Nonlinear Convex Analysis and Optimization
Vol. 4, No. 1, 2025, pp. 1–15
https://bangmodjmcs.com/index.php/ncao

NONLINEAR CONVEX ANALYSIS AND OPTIMIZATION
An International Journal on Numerical, Computation and Applications

The Proximal Point Algorithm for Monotone Vector Fields on Complete Geodesic Spaces

Shuta Sudo

Department of Information Science, Toho University, Miyama, Funabashi, Chiba 274-8510, Japan
7523001s@st.toho-u.ac.jp

ABSTRACT

In this paper, we deal with monotone vector fields defined on geodesic spaces and their zero point approximation method. In an appropriate setting, we can define the resolvent operator for a given monotone vector field, and then that operator has many useful properties which are effective for the fixed point theory. We will show an approximation theorem for a monotone vector field with the canonical proximal point algorithm.

Article History
Received 12 Feb 2025
Revised 3 Apr 2025
Accepted 7 Apr 2025

Keywords:
Geodesic space;
Monotone vector field;
Proximal point algorithm;
Zero point approximation

MSC
47H05; 58C06

1. Introduction

Let A be a set-valued mapping from a Hilbert space H to a subset of H. To find a point $x \in H$ such that $0_H \in Ax$ is called a zero point problem for A. We know that the class of zero point problems includes some nonlinear problems such as convex minimisation problems, equilibrium problems, fixed point problems, and so forth.

For a given maximal monotone operator A on a Hilbert space H and a positive real number r, we can define a mapping J_{rA} by

$$J_{rA}x = (I + rA)^{-1}x$$

for $x \in H$. Note that $(I + rA)^{-1}$ is single-valued even if A is set-valued. We call this mapping J_{rA} the resolvent operator for rA. One of remarkable facts is this: The set of all fixed points of J_{rA} coincides with the set of all zero points of A. On the other hand, the proximal point algorithm is a typical zero point approximation method. Rockafellar [17] has proved the following approximation theorem to find a zero point of a maximal monotone operator:

Theorem 1.1. *[17] Let H be a Hilbert space and A a maximal monotone operator on H, which has a zero point. Let $\{r_n\}$ be a sequence of positive real numbers such that $\inf_{k\in\mathbb{N}} r_k > 0$. For a given initial point $x_1 \in H$, generate a sequence $\{x_n\}$ of H by*

$$x_{n+1} = J_{r_nA}x_n = (I + r_nA)^{-1}x_n$$

Please cite this article as: S. Sudo, The Proximal Point Algorithm for Monotone Vector Fields on Complete Geodesic Spaces, Nonlinear Convex Anal. & Optim., Vol. 4, No. 1, 2025, 1–15. https://doi.org/10.58715/ncao.2025.4.1

for $n \in \mathbb{N}$. Then, $\{x_n\}$ converges weakly to some zero point of A.

Recently, approximation theorems with the proximal point algorithm for convex functions and equilibrium problems have been shown on geodesic spaces having bounded curvature, which are called CAT(κ) spaces. See [2, 8, 12] for instance. Additionally, the notion of monotone operators on Hilbert spaces has been generalised to the framework of geodesic spaces. For instance, Chaipunya, Kohsaka and Kumam [4] have dealt with monotone vector fields on a CAT(0) space using tangent spaces, and recently, the author has proposed a class of monotone vector fields on a CAT(κ) space; see [19]. For related results, refer to [6] for instance.

In this work, we apply the proximal point algorithm with the resolvent operator of a monotone vector field to approximate its zero point. Further, we show an application to an equilibrium problem on CAT(κ) spaces.

2. Preliminaries

Let (M, d) be a metric space and let $D \in]0, \infty]$. For $x, y \in M$ and $l = d(x, y)$, we call a mapping γ_{xy} from $[0, l]$ into M a geodesic from x to y if $\gamma_{xy}(0) = x$, $\gamma_{xy}(l) = y$ and

$$d(\gamma_{xy}(s), \gamma_{xy}(t)) = |s - t|$$

for $s, t \in [0, l]$. We say M is uniquely D-geodesic if for $x, y \in M$ with $d(x, y) < D$, there is a unique geodesic from x to y. In a uniquely D-geodesic space M, for $x, y \in M$ with $d(x, y) < D$ and $t \in [0, 1]$, we define their convex combination by

$$tx \oplus (1 - t)y = \gamma_{xy}((1 - t)d(x, y)).$$

Let C be a subset of a uniquely D-geodesic space M such that $d(u, v) < D$ for any $u, v \in C$. We say C is convex if

$$tx \oplus (1 - t)y \in C$$

for $x, y \in C$ and $t \in [0, 1]$.

To define a CAT(κ) space, we first define a function c_κ from $\mathbb{R}$ to $[0, \infty[$ by

$$c_\kappa(a) = \frac{1}{2}a^2 + \sum_{n=2}^{\infty} \frac{(-\kappa)^{n-1}a^{2n}}{(2n)!} = \begin{cases} \dfrac{1}{\kappa}\left(1 - \cos\left(\sqrt{\kappa}a\right)\right) & (\kappa > 0); \\ \dfrac{1}{2}a^2 & (\kappa = 0); \\ \dfrac{1}{-\kappa}\left(\cosh\left(\sqrt{-\kappa}a\right) - 1\right) & (\kappa < 0) \end{cases}$$

for $a \in \mathbb{R}$. Then, for $a \in \mathbb{R}$,

$$c'_\kappa(a) = a + \sum_{n=2}^{\infty} \frac{(-\kappa)^{n-1}a^{2n-1}}{(2n-1)!} = \begin{cases} \dfrac{\sin\left(\sqrt{\kappa}a\right)}{\sqrt{\kappa}} & (\kappa > 0); \\ a & (\kappa = 0); \\ \dfrac{\sinh\left(\sqrt{-\kappa}a\right)}{\sqrt{-\kappa}} & (\kappa < 0) \end{cases}$$

and

$$c_\kappa''(a) = 1 + \sum_{n=2}^{\infty} \frac{(-\kappa)^{n-1} a^{2n-2}}{(2n-2)!} = \begin{cases} \cos\left(\sqrt{\kappa}a\right) & (\kappa > 0); \\ 1 & (\kappa = 0); \\ \cosh\left(\sqrt{-\kappa}a\right) & (\kappa < 0). \end{cases}$$

It hold from the definition of c_κ that $c_\kappa(0) = c_\kappa'(0) = 0$ and $c_\kappa''(0) = 1$. Remark that

$$(1 - c_\kappa''(a))c_\kappa(b) = c_\kappa(a)(1 - c_\kappa''(b))$$

for $a, b \in \mathbb{R}$.

We denote the diameter of model spaces by D_κ, and define it by

$$D_\kappa = \begin{cases} \dfrac{\pi}{\sqrt{\kappa}} & (\kappa > 0); \\ \infty & (\kappa \leq 0). \end{cases}$$

Note that c_κ is increasing on $[0, D_\kappa[$. For a metric space (M, d) and a real number κ, we define a function ϕ_κ from M^2 to $\mathbb{R}$ by

$$\phi_\kappa(x, y) = c_\kappa(d(x, y))$$

for $x, y \in M$. We know the following properties of ϕ_κ:

- $\phi_\kappa(x, y) \geq 0$ for $x, y \in M$;
- $\phi_\kappa(x, y) = 0$ if and only if $x = y$ for $x, y \in M$ with $d(x, y) < 2D_\kappa$;
- $\phi_\kappa(x, y) = \phi_\kappa(y, x)$ for $x, y \in M$.

We define a coefficient adjuster $(\cdot)_l^\kappa$ on $[0, 1]$ by

$$(t)_l^\kappa = \begin{cases} \dfrac{c_\kappa'(tl)}{c_\kappa'(l)} & (l \in]0, D_\kappa[); \\ t & (l = 0) \end{cases}$$

for $t \in [0, 1]$.

Now, we define a CAT(κ) space. It is usually defined with a notion of model spaces and their triangles. However, we can define a CAT(κ) space with the following equivalent condition to the definition: That is, a uniquely D_κ-geodesic space M for $\kappa \in \mathbb{R}$ is a CAT(κ) space if and only if

$$\begin{aligned} \phi_\kappa(tx \oplus (1-t)y, z) \leq{}& (t)_l^\kappa \phi_\kappa(x, z) + (1-t)_l^\kappa \phi_\kappa(y, z) \\ &- (t)_l^\kappa \phi_\kappa(x, tx \oplus (1-t)y) - (1-t)_l^\kappa \phi_\kappa(y, tx \oplus (1-t)y) \end{aligned}$$

for $x, y, z \in M$ with $d(y, z) + d(z, x) + l < 2D_\kappa$ and $t \in [0, 1]$, where $l = d(x, y)$. We call this inequality Stewart's inequality. For more details about Stewart's inequality, see [13]. We say that a CAT(κ) space M is admissible [12] if

$$d(u, v) < \frac{D_\kappa}{2}$$

for any $u, v \in M$. CAT(κ) spaces are always admissible when $\kappa \leq 0$. If M is admissible, then

$$c''_\kappa(d(u,v)) > 0$$

for $u, v \in M$.

Let M be a metric space and T a mapping on M. We call a point $x \in M$ a fixed point of T if $Tx = x$, and denote the set of all fixed points of T by

$$\operatorname{Fix} T = \{x \in M \mid Tx = x\}.$$

Further, we say T is quasinonexpansive if $\operatorname{Fix} T$ is nonempty and

$$d(Tx, y) \leq d(x, y)$$

for $x \in M$ and $y \in \operatorname{Fix} T$. If M is an admissible CAT(κ) space and T is quasinonexpansive, then its fixed point set is closed and convex. For the sake of completeness, we give a poof.

Proposition 2.1. *Let M be an admissible* CAT(κ) *space and T a quasinonexpansive mapping on M. Then,* $\operatorname{Fix} T$ *is closed and convex.*

Proof. We first show that $\operatorname{Fix} T$ is closed and convex. Take a sequence $\{x_n\}$ of $\operatorname{Fix} T$ converging to $x \in M$. Then, since T is quasinonexpansive,

$$d(Tx, x) \leq d(Tx, x_n) + d(x_n, x) \leq 2d(x_n, x).$$

Letting $n \to \infty$, we have $d(Tx, x) \leq 0$, and hence x is a fixed point of T. Thus, $\operatorname{Fix} T$ is closed.

Let $x, y \in \operatorname{Fix} T$ and $t \in [0,1]$. Then, for

$$w = tx \oplus (1-t)y$$

and $l = d(x, y)$, from Stewart's inequality of M and the quasinonexpansiveness of T,

$$\begin{aligned}
\phi_\kappa(Tw, w) &\leq (t)_l^\kappa \phi_\kappa(Tw, x) + (1-t)_l^\kappa \phi_\kappa(Tw, y) - (t)_l^\kappa \phi_\kappa(x, w) - (1-t)_l^\kappa \phi_\kappa(y, w) \\
&\leq (t)_l^\kappa \phi_\kappa(w, x) + (1-t)_l^\kappa \phi_\kappa(w, y) - (t)_l^\kappa \phi_\kappa(x, w) - (1-t)_l^\kappa \phi_\kappa(y, w) \\
&= 0.
\end{aligned}$$

Therefore, w is a fixed point of T, and hence $\operatorname{Fix} T$ is convex. ∎

Let C be a nonempty closed convex subset of an admissible complete CAT(κ) space M. For $x \in M$, there exists a unique point $y_x \in C$ such that

$$d(x, y_x) = \inf_{y \in C} d(x, y).$$

We call a mapping $P_C \colon x \mapsto y_x$ the metric projection onto C. The metric projection P_C is quasinonexpansive with the fixed point set $\operatorname{Fix} P_C = C$. That is,

$$d(P_C x, y) \leq d(x, y)$$

for $x \in M$ and $y \in C$. For more details, see [2, 5].

Let M be a metric space and $\{x_n\}$ a bounded sequence of M. We call a point $w \in M$ an asymptotic centre of $\{x_n\}$ if

$$\limsup_{n\to\infty} d(x_n, w) = \inf_{y\in M} \limsup_{n\to\infty} d(x_n, y).$$

We say $\{x_n\}$ Δ-converges to a Δ-limit $x \in M$ if x is a unique asymptotic centre of any subsequence of $\{x_n\}$. A sequence $\{x_n\}$ of an admissible CAT(κ) space M is said to be κ-bounded if

$$\inf_{y\in M} \limsup_{n\to\infty} d(x_n, y) < \frac{D_\kappa}{2}.$$

Every κ-bounded sequence is bounded in the usual sense. Moreover, it is well known that a κ-bounded sequence of an admissible complete CAT(κ) space has a unique asymptotic centre; refer to [2, 5, 15] for example.

3. Tangent spaces and monotone vector fields

In what follows, we define tangent spaces and a metric on a CAT(κ) space corresponding to Riemannian metrics. For more details, see [14] and references therein.

We first define the Alexandrov angle. Let M be an admissible CAT(κ) space, and let $p, x, y \in M$. We define the Alexandrov angle A_p at p by

$$A_p(x, y) = \lim_{t\to 0+} \arccos\left(1 - \frac{d(\gamma_{px}(t), \gamma_{py}(t))^2}{2t^2}\right) \in [0, \pi]$$

if $p \neq x$ and $p \neq y$; $A_p(x, p) = A_p(p, x) = \pi/2$ if $p \neq x$; $A_p(p, p) = 0$. For more details about the Alexandrov angles, refer to [3, Proposition 1.14 in Chapter I.1 and Proposition 3.1 in Chapter II.3] for instance.

Let M be an admissible CAT(κ) space, and let $p \in M$. We define an equivalence relation $\sim_p$ on M by $x \sim_p y$ if

$$A_p(x, y) = 0.$$

For $x \in M$, we denote an equivalence class of x by

$$[x]_p = \{z \in M \mid x \sim_p z\}.$$

Notice that $[p]_p = \{p\}$ since $A_p(p, x) = \pi/2$ if $p \neq x$. Further, let

$$D_pM = M/{\sim_p} = \{[x]_p \mid x \in M\}.$$

Then, (D_pM, A_p) is a metric space, where the distance A_p is defined by

$$A_p([x]_p, [y]_p) = A_p(x, y)$$

for $[x]_p, [y]_p \in D_pM$. We next define a function ζ from D_pM to $\{0, 1\}$ by

$$\zeta([x]_p) = \begin{cases} 0 & ([x]_p = [p]_p); \\ 1 & ([x]_p \neq [p]_p) \end{cases}$$

for $[x]_p \in D_pM$. We define an equivalence relation $\simeq_p$ on a Cartesian product

$$[0, \infty[\times D_pM$$

by $(r_1, [x]_p) \simeq_p (r_2, [y]_p)$ if one of the following conditions is satisfied:

- $r_1\zeta([x]_p) = r_2\zeta([y]_p) = 0$;
- $r_1\zeta([x]_p) = r_2\zeta([y]_p) > 0$ and $[x]_p = [y]_p$.

Now, we define a tangent space on M. We define a set T_pM by

$$T_pM = ([0,\infty[\, \times D_pM)/\simeq_p.$$

For the simplicity, denote an element $[(r,[x]_p)]_{\simeq_p} \in T_pM$ by $r[x]_p$. Particularly, we denote $0[p]_p$ by 0_p. Furthermore, we let T_pM equip a distance function d_p defined by

$$d_p(r[x]_p, s[y]_p) = \sqrt{r^2\zeta([x]_p) + s^2\zeta([y]_p) - 2rs\zeta([x]_p)\zeta([y]_p)\cos A_p(x,y)}$$

for $r[x]_p, s[y]_p \in T_pM$. We call this metric space (T_pM, d_p) the tangent space of M at p. Set

$$TM = \bigsqcup_{p\in M} T_pM = \bigcup_{p\in M} \{(v_p, p) \mid v_p \in T_pM\},$$

and call it the tangent bundle of M.

Let M be an admissible CAT(κ) space, and let $p \in M$. For $v_p = r[v]_p \in T_pM$ and $t \in [0,\infty[$, we denote a point $(tr)[v]_p$ in T_pM by tv_p. Particularly, for $t > 0$, we denote a point $(r/t)[v]_p$ by v_p/t. We define a logarithmic mapping $\log_p$ from M to T_pM by

$$\log_p x = d(p,x)[x]_p \in T_pM$$

for $x \in M$. This logarithmic mapping is a generalisation of the inverse mapping of the exponential mapping on Riemannian manifolds. We further define another logarithmic mapping $\log_{\kappa,p}$ from M to T_pM by

$$\log_{\kappa,p} x = c_\kappa'(d(p,x))[x]_p \in T_pM$$

for $x \in M$. We define a function $g_p\colon T_pM \times T_pM \to \mathbb{R}$ by

$$g_p(u_p, v_p) = \frac{d_p(u_p,0_p)^2 + d_p(v_p,0_p)^2 - d_p(u_p,v_p)^2}{2}$$

for $u_p, v_p \in T_pM$. We call $g = \{g_p\}_{p\in M}$ a metric on M. The following hold:

- $g_p(v_p, v_p) \geq 0$ for $v_p \in T_pM$;
- $g_p(u_p, v_p) = g_p(v_p, u_p)$ for $u_p, v_p \in T_pM$;
- $g_p(v_p, 0_p) = 0$ for $v_p \in T_pM$;
- $tg_p(u_p, v_p) = g_p(u_p, tv_p)$ for $u_p, v_p \in T_pM$ and $t \geq 0$;
- $d(x,y)^2 = g_x(\log_x y, \log_x y) = g_y(\log_y x, \log_y x)$ for $x, y \in M$;
- $c_\kappa'(d(x,y))^2 = g_x(\log_{\kappa,x} y, \log_{\kappa,x} y) = g_y(\log_{\kappa,y} x, \log_{\kappa,y} x)$ for $x, y \in M$.

Theorem 3.1. *[14] For an admissible* CAT(κ) *space M and $p, x, y \in M$,*

$$\begin{aligned} g_p(\log_{\kappa,p} x, \log_{\kappa,p} y) &\geq \phi_\kappa(p,x) + c_\kappa''(d(p,x))\phi_\kappa(p,y) - \phi_\kappa(x,y) \\ &\geq \phi_\kappa(p,x) - \phi_\kappa(x,y). \end{aligned}$$

We next introduce a notion of monotone vector fields. Let M be an admissible CAT(κ) space and A a set-valued mapping from M to a subset of the tangent bundle TM. We call A a set-valued vector field if $Ax \subset T_xM$ for $x \in M$. Henceforth, suppose that A is a set-valued vector field on M. We denote the domain of A by

$$\operatorname{Dom} A = \{x \in M \mid Ax \neq \emptyset\}.$$

We denote the graph of A by

$$\operatorname{Gph} A = \{(x, v_x) \in \operatorname{Dom} A \times TM \mid v_x \in Ax\}.$$

We call a point $x \in M$ a zero point of A if $0_x \in Ax$, denote the set of all zero points of A by

$$\operatorname{Zero} A = \{x \in M \mid 0_x \in Ax\}.$$

For $r > 0$, we define a set-valued vector field rA on M by

$$rAx = \{rv_x \in T_xM \mid v_x \in Ax\}$$

for $x \in M$. Notice that $\operatorname{Dom}(rA) = \operatorname{Dom} A$ and $\operatorname{Zero}(rA) = \operatorname{Zero} A$. We say that A is monotone if

$$g_x(\log_x y, u_x) + g_y(\log_y x, v_y) \leq 0$$

for $(x, u_x), (y, v_y) \in \operatorname{Gph} A$. If A is monotone, then so is rA for $r > 0$. Further, we immediately obtain the following:

Proposition 3.2. *Let M be an admissible* CAT(κ) *space and A a set-valued vector field on M. Then, A is monotone if and only if*

$$g_x(\log_{\kappa,x} y, u_x) + g_y(\log_{\kappa,y} x, v_y) \leq 0$$

for $(x, u_x), (y, v_y) \in \operatorname{Gph} A$.

For a monotone vector field A on an admissible CAT(κ) space M, we consider a condition as follows: For fixed $x \in M$, there exists $z \in M$ such that

$$\log_{\kappa,z} x \in Az.$$

In this case, such a point is unique since A is monotone, and therefore we define a mapping J_A on M by

$$\{J_A x\} = \left\{z \in M \;\middle|\; \log_{\kappa,z} x \in Az\right\}$$

for $x \in M$. We call the mapping J_A the resolvent operator of A. We say that A is resolvably monotone if it is monotone, and

$$\left\{z \in M \;\middle|\; \frac{\log_{\kappa,z} x}{r} \in Az\right\} \neq \emptyset$$

for any $r > 0$ and any $x \in M$. If A is resolvably monotone, then so is rA for $r > 0$. In this case, we can define the resolvent operator J_{rA} of rA for $r > 0$.

Let M be an admissible CAT(κ) space and A a resolvably monotone vector field on M. Then, for $r > 0$, we know that

$$\operatorname{Zero} A = \operatorname{Fix} J_{rA}.$$

Furthermore, J_A is geodesically nonspreading, that is,

$$\phi_\kappa(J_Ax, J_Ay) + \phi_\kappa(J_Ay, J_Ax) \le \phi_\kappa(J_Ax, y) + \phi_\kappa(J_Ay, x)$$

for $x, y \in M$. Such a mapping is also said to be metrically nonspreading [16] if $\kappa = 0$; spherically nonspreading of sum type [7, 10] if $\kappa = 1$; hyperbolically nonspreading [9] if $\kappa = -1$. If A has a zero point, then for $x \in M$ and $y \in \operatorname{Fix} J_A$,

$$\begin{aligned}\phi_\kappa(J_Ax, y) + \phi_\kappa(y, J_Ax) &= \phi_\kappa(J_Ax, J_Ay) + \phi_\kappa(J_Ay, J_Ax)\\ &\le \phi_\kappa(J_Ax, y) + \phi_\kappa(J_Ay, x)\\ &= \phi_\kappa(J_Ax, y) + \phi_\kappa(y, x),\end{aligned}$$

and therefore

$$d(J_Ax, y) \le d(x, y).$$

It means that J_A is quasinonexpansive. Thus, $\operatorname{Zero} A$ is closed and convex. For more details about monotone vector fields, refer to [19].

In what follows, we see an equilibrium problem on geodesic spaces as an example of monotone vector fields.

Let M be an admissible CAT(κ) space. We say that a nonempty closed convex subset K has the convex hull finite property [10, 18] if every continuous mapping on $\operatorname{cl}\operatorname{co} E$ has a fixed point for every finite subset E of K, where $\operatorname{cl}\operatorname{co} E$ is the closed convex hull of E. If K is compact, then it enjoys the convex hull finite property according to [1].

Let K be a nonempty closed convex subset of an admissible CAT(κ) space M. In this work, for a function f from K^2 to $\mathbb{R}$, we consider the following equilibrium problem: To find a point $x \in K$ such that

$$\inf_{y \in K} f(x, y) \ge 0.$$

We call such a point an equilibrium point of f, and we denote the set of all equilibrium points of f by $\operatorname{Equil} f$. We further assume the following conditions:

(E1) For $x \in K$, $f(x, x) = 0$;

(E2) for $x, y \in K$, $f(x, y) + f(y, x) \le 0$;

(E3) for $x \in K$, a real function $f(x, \cdot)$ on K is lower semicontinuous and

$$f(x, ty_1 \oplus (1-t)y_2) \le tf(x, y_1) + (1-t)f(x, y_2)$$

for $y_1, y_2 \in K$ and $t \in [0, 1]$;

(E4) for $x, y \in K$, $\limsup_{t \to 0+} f(ty \oplus (1-t)x, y) \le f(x, y)$.

Then, the following holds:

Theorem 3.3. *[19] Let M be an admissible complete* CAT(κ) *space and K a nonempty closed convex subset of M having the convex hull finite property. For a function f from K^2 to $\mathbb{R}$ satisfying the four conditions* (E1) *to* (E4)*, define a set-valued vector field A_f on M by*

$$A_f x = \left\{ v_x \in T_x M \;\middle|\; 0 \le \inf_{y \in M} \left(f(x, y) - g_x(\log_x y, v_x) \right) \right\}$$

if $x \in K$; $A_f x = \emptyset$ if $x \notin K$. Assume that Equil f *is nonempty. Then, the following hold:*

(i) The set-valued vector field A_f is resolvably monotone;

(ii) for $r > 0$ and $x \in M$,

$$\{J_{rA_f}x\} = \left\{ z \in K \;\middle|\; \inf_{y \in K} \left(f(z, y) + \frac{1}{r}\phi_\kappa(y, x) \right) - \frac{1}{r}\phi_\kappa(z, x) \geq 0 \right\};$$

(iii) $\operatorname{Equil} f = \operatorname{Zero} A_f$*, and* $K = \operatorname{cl} \operatorname{Dom} A_f$*.*

4. The proximal point algorithm for a monotone vector field

In this section, we show that a zero point approximation theorem with the proximal point algorithm. At first, we prove the following lemma. Termkaew, Chaipunya and Kohsaka [20] have shown this in the case where $\kappa = 0$.

Lemma 4.1. *Let M be an admissible complete* $\mathrm{CAT}(\kappa)$ *space and C a nonempty closed convex subset of M. Let $\{x_n\}$ be a sequence of M such that*

$$d(x_{n+1}, p) \leq d(x_n, p)$$

for any $p \in C$ and $n \in \mathbb{N}$. Then, a sequence $\{P_C x_n\}$ converges to a point in C.

Proof. We show a sequence $\{P_C x_n\}$ is a Cauchy one. Henceforth, we denote P_C by P for the simplicity. From the definition of the metric projection P and the assumption of $\{x_n\}$, we have

$$d(Px_{n+1}, x_{n+1}) \leq d(Px_n, x_{n+1}) \leq d(Px_n, x_n)$$

for $n \in \mathbb{N}$, and thus $\{\phi_\kappa(Px_n, x_n)\}$ is convergent. Note that there exists a nonnegative real sequence $\{a_n\}$ converging to 0 such that

$$|\phi_\kappa(Px_m, x_m) - \phi_\kappa(Px_n, x_n)| \leq a_n$$

for $m, n \in \mathbb{N}$ with $m \geq n$. Moreover, there exists a positive real number c such that

$$c \leq \inf_{n \in \mathbb{N}} c''_\kappa(d(Px_n, x_n)).$$

Actually, we should take c as

$$c = \begin{cases} c''_\kappa(d(Px_1, x_1)) & (\kappa > 0); \\ 1 & (\kappa \leq 0). \end{cases}$$

Fix $m, n \in \mathbb{N}$ with $m \geq n$ arbitrarily. Let $l = d(Px_n, Px_m)$ and $t \in]0, 1[$. From the definition of the metric projection P and Stewart's inequality of M, we have

$$\begin{aligned} \phi_\kappa(x_m, Px_m) &\leq \phi_\kappa(x_m, tPx_n \oplus (1-t)Px_m) \\ &\leq (t)_l^\kappa \phi_\kappa(x_m, Px_n) + (1-t)_l^\kappa \phi_\kappa(x_m, Px_m) - (t)_l^\kappa c_\kappa((1-t)l), \end{aligned}$$

and hence

$$\frac{1 - (1-t)_l^\kappa}{(t)_l^\kappa} \phi_\kappa(x_m, Px_m) \leq \phi_\kappa(x_m, Px_n) - c_\kappa((1-t)l). \qquad (*)$$

From l'Hôpital's rule, if $l \neq 0$, then

$$\lim_{t\to 0+} \frac{1-(1-t)_l^\kappa}{(t)_l^\kappa} = \lim_{t\to 0+} \frac{c_\kappa'(l) - c_\kappa'((1-t)l)}{c_\kappa'(tl)} = \lim_{t\to 0+} \frac{l \cdot c_\kappa''((1-t)l)}{l \cdot c_\kappa''(tl)} = c_\kappa''(l).$$

If $l = 0$, then

$$\frac{1-(1-t)_l^\kappa}{(t)_l^\kappa} = \frac{1-(1-t)}{t} = 1 = c_\kappa''(l).$$

Therefore, letting $t \to 0+$ for the equation $(*)$, we have

$$c_\kappa''(l)\phi_\kappa(x_m, Px_m) \leq \phi_\kappa(x_m, Px_n) - \phi_\kappa(Px_n, Px_m).$$

Then,

$$\begin{aligned} 0 &\leq \phi_\kappa(Px_n, x_m) - c_\kappa''(l)\phi_\kappa(Px_m, x_m) - \phi_\kappa(Px_n, Px_m) \\ &= \phi_\kappa(Px_n, x_m) - \phi_\kappa(Px_m, x_m) + \left(1 - c_\kappa''(l)\right)\phi_\kappa(Px_m, x_m) - \phi_\kappa(Px_n, Px_m) \\ &= \phi_\kappa(Px_n, x_m) - \phi_\kappa(Px_m, x_m) - c_\kappa''(d(Px_m, x_m))\phi_\kappa(Px_n, Px_m) \\ &\leq \phi_\kappa(Px_n, x_m) - \phi_\kappa(Px_m, x_m) - c \cdot \phi_\kappa(Px_n, Px_m), \end{aligned}$$

and hence

$$\phi_\kappa(Px_n, Px_m) \leq \frac{\phi_\kappa(Px_n, x_m) - \phi_\kappa(Px_m, x_m)}{c}.$$

On the other hand, from the assumption of $\{x_n\}$, we have

$$d(Px_n, x_m) \leq d(Px_n, x_{m-1}) \leq \cdots \leq d(Px_n, x_n).$$

Hence,

$$\begin{aligned} \phi_\kappa(Px_n, Px_m) &\leq \frac{\phi_\kappa(Px_n, x_m) - \phi_\kappa(Px_m, x_m)}{c} \leq \frac{\phi_\kappa(Px_n, x_n) - \phi_\kappa(Px_m, x_m)}{c} \\ &\leq \frac{|\phi_\kappa(Px_m, x_m) - \phi_\kappa(Px_n, x_n)|}{c} \leq \frac{a_n}{c}. \end{aligned}$$

It means that $\{Px_n\}$ is a Cauchy sequence, which completes the proof. ∎

We further have known the following result:

Theorem 4.2. *[8, 11, 12] For an admissible complete* CAT(κ) *space M, let $\{x_n\}$ be a κ-bounded sequence of M and $\{b_n\}$ a positive real sequence such that $\sum_{i=1}^\infty b_i = \infty$. Then, a function h on M defined by*

$$h(y) = \limsup_{n\to\infty} \frac{1}{\sum_{j=1}^n b_j} \sum_{i=1}^n b_i \phi_\kappa(x_i, y)$$

for $y \in M$ has a unique minimiser.

Now, we prove the following theorem:

Theorem 4.3. *Let M be an admissible complete* CAT(κ) *space and A a resolvably monotone vector field on M. Let $\{r_n\}$ be a sequence of positive real numbers whose sum is divergent to ∞. For a given initial point $x_1 \in M$, generate a sequence $\{x_n\}$ of M by*

$$x_{n+1} = J_{r_n A} x_n$$

for $n \in \mathbb{N}$. Then, the following hold:

(i) The resolvably monotone vector field A has a zero point if and only if the generated sequence $\{x_n\}$ is κ-bounded;

(ii) if A has a zero point and $\inf_{k\in\mathbb{N}} r_k > 0$, then the generated sequence $\{x_n\}$ Δ-converges to a zero point of A, which equals to

$$\lim_{n\to\infty} P_{\mathrm{Zero}\,A} x_n.$$

Proof. We first show (i). If A has a zero point, then the resolvent operator $J_{r_n A}$ is quasinonexpansive for $n \in \mathbb{N}$. Thus, for $w \in \mathrm{Zero}\,A$ and $n \in \mathbb{N}$, we have

$$d(x_{n+1}, w) = d(J_{r_n A} x_n, w) \le d(x_n, w),$$

and therefore

$$\inf_{y\in M} \limsup_{n\to\infty} d(x_n, y) \le \limsup_{n\to\infty} d(x_n, w) \le d(x_1, w) < \frac{D_\kappa}{2},$$

which means that $\{x_n\}$ is κ-bounded. We inversely assume that $\{x_n\}$ is κ-bounded. We define a function h on M by

$$h(y) = \limsup_{n\to\infty} \frac{1}{\sum_{j=1}^n r_j} \sum_{i=1}^n r_i \phi_\kappa(x_{i+1}, y)$$

for $y \in M$. From Theorem 4.2, this function h has a unique minimiser. Let $w \in M$ be its unique minimiser. Fix $i \in \mathbb{N}$ arbitrarily. From the definition of the resolvent operators $J_{r_i A}$ and J_A, we know that

$$(J_A w, \log_{\kappa, J_A w} w), \left(x_{i+1}, \frac{\log_{\kappa, x_{i+1}} x_i}{r_i}\right) \in \mathrm{Gph}\,A.$$

From the monotonicity of A, we obtain

$$\begin{aligned} 0 &\ge g_{J_A w}(\log_{\kappa, J_A w} x_{i+1}, \log_{\kappa, J_A w} w) + \frac{g_{x_{i+1}}(\log_{\kappa, x_{i+1}} J_A w, \log_{\kappa, x_{i+1}} x_i)}{r_i} \\ &\ge \phi_\kappa(J_A w, x_{i+1}) - \phi_\kappa(x_{i+1}, w) + \frac{\phi_\kappa(x_{i+1}, J_A w) - \phi_\kappa(J_A w, x_i)}{r_i}, \end{aligned}$$

and hence

$$r_i \phi_\kappa(x_{i+1}, J_A w) \le r_i \phi_\kappa(x_{i+1}, w) + \phi_\kappa(x_i, J_A w) - \phi_\kappa(x_{i+1}, J_A w)$$

for $i \in \mathbb{N}$. Fix $n \in \mathbb{N}$. Summing up this inequality with respect to $i = 1, 2, \dots, n$, we have

$$\sum_{i=1}^n r_i \phi_\kappa(x_{i+1}, J_A w) \le \sum_{i=1}^n r_i \phi_\kappa(x_{i+1}, w) + \phi_\kappa(x_1, J_A w) - \phi_\kappa(x_{n+1}, J_A w).$$

Dividing both sides by $\sum_{j=1}^n r_j$, and letting $n \to \infty$, we have

$$\begin{aligned} h(J_A w) &= \limsup_{n\to\infty} \frac{1}{\sum_{j=1}^n r_j} \sum_{i=1}^n r_i \phi_\kappa(x_{i+1}, J_A w) \\ &\le \limsup_{n\to\infty} \frac{1}{\sum_{j=1}^n r_j} \sum_{i=1}^n r_i \phi_\kappa(x_{i+1}, w) = h(w). \end{aligned}$$

Since w is a unique minimiser of h, we obtain $J_A w = w$, which means that w is a zero point of A. Therefore, A has a zero point.

We next show (ii). From the assumption and (i), the sequence $\{x_n\}$ is κ-bounded. Since $J_{r_n A}$ is quasinonexpansive for $n \in \mathbb{N}$, for any $p \in \operatorname{Zero} A$ and $n \in \mathbb{N}$, we have

$$d(x_{n+1}, p) = d(J_{r_n A} x_n, p) \le d(x_n, p).$$

From Lemma 4.1, the sequence $\{P_{\operatorname{Zero} A} x_n\}$ converges to some zero point x_0 of A. In particular, for $n \in \mathbb{N}$, we know that

$$d(x_{n+1}, x_0) \le d(x_n, x_0).$$

Therefore, the real sequence $\{d(x_n, x_0)\}$ is convergent. Note that there exists a positive real number c such that

$$c \le \inf_{n\in\mathbb{N}} c''_\kappa(d(x_n, x_0)).$$

On the other hand, from the definition of the resolvent operator $J_{r_n A}$, we have

$$(x_0, 0_{x_0}), \left(x_{n+1}, \frac{\log_{\kappa, x_{n+1}} x_n}{r_n} \right) \in \operatorname{Gph} A.$$

Since A is monotone, we get

$$\begin{aligned} 0 &\ge g_{x_{n+1}} \left(\log_{\kappa, x_{n+1}} x_0, \frac{\log_{\kappa, x_{n+1}} x_n}{r_n} \right) + g_{x_0}(\log_{\kappa, x_0} x_{n+1}, 0_{x_0}) \\ &= \frac{g_{x_{n+1}}(\log_{\kappa, x_{n+1}} x_0, \log_{\kappa, x_{n+1}} x_n)}{r_n} \\ &\ge \frac{\phi_\kappa(x_{n+1}, x_0) + c''_\kappa(d(x_{n+1}, x_0))\phi_\kappa(x_{n+1}, x_n) - \phi_\kappa(x_0, x_n)}{r_n} \\ &\ge \frac{\phi_\kappa(x_{n+1}, x_0) + c \cdot \phi_\kappa(x_{n+1}, x_n) - \phi_\kappa(x_0, x_n)}{r_n}. \end{aligned}$$

Therefore,

$$\phi_\kappa(J_{r_n A} x_n, x_n) \le \frac{\phi_\kappa(x_n, x_0) - \phi_\kappa(x_{n+1}, x_0)}{c}.$$

Letting $n \to \infty$, we obtain

$$\lim_{n\to\infty} d(J_{r_n A} x_n, x_n) = 0.$$

Take a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ arbitrarily, and let $w \in M$ be a unique asymptotic centre of $\{x_{n_i}\}$. Set

$$w_i = J_{r_{n_i} A} x_{n_i}$$

for $i \in \mathbb{N}$. Remark that

$$\lim_{i\to\infty} d(w_i, x_{n_i}) = 0.$$

Then, w is a unique asymptotic centre of $\{w_i\}$. Indeed, for arbitrary $v \in M$, we get

$$\begin{aligned}
\limsup_{i\to\infty} d(w_i, w) &\le \limsup_{i\to\infty} \left(d(w_i, x_{n_i}) + d(x_{n_i}, w)\right) = \limsup_{i\to\infty} d(x_{n_i}, w) \\
&\le \limsup_{i\to\infty} d(x_{n_i}, v) \\
&\le \limsup_{i\to\infty} \left(d(w_i, v) + d(w_i, x_{n_i})\right) = \limsup_{i\to\infty} d(w_i, v).
\end{aligned}$$

Now, we prove that w is a zero point of A. Since

$$(J_A w, \log_{\kappa, J_A w} w), \left(w_i, \frac{\log_{\kappa, w_i} x_{n_i}}{r_{n_i}}\right) \in \operatorname{Gph} A$$

and A is monotone, from Theorem 3.1,

$$\begin{aligned}
0 &\ge \frac{g_{w_i}(\log_{\kappa, w_i} J_A w, \log_{\kappa, w_i} x_{n_i})}{r_{n_i}} + g_{J_A w}(\log_{\kappa, J_A w} w_i, \log_{\kappa, J_A w} w) \\
&\ge \frac{\phi_\kappa(w_i, J_A w) - \phi_\kappa(J_A w, x_{n_i})}{r_{n_i}} + \phi_\kappa(J_A w, w_i) - \phi_\kappa(w_i, w).
\end{aligned}$$

Hence,

$$\begin{aligned}
\phi_\kappa(w_i, J_A w) &\le \phi_\kappa(w_i, w) + \frac{\phi_\kappa(J_A w, x_{n_i}) - \phi_\kappa(w_i, J_A w)}{r_{n_i}} \\
&\le \phi_\kappa(w_i, w) + \frac{|c_\kappa(d(J_A w, x_{n_i})) - c_\kappa(d(w_i, J_A w))|}{\inf_{k\in\mathbb{N}} r_k}.
\end{aligned} \tag{$**$}$$

Remark that c_κ is uniformly continuous on a compact interval, and

$$\lim_{i\to\infty} |d(J_A w, x_{n_i}) - d(w_i, J_A w)| \le \lim_{i\to\infty} d(w_i, x_{n_i}) = 0.$$

Therefore, letting $i \to \infty$ for the equation ($**$), we have

$$\limsup_{i\to\infty} \phi_\kappa(w_i, J_A w) \le \limsup_{i\to\infty} \phi_\kappa(w_i, w).$$

Hence, $J_A w = w$ since w is a unique asymptotic centre of $\{w_i\}$, and therefore w is a zero point of A since $\operatorname{Fix} J_A = \operatorname{Zero} A$. Then,

$$\begin{aligned}
\limsup_{i\to\infty} d(x_{n_i}, x_0) &\le \limsup_{i\to\infty} \left(d(x_{n_i}, P_{\operatorname{Zero} A} x_{n_i}) + d(P_{\operatorname{Zero} A} x_{n_i}, x_0)\right) \\
&= \limsup_{i\to\infty} d(x_{n_i}, P_{\operatorname{Zero} A} x_{n_i}) \le \limsup_{i\to\infty} d(x_{n_i}, w),
\end{aligned}$$

which implies that $x_0 = w$ since w is a unique asymptotic centre of $\{x_{n_i}\}$. Therefore, the generated sequence $\{x_n\}$ Δ-converges to x_0, which completes the proof. ∎

As a direct consequence of Theorem 4.3, we obtain the following:

Theorem 4.4. *Let M be an admissible complete* $\mathrm{CAT}(\kappa)$ *space and K a nonempty closed convex subset of M having the convex hull finite property. Let f be a function from K^2 to $\mathbb{R}$ satisfying the four conditions* (E1) *to* (E4), *and suppose that* $\mathrm{Equil}\, f$ *is nonempty. Let $\{r_n\}$ be a sequence of positive real numbers such that* $\inf_{k\in\mathbb{N}} r_k > 0$. *For a given initial point $x_1 \in M$, generate a sequence $\{x_n\}$ of M as follows:*

$$x_{n+1} = \left\{ z \in K \;\middle|\; \inf_{y\in K} \left(f(z,y) + \frac{1}{r_n}\phi_\kappa(y,x_n) \right) - \frac{1}{r_n}\phi_\kappa(z,x_n) \geq 0 \right\}$$

for $n \in \mathbb{N}$. Then, the generated sequence $\{x_n\}$ Δ-converges to an equilibrium point of f, which equals to

$$\lim_{n\to\infty} P_{\mathrm{Equil}\, f}\, x_n.$$

Conclusion

In this work, we obtain a convergence theorem to find zero points of monotone set-valued vector fields. In the setting of Hilbert or Banach spaces, we have got some other iterative scheme to generate iterative sequence such as Mann's one, Halpern's one and projection methods. The main result of this paper gives us a Δ-convergence theorem, and it is not convergent strongly in general. We ought to obtain convergence theorem with the above typical schemes using some techniques in this paper.

References

[1] D. Ariza-Ruiz, C. Li and G. López-Acedo, The Schauder fixed point theorem in geodesic spaces, J. Math. Anal. Appl., 417 (2014), 345–360.

[2] M. Bačák, Convex Analysis and Optimization in Hadamard Spaces, De Gruyter, Berlin, 2014.

[3] M.R. Bridson and Haefliger, Metric Spaces of Non-positive Curvature, Springer-Verlag, Berlin, 1999.

[4] P. Chaipunya, F. Kohsaka and P. Kumam, Monotone vector fields and generation of nonexpansive semigroups in complete CAT(0) spaces, Numer. Funct. Anal. Optim., 42 (2021), 989–1018.

[5] R. Espínola and A. Fernández-León, CAT(k)-spaces, weak convergence and fixed points, J. Math. Anal. Appl. 353 (2009), 410–427.

[6] A.Y. Inuwa, P. Chaipunya, P. Kumam and S. Salisu, Equilibrium problems and proximal algorithm using tangent space products, Carpathian J. Math., 40 (2024), 65–76.

[7] T. Kajimura and Y. Kimura, A new resolvent for convex functions in complete geodesic spaces, RIMS Kôkyûroku, 2112 (2019), 141–147.

[8] T. Kajimura and Y. Kimura, The proximal point algorithm in complete geodesic spaces with negative curvature, Adv. Theory Nonlinear Anal. Appl., 3 (2019), 192–200.

[9] T. Kajimura and Y. Kimura, Resolvents of convex functions in complete geodesic metric spaces with negative curvature, J. Fixed Point Theory Appl., 21 (2019), 1–15.

[10] Y. Kimura, Resolvents of equilibrium problems on a complete geodesic space with curvature bounded above, Carpathian J. Math., 37 (2021), 463–476.

[11] Y. Kimura and F. Kohsaka, Two modified proximal point algorithm for convex functions in Hadamard spaces, Linear Nonlinear Anal., 2 (2016), 69–86.

[12] Y. Kimura and F. Kohsaka, The proximal point algorithm in geodesic spaces with curvature bounded above, Linear Nonlinear Anal., 3 (2017), 133–148.

[13] Y. Kimura and S. Sudo, New type parallelogram laws in Banach spaces and geodesic spaces with curvature bounded above, Arab. J. Math., 12 (2023), 389–412.

[14] Y. Kimura and S. Sudo, Tangent spaces and a metric on geodesic spaces, RIMS Kôkyûroku, 2240 (2023), 7–19.

[15] W.A. Kirk and B. Panyanak, A concept of convergence in geodesic spaces, Nonlinear Anal., 68 (2008), 3689–3696.

[16] F. Kohsaka, Fixed points of metrically nonspreading mappings in Hadamard spaces, Appl. Anal. Optim., 3 (2019), 213–230.

[17] R.T. Rockafellar, Monotone operators and the proximal point algorithm, SIAM J. Control Optimization, 14 (1976), 877–898.

[18] S. Shabanian and S.M. Vaezpour, A minimax inequality and its applications to fixed point theorems in CAT(0) spaces, Fixed Point Theory Appl., (2011), 2011:61, 1–9.

[19] S. Sudo, Monotone vector fields on a geodesic space with curvature bounded above by a general real number, Lett. Nonlinear Anal. Appl., 3 (2025), 5–27.

[20] S. Termkaew, P. Chaipunya and F. Kohsaka, Mean ergodic theorems for a sequence of nonexpansive mappings in complete CAT(0) spaces and its applications, Open Math., 21 (2023), 1–22.