

For Υ, the Fubini-Tonelli integral theorem, which ensures the validity of interchanging the order of integration, combined with the fact that h is increasing with $\lim_{x\to 0} h(x) = 0$ and $\lim_{x\to+\infty} h(x) = +\infty$, gives

$$\Upsilon = \theta^{-1}\int_0^{+\infty}\int_{h^{-1}[y/(2\theta)]}^{h^{-1}(y/\theta)} \varphi[f(y)]g(x)dxdy = \theta^{-1}\int_0^{+\infty} \varphi[f(y)]\left[\int_{h^{-1}[y/(2\theta)]}^{h^{-1}(y/\theta)} g(x)dx\right]dy.$$

The assumption in Equation (2.6) says that there exist $\theta \in (0, +\infty)$ and $\kappa \in (0, \alpha^{-1})$ such that, for any $y \in (0, +\infty)$,

$$\int_{h^{-1}[y/(2\theta)]}^{h^{-1}(y/\theta)} g(x)dx \leq \theta\kappa g(y)h(y).$$

This implies that

$$\Upsilon \leq \kappa \int_0^{+\infty} \varphi[f(y)]g(y)h(y)dy. \tag{3.5}$$

For Ω, a suitable fractional-type decomposition of the integrand gives

$$\begin{aligned}
\Omega &= \theta^{-1} \int_0^{+\infty} \int_{\theta h(x)}^{2\theta h(x)} \psi[h(x)-y]g(x) \times \frac{\varphi[f(x)-f(y)]}{\psi[h(x)-y]} dydx \\
&\leq \theta^{-1} \left[\sup_{x\in(0,+\infty)} \sup_{y\in(\theta h(x),2\theta h(x))} \{\psi[h(x)-y]g(x)\} \right] \int_0^{+\infty} \int_{\theta h(x)}^{2\theta h(x)} \frac{\varphi[f(x)-f(y)]}{\psi[h(x)-y]} dydx \\
&\leq \theta^{-1} \left[\sup_{x\in(0,+\infty)} \sup_{y\in(\theta h(x),2\theta h(x))} \{\psi[h(x)-y]g(x)\} \right] \int_0^{+\infty} \int_0^{+\infty} \frac{\varphi[f(x)-f(y)]}{\psi[h(x)-y]} dydx.
\end{aligned}$$

Since ψ is continuous and monotonic, we have

$$\begin{aligned}
&\sup_{x\in(0,+\infty)} \sup_{y\in(\theta h(x),2\theta h(x))} \{\psi[h(x)-y]g(x)\} \\
&= \sup_{x\in(0,+\infty)} \max\left[\psi[h(x)-\theta h(x)]g(x), \psi[h(x)-2\theta h(x)]g(x)\right].
\end{aligned}$$

It follows from the assumption in Equation (2.5) that

$$\psi[h(x)-\theta h(x)] = \psi[(1-\theta)h(x)] \leq \beta\psi(1-\theta)\psi[h(x)]$$

and

$$\psi[h(x)-2\theta h(x)] = \psi[(1-2\theta)h(x)] \leq \beta\psi(1-2\theta)\psi[h(x)].$$

This combined with the assumption in Equation (2.7) gives

$$\begin{aligned}
&\sup_{x\in(0,+\infty)} \max\left[\psi[h(x)-\theta h(x)]g(x), \psi[h(x)-2\theta h(x)]g(x)\right] \\
&\leq \beta \max\left[\psi(1-\theta), \psi(1-2\theta)\right] \left\{ \sup_{x\in(0,+\infty)} [\psi[h(x)]g(x)] \right\} \\
&\leq \beta \max\left[\psi(1-\theta), \psi(1-2\theta)\right] \zeta.
\end{aligned}$$

We therefore obtain

$$\Omega \leq \theta^{-1}\beta \max\left[\psi(1-\theta), \psi(1-2\theta)\right] \zeta \int_0^{+\infty} \int_0^{+\infty} \frac{\varphi[f(x)-f(y)]}{\psi[h(x)-y]} dydx. \tag{3.6}$$

Combining Equations (3.4), (3.5) and (3.6) together, we get

$$\begin{aligned}
&\int_0^{+\infty} \varphi[f(x)]g(x)h(x)dx \leq \kappa\alpha \int_0^{+\infty} \varphi[f(x)]g(x)h(x)dx \\
&+ \alpha\theta^{-1}\beta \max\left[\psi(1-\theta), \psi(1-2\theta)\right] \zeta \int_0^{+\infty} \int_0^{+\infty} \frac{\varphi[f(x)-f(y)]}{\psi[h(x)-y]} dydx.
\end{aligned}$$

This gives

$$(1-\kappa\alpha)\int_0^{+\infty}\varphi[f(x)]g(x)h(x)dx$$
$$\leq \alpha\theta^{-1}\beta\max\left[\psi(1-\theta),\psi(1-2\theta)\right]\zeta\int_0^{+\infty}\int_0^{+\infty}\frac{\varphi[f(x)-f(y)]}{\psi[h(x)-y]}dydx.$$

Since $\kappa\in(0,\alpha^{-1})$, we can isolate the main integral of interest as follows:

$$\int_0^{+\infty}\varphi[f(x)]g(x)h(x)dx \leq (1-\kappa\alpha)^{-1}\alpha\theta^{-1}\beta\max\left[\psi(1-\theta),\psi(1-2\theta)\right]\zeta\times$$
$$\int_0^{+\infty}\int_0^{+\infty}\frac{\varphi[f(x)-f(y)]}{\psi[h(x)-y]}dydx$$
$$= D_{\alpha,\beta,\theta,\kappa,\zeta}\int_0^{+\infty}\int_0^{+\infty}\frac{\varphi[f(x)-f(y)]}{\psi[h(x)-y]}dydx,$$

where $D_{\alpha,\beta,\theta,\kappa,\zeta}$ is indicated in Equation (2.8). This concludes the proof of Theorem 2.2. ∎

As previously noted, the proofs of the theorems are entirely self-contained and do not rely on auxiliary lemmas. Their structure may also inspire further extensions and improvements, representing an additional contribution of this work.

4. Conclusion and Future Work

In this article, we have presented two theorems that generalize the fractional Hardy integral inequality in several ways. Notably, these results introduce new intermediate functions and parameters that significantly increase the flexibility and applicability of this inequality. The self-contained proofs ensure both clarity and completeness, making our results accessible for further theoretical exploration. Potential future directions include refining the obtained inequalities for different function classes beyond sub-additive and sub-multiplicative functions, extending the results to integral operators, and exploring discrete analogues. In addition, future work could address the numerical aspects, where fractional Hardy-type integral inequalities could provide rigorous bounds for approximation methods.

Competing Interests

The authors declare that they have no competing interests.

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