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Approximating Endpoints of Multi-Valued Nonexpansive Mappings in Uniformly Convex Hyperbolic Spaces Using a Novel Iteration Process

Thanomsak Laokul

Department of Mathematics and Computing Science, Mahidol Wittayanusorn School, Nakorn Pathom 73170, Thailand
thanom.kul@mwit.ac.th

ABSTRACT

This paper introduces a modified iterative process for approximating the endpoints of multi-valued nonexpansive mappings in 2-uniformly convex hyperbolic spaces, thereby extending the framework of uniformly convex Banach spaces. We establish a Δ-convergence theorem and, under suitable conditions, prove strong convergence results. Our results generalize and enhance the iterative framework introduced by Makbule Kaplan Özekes, broadening its applicability in uniformly convex hyperbolic spaces.

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1. Introduction

Let C be a nonempty subset of a metric space (X, d). A mapping $T : C \to 2^X$, where 2^X represents the collection of all nonempty subsets of X, is called a multi-valued mapping. If $T(x)$ is a singleton for every $x \in C$, the mapping T is said to be single-valued. An element $x \in C$ is defined as a *fixed point* of T if $x \in T(x)$. Moreover, if x is a fixed point and $T(x) = \{x\}$, then x is called an *endpoint* (or *stationary point*) of T. Clearly, the set of endpoints is a subset of the set of fixed points, and the two coincide when T is single-valued. Denote the set of fixed points by $\mathrm{Fix}(T)$ and the set of endpoints by $\mathrm{End}(T)$. Since an endpoint satisfies a stricter condition than a fixed point, it follows that $\mathrm{End}(T) \subseteq \mathrm{Fix}(T)$.

Fixed point theorems, which establish the existence and properties of fixed points, play a fundamental role in various mathematical applications, particularly in optimization and nonlinear analysis. Additionally, endpoint theory has been extensively utilized in optimization. For instance, Corley [2] established an equivalence between cone maximization and endpoint problems, whereas Tarafdar and Yuan [16] applied endpoint theorems to prove the existence of Pareto optima in ordered Banach spaces.

The existence of endpoints for nonexpansive mappings was first addressed by Panyanak

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[12] in 2015 and later extended to more general settings by Espínola et al. [3] and Kudtha and Panyanak [9]. Moreover, numerous applications of endpoint theory have been reported in the literature (see, e.g., [4, 7, 17, 18]).

Recently, Makbule Kaplan Özekes [5] introduced an iterative scheme in Banach spaces for approximating endpoints of multi-valued nonexpansive mappings. Given a nonempty subset C of a metric space and a nonexpansive multi-valued mapping $T : C \to \mathcal{K}(C)$, the iterative process is defined as follows: for initial $x_1 \in C$ and real sequences $\alpha_n, \beta_n, \gamma_n \in [a, b] \subset (0, 1)$,

$$z_n = (1 - \gamma_n)x_n + \gamma_n v_n, \quad n \in \mathbb{N},$$

where $v_n \in T(x_n)$ such that $\|x_n - v_n\| = R(x_n, T(x_n))$, and

$$y_n = (1 - \beta_n)v_n + \beta_n w_n,$$

where $w_n \in T(z_n)$ such that $\|z_n - w_n\| = R(z_n, T(z_n))$, and

$$x_{n+1} = (1 - \alpha_n)v_n + \alpha_n u_n,$$

where $u_n \in T(y_n)$ such that $\|y_n - u_n\| = R(y_n, T(y_n))$.

Kaplan Özekes subsequently proved both weak and strong endpoint convergence theorems for this iteration in uniformly convex Banach spaces.

In this paper, we propose modified iteration processes for approximating the endpoints of multi-valued nonexpansive mappings in 2-uniformly convex hyperbolic spaces, a natural generalization of uniformly convex Banach spaces. We establish a Δ-convergence theorem for the iterative sequence and, under suitable conditions, prove strong convergence results.

2. Preliminaries

Throughout this paper, $\mathbb{N}$ denotes the set of natural numbers and $\mathbb{R}$ denotes the set of real numbers. Given a metric space (X, d), the *distance* from a point x to a nonempty subset C of X, is defined as

$$\text{dist}(x, C) := \inf\{d(x, y) : y \in C\},$$

and the *radius* of C relative to x is given by

$$R(x, C) := \sup\{d(x, y) : y \in C\}.$$

Let $\mathcal{K}(X)$ denote the collection of all nonempty compact subsets of X. It follows from [13] that for a multi-valued mapping $T : C \to \mathcal{K}(C)$, the following properties hold:

(i) $x \in \text{Fix}(T)$ if and only if $\text{dist}(x, T(x)) = 0$.

(ii) $x \in \text{End}(T)$ if and only if $R(x, T(x)) = 0$.

The *Pompeiu-Hausdorff distance* is a fundamental concept in mathematical analysis and topology, particularly in the study of metric spaces. It provides a way to measure the distance between two nonempty subsets of a metric space, thereby generalizing the notion of pointwise distance to set-valued functions. Given two nonempty subsets A, B of a metric space (X, d), the Hausdorff distance is defined as

$$H(A, B) := \max\left\{\sup_{a\in A} \text{dist}(a, B), \sup_{b\in B} \text{dist}(b, A)\right\}.$$

A mapping $T : C \to \mathcal{K}(C)$ is *nonexpansive* if

$$H(T(x), T(y)) \leq d(x, y) \quad \text{for all} \quad x, y \in C.$$

According to [13, Lemma 2.2(iii)], if T is nonexpansive, the function $g : C \to \mathbb{R}$ defined by $g(x) := R(x, T(x))$ is continuous.

Hyperbolic spaces, as introduced by Kohlenbach [8], provide a framework that extends the notion of convexity beyond linear settings.

Definition 2.1. A *hyperbolic space* (X, d, W) is a metric space together with a function $W : X \times X \times [0, 1] \to X$ such that for all $x, y, z, w \in X$ and $s, t \in [0, 1]$, we have

(W1) $d(z, W(x, y, t)) \leq (1 - t)d(z, x) + td(z, y)$;

(W2) $d\left(W(x, y, t), W(x, y, s)\right) = |t - s|d(x, y)$;

(W3) $W(x, y, t) = W(y, x, 1 - t)$;

(W4) $d(W(x, z, t), W(y, w, t)) \leq (1 - t)d(x, y) + td(z, w)$.

If $x, y \in X$ and $t \in [0, 1]$, then we use the notation $(1 - t)x \oplus ty$ for $W(x, y, t)$. A nonempty subset C of X is said to be *convex* if $\{(1 - t)x \oplus ty : t \in [0, 1]\} \subseteq C$ for all $x, y \in C$.

Uniform convexity can also be defined in the context of hyperbolic spaces. Leustean [11] introduced the notion of uniformly convex hyperbolic spaces, which is defined as follows: The hyperbolic space X is said to be *uniformly convex* if for any $r \in (0, \infty)$ and $\varepsilon \in (0, 2]$ there exists $\delta \in (0, 1]$ such that for all $x, y, z \in X$ with $d(x, z) \leq r$, $d(y, z) \leq r$ and $d(x, y) \geq r\varepsilon$, we have

$$d\left(\frac{1}{2}x \oplus \frac{1}{2}y, z\right) \leq (1 - \delta)r.$$

A function $\eta : (0, \infty) \times (0, 2] \to (0, 1]$ providing such a $\delta := \eta(r, \varepsilon)$ for given $r \in (0, \infty)$ and $\varepsilon \in (0, 2]$ is called *a modulus of uniform convexity.* We call η *monotone* if it is a nonincreasing function of r (for every fixed ε).

The notion of p-uniform convexity has been extensively explored by Xu [19], while its nonlinear variant for $p = 2$ was investigated by Khan and Khamsi [6]. We now present the definition of a 2-uniformly convex hyperbolic space.

Definition 2.2. Let X be a uniformly convex hyperbolic space. For each $r \in (0, \infty)$ and $\varepsilon \in (0, 2]$, we define

$$\Psi(r, \varepsilon) := \inf \left\{ \frac{1}{2}d^2(x, z) + \frac{1}{2}d^2(y, z) - d^2(\frac{1}{2}x \oplus \frac{1}{2}y, z) \right\},$$

where the infimum is taken over all $x, y, z \in X$ such that $d(x, z) \leq r$, $d(y, z) \leq r$, and $d(x, y) \geq r\varepsilon$. We say that X is 2-*uniformly convex* if

$$c_M := \inf \left\{ \frac{\Psi(r, \varepsilon)}{r^2\varepsilon^2} : r \in (0, \infty), \varepsilon \in (0, 2] \right\} > 0.$$

According to [14, Example 2.9], every uniformly convex Banach space is also a 2-uniformly convex hyperbolic space. In [10, Theorem 2.2], the authors prove that if X is a 2-uniformly convex hyperbolic space, then for all $x, y, z \in X$ and $t \in [0, 1]$,

$$d^2((1 - t)x \oplus ty, z) \leq (1 - t)d^2(x, z) + td^2(y, z) - 4c_M t(1 - t)d^2(x, y). \tag{2.1}$$

Henceforth, X denotes a complete 2-uniformly convex hyperbolic space with a monotone modulus of uniform convexity.

In nonlinear analysis and metric fixed point theory, the asymptotic center is a key concept for analyzing the limiting behavior of sequences, offering insights into the stability and convergence of iterative processes, particularly in fixed point approximation and optimization. Let C be a nonempty subset of X, and let $\{x_n\}$ be a bounded sequence in X. The *asymptotic radius* of $\{x_n\}$ relative to C is defined by

$$r(C,\{x_n\}) = \inf\left\{\limsup_{n\to\infty} d(x_n,x) : x \in C\right\}.$$

The *asymptotic center* of $\{x_n\}$ relative to C is defined by

$$A(C,\{x_n\}) = \left\{x \in C : \limsup_{n\to\infty} d(x_n,x) = r(C,\{x_n\})\right\}.$$

According to [11, Proposition 3.3], any bounded sequence $\{x_n\}$ in X has a unique asymptotic center relative to any nonempty closed convex subset C of X.

We now define Δ-convergence and outline its key properties, which are essential to this study.

Definition 2.3. Let C be a nonempty closed convex subset of X and $x \in C$. A bounded sequence $\{x_n\}$ in X is said to be $\Delta-$*converges* to x if $A(C,\{u_n\}) = \{x\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case we write $x_n \xrightarrow{\Delta} x$ and call x the $\Delta-$*limit* of $\{x_n\}$.

Definition 2.4. Let C be a nonempty closed convex subset of X and $T : C \to \mathcal{K}(C)$ and let I be the identity mapping on C. We say that $I - T$ is semiclosed if for any sequence $\{x_n\}$ in C such that $x_n \xrightarrow{\Delta} x$ and $R(x_n, T(x_n)) \to 0$, one has $T(x) = \{x\}$.

As shown in [14, Theorem 3.1], for a multi-valued nonexpansive mapping T on a closed convex set, $I - T$ is semiclosed. The following facts, as shown in [14, Lemma 2.13], are also needed.

Lemma 2.5. *Let C be a nonempty subset of X and $T : C \to \mathcal{K}(C)$. Then the following statements hold.*

1. *If C is convex and T is nonexpansive, then $End(T)$ is convex.*
2. *If C is closed and convex and $I - T$ is semiclosed, then $End(T)$ is closed.*

Lemma 2.6. [14, Lemma 4.3] *Let C be a nonempty closed convex subset of X, and let $T : C \to \mathcal{K}(C)$ be a mapping such that $I - T$ is semiclosed. Suppose $\{x_n\}$ is a bounded sequence in C such that $\lim_{n\to\infty} R(x_n, T(x_n)) = 0$ and $\{d(x_n, q)\}$ converges for all $q \in End(T)$, then $\omega_w(x_n) \subseteq End(T)$. Here $\omega_w(x_n) := \bigcup A(C,\{u_n\})$ where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$. Moreover, $\omega_w(x_n)$ consists of exactly one point.*

3. Main Results

First, we define a modified iteration process, analogous to the version in [5], within the setting of hyperbolic spaces as follows: Let C be a nonempty subset of X, and $\alpha_n, \beta_n, \gamma_n \in [a, b] \subset (0, 1)$ be real sequences, and $T : C \to \mathcal{K}(C)$ be a multi-valued mapping. For $x_1 \in C$,

$$z_n = (1 - \gamma_n)x_n \oplus \gamma_n v_n, \quad n \in \mathbb{N},$$

where $v_n \in T(x_n)$ such that $d(x_n, v_n) = R\left(x_n, T(x_n)\right)$, and

$$y_n = (1 - \beta_n)v_n \oplus \beta_n w_n, \tag{3.1}$$

where $w_n \in T(z_n)$ such that $d(z_n, w_n) = R\left(z_n, T(z_n)\right)$, and

$$x_{n+1} = (1 - \alpha_n)v_n \oplus \alpha_n u_n,$$

where $u_n \in T(y_n)$ such that $d(y_n, u_n) = R\left(y_n, T(y_n)\right)$.

A sequence $\{x_n\}$ in X is said to be *Fejér monotone* with respect to C if

$$d(x_{n+1}, p) \leq d(x_n, p) \text{ for all } p \in C \text{ and } n \in \mathbb{N}.$$

The following lemma establishes the Fejér monotonicity of the sequence defined in (3.1) with respect to the endpoint set of a multi-valued nonexpansive mapping.

Lemma 3.1. *Let C be a nonempty convex subset of X and let $T : C \to \mathcal{K}(C)$ be a multi-valued nonexpansive mapping with End$(T) \neq \emptyset$. If $\{x_n\}$ be the sequence as defined by (3.1), then $\{x_n\}$ is Fejér monotone with respect to End(T).*

Proof. Let $p \in \text{End}(T)$. Then, for each $n \in \mathbb{N}$, we have

$$\begin{aligned} d(z_n, p) &\leq (1 - \gamma_n)d(x_n, p) + \gamma_n d(v_n, p) \\ &\leq (1 - \gamma_n)d(x_n, p) + \gamma_n H(T(x_n), T(p)) \\ &\leq (1 - \gamma_n)d(x_n, p) + \gamma_n d(x_n, p) \\ &\leq d(x_n, p). \end{aligned}$$

This implies that

$$\begin{aligned} d(y_n, p) &\leq (1 - \beta_n)d(v_n, p) + \beta_n d(w_n, p) \\ &\leq (1 - \beta_n)H(T(x_n), T(p)) + \beta_n H(T(z_n), T(p)) \\ &\leq (1 - \beta_n)d(x_n, p) + \beta_n d(z_n, p) \\ &\leq (1 - \beta_n)d(x_n, p) + \beta_n d(x_n, p) \\ &\leq d(x_n, p). \end{aligned}$$

Hence,

$$\begin{aligned} d(x_{n+1}, p) &\leq (1 - \alpha_n)d(v_n, p) + \alpha_n d(u_n, p) \\ &\leq (1 - \alpha_n)H(T(x_n), T(p)) + \alpha_n H(T(y_n), T(p)) \\ &\leq (1 - \alpha_n)d(x_n, p) + \alpha_n d(y_n, p) \\ &\leq (1 - \alpha_n)d(x_n, p) + \alpha_n d(x_n, p) \\ &\leq d(x_n, p). \end{aligned}$$

Thus, the sequence $\{x_n\}$ is Fejér monotone with respect to End(T). ∎

Now, we prove Δ-convergence theorem.

Theorem 3.2. *Let C be a nonempty closed convex subset of X and let $T : C \to \mathcal{K}(C)$ be a multi-valued nonexpansive mapping with $\mathrm{End}(T) \neq \emptyset$. If $\{x_n\}$ be the sequence defined by (3.1), then $\{x_n\}$ Δ-converges to an endpoint of T.*

Proof. Let $p \in \mathrm{End}(T)$. By applying (2.1), we obtain

$$\begin{aligned} d^2(z_n, p) &\leq (1-\gamma_n)d^2(x_n, p) + \gamma_n d^2(v_n, p) - 4c_M\gamma_n(1-\gamma_n)d^2(x_n, v_n) \\ &\leq (1-\gamma_n)d^2(x_n, p) + \gamma_n H^2(T(x_n), T(p)) - 4c_M\gamma_n(1-\gamma_n)d^2(x_n, v_n) \\ &\leq (1-\gamma_n)d^2(x_n, p) + \gamma_n d^2(x_n, p) - 4c_M\gamma_n(1-\gamma_n)d^2(x_n, v_n) \\ &\leq d^2(x_n, p) - 4c_M\gamma_n(1-\gamma_n)d^2(x_n, v_n), \end{aligned}$$

which yields

$$\begin{aligned} d^2(y_n, p) &\leq (1-\beta_n)d^2(v_n, p) + \beta_n d^2(w_n, p) - 4c_M\beta_n(1-\beta_n)d^2(v_n, w_n) \\ &\leq (1-\beta_n)H^2(T(x_n), T(p)) + \beta_n H^2(T(z_n), T(p)) - 4c_M\beta_n(1-\beta_n)d^2(v_n, w_n) \\ &\leq (1-\beta_n)d^2(x_n, p) + \beta_n d^2(z_n, p) - 4c_M\beta_n(1-\beta_n)d^2(v_n, w_n) \\ &\leq (1-\beta_n)d^2(x_n, p) + \beta_n d^2(z_n, p) \\ &\leq (1-\beta_n)d^2(x_n, p) + \beta_n \left(d^2(x_n, p) - 4c_M\gamma_n(1-\gamma_n)d^2(x_n, v_n)\right) \\ &\leq d^2(x_n, p) - 4c_M\beta_n\gamma_n(1-\gamma_n)d^2(x_n, v_n). \end{aligned}$$

We have

$$\begin{aligned} d^2(x_{n+1}, p) &\leq (1-\alpha_n)d^2(v_n, p) + \alpha_n d^2(u_n, p) - 4c_M\alpha_n(1-\alpha_n)d^2(u_n, v_n) \\ &\leq (1-\alpha_n)H^2(T(x_n), T(p)) + \alpha_n H^2(T(y_n), T(p)) - 4c_M\alpha_n(1-\alpha_n)d^2(u_n, v_n) \\ &\leq (1-\alpha_n)d^2(x_n, p) + \alpha_n d^2(y_n, p) - 4c_M\alpha_n(1-\alpha_n)d^2(u_n, v_n) \\ &\leq (1-\alpha_n)d^2(x_n, p) + \alpha_n d^2(y_n, p) \\ &\leq (1-\alpha_n)d^2(x_n, p) + \alpha_n \left(d^2(x_n, p) - 4c_M\beta_n\gamma_n(1-\gamma_n)d^2(x_n, v_n)\right) \\ &\leq d^2(x_n, p) - 4c_M\alpha_n\beta_n\gamma_n(1-\gamma_n)d^2(x_n, v_n). \end{aligned}$$

Consequently, the following inequality holds

$$d^2(x_{n+1}, p) \leq d^2(x_n, p) - 4c_M\alpha_n\beta_n\gamma_n(1-\gamma_n)d^2(x_n, v_n).$$

Since $4c_M > 0$, it follows that

$$a^3(1-b)d^2(x_n, v_n) \leq \alpha_n\beta_n\gamma_n(1-\gamma_n)d^2(x_n, v_n) \leq \frac{d^2(x_n, p) - d^2(x_{n+1}, p)}{4c_M}.$$

By Lemma 3.1, the sequence $\{x_n\}$ is Fejér monotone with respect to $\mathrm{End}(T)$, and therefore the $\lim_{n\to\infty} d^2(x_n, p)$ exist. Consequently, summing over n yields

$$\sum_{n=1}^{\infty} a^3(1-b)d^2(x_n, v_n) \leq \sum_{n=1}^{\infty} \alpha_n\beta_n\gamma_n(1-\gamma_n)d^2(x_n, v_n) < \infty.$$

This implies that $\lim_{n\to\infty} d^2(x_n, v_n) = 0$, and hence

$$\lim_{n\to\infty} R(x_n, T(x_n)) = \lim_{n\to\infty} d(x_n, v_n) = 0. \tag{3.2}$$

Furthermore, by Lemma 3.1, the sequence $\{d(x_n, q)\}$ converges for all $q \in \mathrm{End}(T)$. and by Lemma 2.6, $\omega_w(x_n)$ consists of exactly one point and is contained in $\mathrm{End}(T)$. This shows that $\{x_n\}$ Δ-converges to an endpoint of T. ∎

Next, we establish strong convergence theorems by imposing additional conditions. Recall that a multi-valued mapping $T : C \to \mathcal{K}(C)$ is said to satisfy *condition (J)* if there exists a nondecreasing function $h : [0, \infty) \to [0, \infty)$ with $h(0) = 0$ and $h(r) > 0$ for every $r \in (0, \infty)$ such that

$$R(x, T(x)) \geq h(\mathrm{dist}(x, \mathrm{End}(T))) \text{ for all } x \in C.$$

Furthermore, the mapping $T : C \to \mathcal{K}(C)$ is called *semicompact* if for any sequence $\{x_n\}$ in C satisfying

$$\lim_{n\to\infty} R(x_n, T(x_n)) = 0,$$

there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $q \in C$ such that $\lim_{k\to\infty} x_{n_k} = q$.

Additionally, we require the following facts.

Lemma 3.3 ([1], Lemma 3.4)**.** *Let C be a nonempty closed subset of X and $\{x_n\}$ be a Fejér monotone sequence with respect to C. Then $\{x_n\}$ converges strongly to an element of C if and only if $\lim_{n\to\infty} dist(x_n, C) = 0$.*

Lemma 3.4 ([15], Lemma 2)**.** *Let $\{\alpha_n\}$, $\{\beta_n\}$ be two real sequences in $[0, 1)$ such that $\beta_n \to 0$ and $\sum \alpha_n\beta_n = \infty$. Let $\{\gamma_n\}$ be a nonnegative real sequence such that*

$$\sum_{n=1}^{\infty} \alpha_n\beta_n(1 - \beta_n)\gamma_n < \infty.$$

Then, $\{\gamma_n\}$ has a subsequence which converges to zero.

Theorem 3.5. *Let C be a nonempty closed convex subset of X and let $T : C \to \mathcal{K}(C)$ be a multi-valued nonexpansive mapping with $End(T) \neq \emptyset$. Assume that the sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \in [a, b] \subset (0, 1)$ such that $\gamma_n \to 0$ as $n \to \infty$ and $\sum_{n=1}^{\infty} \alpha_n\beta_n\gamma_n = \infty$. Let $\{x_n\}$ be the sequence defined by (3.1). If T is semicompact, then $\{x_n\}$ converges strongly to an endpoint of T.*

Proof. It follows from the proof of Theorem 3.2 that

$$\sum_{n=1}^{\infty} \alpha_n\beta_n\gamma_n(1 - \gamma_n)d^2(x_n, v_n) < \infty.$$

By Lemma 3.4, there exists a subsequence $\{d^2(x_{n_k}, v_{n_k})\}$ of $\{d^2(x_n, v_n)\}$ such that $\lim_{k\to\infty} d^2(x_{n_k}, v_{n_k}) = 0$, which implies

$$\lim_{k\to\infty} R(x_{n_k}, T(x_{n_k})) = \lim_{k\to\infty} d(x_{n_k}, v_{n_k}) = 0. \tag{3.3}$$

Since T is semicompact, by passing to a subsequence, we may assume that $x_{n_k} \to q \in C$. Moreover, because T is continuous, we have

$$\begin{aligned}\operatorname{dist}(q, T(q)) &\leq d(q, x_{n_k}) + \operatorname{dist}(x_{n_k}, T(x_{n_k})) + H(T(x_{n_k}), T(q)) \\ &\leq 2d(x_{n_k}, q) + \operatorname{dist}(x_{n_k}, T(x_{n_k})) \\ &\leq 2d(x_{n_k}, q) + R(x_{n_k}, T(x_{n_k})) \to 0 \quad \text{as } k \to \infty.\end{aligned}$$

Thus, $q \in T(q)$. Let $w \in T(q)$. For each k, there exists w_{n_k} in $T(x_{n_k})$ such that

$$d(w, w_{n_k}) = \operatorname{dist}(w, T(x_{n_k})).$$

Then, using (3.3) and the continuity of T, we obtain

$$\begin{aligned}d(q, w) &\leq d(q, x_{n_k}) + d(x_{n_k}, v_{n_k}) + d(v_{n_k}, w) \\ &\leq d(q, x_{n_k}) + R(x_{n_k}, T(x_{n_k})) + H((Tx_{n_k}), T(q)) \\ &\leq 2d(x_{n_k}, q) + R(x_{n_k}, T(x_{n_k})) \to 0 \quad \text{as} \quad k \to \infty.\end{aligned}$$

Hence, $w = q$ for every $w \in T(q)$, which implies $q \in \operatorname{End}(T)$. Finally, by Lemma 3.1 the sequence $\{d(x_n, q)\}$ converges. Therefore, q is the strong limit of $\{x_n\}$. ∎

Theorem 3.6. *Let C be a nonempty closed convex subset of X, and let $T : C \to \mathcal{K}(C)$ be a multi-valued nonexpansive mapping. Assume that the sequence $\{x_n\}$ is defined by (3.1). If T satisfies condition (J), then $\{x_n\}$ converges strongly to an endpoint of T.*

Proof. Since T is nonexpansive, the set $\operatorname{End}(T)$ is closed. Moreover, because T satisfies condition (J), it follows from (3.2) that

$$\lim_{n\to\infty} \operatorname{dist}(x_n, \operatorname{End}(T)) = 0.$$

Furthermore, by Lemma 3.1, the sequence $\{x_n\}$ is Fejér monotone with respect to $\operatorname{End}(T)$. Consequently, the desired conclusion follows from Lemma 3.3. ∎

The following examples illustrate the validity of the main theorems.

Example 3.7. Let $X = \mathbb{R}$ and $C = [3, 8]$. We define a multi-valued mapping $T : C \to \mathcal{K}(C)$ by

$$T(x) = [3, x] \quad \text{for each } x \in C.$$

Clearly, T is nonexpansive with $\operatorname{End}(T) = \{3\}$. We now prove that T satisfies condition (J). Let $h(t) = t$. For $x \in C = [3, 8]$, we have

$$\begin{aligned}h\left(\operatorname{dist}(x, \operatorname{End}(T))\right) &= h\left(\operatorname{dist}(x, \{3\})\right) \\ &= h\left(\left|x-3\right|\right) \\ &= \left|x-3\right| \\ &= \sup\left\{\left|x-y\right| \,:\, y \in [3, x]\right\} \\ &= R\left(x, [3, x]\right) \\ &= R\left(x, T(x)\right).\end{aligned}$$

This completes the verification that T satisfies condition (J). Next, we show that the sequence $\{x_n\}$ defined by (3.1) converges strongly to 3. For this demonstration, we choose $\alpha_n, \beta_n, \gamma_n = \frac{1}{2}$ for all $n \in \mathbb{N}$. Let $x_1 = 5$, then $T(x_1) = [3, 5]$. Select $v_1 \in [3, 5]$ such that

$$\left|5 - v_1\right| = R\left(5, T(5)\right) = R\left(5, [3, 5]\right) = \sup\{\left|5 - y\right| \,:\, y \in [3, 5]\} = \left|5 - 3\right|.$$

Thus, we have $v_1 = 3$. Consequently,

$$z_1 = \left(1 - \gamma_1\right) x_1 + \gamma_1 v_1 = \frac{1}{2}(5) + \frac{1}{2}(3) = 4.$$

Next, since $T(z_1) = T(4) = [3, 4]$, choose $w_1 \in T(z_1)$ satisfying

$$\left|4 - w_1\right| = R\left(4, T(4)\right) = R\left(4, [3, 4]\right) = \sup\{\left|4 - y\right| \,:\, y \in [3, 4]\} = \left|4 - 3\right|.$$

Hence, $w_1 = 3$ and

$$y_1 = \left(1 - \beta_1\right) v_1 + \beta_1 w_1 = \frac{1}{2}(3) + \frac{1}{2}(3) = 3.$$

Now, $T(y_1) = [3, 3]$, implies $u_1 = 3$, and therefore,

$$x_2 = \left(1 - \alpha_1\right) v_1 + \alpha_1 u_1 = \frac{1}{2}(3) + \frac{1}{2}(3) = 3.$$

Repeating this process shows that $x_n = 3$ for all $n \geq 2$, so the sequence $\{x_n\}$ converges strongly to $3 \in \mathrm{End}(T)$.

Example 3.8. Let $X = \mathbb{R}^2$ and define the set $C = \left\{(x, y) \in X \,:\, x^2 + y^2 \leq 1\right\}$. We then define a multi-valued mapping $T : C \to 2^C$ by

$$T(x, y) = \{(tx, -ty) : 0 \leq t \leq 1\} \quad \text{for all } (x, y) \in C.$$

We first note that T is semicompact. Next, we show that T is nonexpansive. For any two points (x_1, y_1) and (x_2, y_2) in C, their images under the multi-value mapping T are given by

$$T(x_1, y_1) = \{(tx_1, -ty_1) : 0 \leq t \leq 1\},$$

$$T(x_2, y_2) = \{(sx_2, -sy_2) : 0 \leq s \leq 1\}.$$

Select arbitrary points $v_1 = (tx_1, -ty_1) \in T(x_1, y_1)$ and $v_2 = (sx_2, -sy_2) \in T(x_2, y_2)$ with $0 \leq t, s \leq 1$. Their distance is

$$d(v_1, v_2) = \sqrt{(tx_1 - sx_2)^2 + (-ty_1 + sy_2)^2}.$$

Expanding the square, we obtain

$$d(v_1, v_2) = \sqrt{t^2\left(x_1^2 + y_1^2\right) + s^2\left(x_2^2 + y_2^2\right) - 2ts\left(x_1 x_2 + y_1 y_2\right)}.$$

Taking the supremum over t and s in $[0, 1]$, we observe:

- The farthest distance occurs when $t = s = 1$, which gives

$$d(v_1, v_2) = \sqrt{\left(x_1 - x_2\right)^2 + \left(y_1 - y_2\right)^2} = d((x_1, y_1), (x_2, y_2)).$$

- The closest distance occurs when $t = s = 0$, yielding $d(v_1, v_2) = 0$.

Since the Hausdorff distance reflects the worst-case (i.e., maximum) distance, we have

$$H\left(T(x_1, y_1), T(x_2, y_2)\right) = d((x_1, x_1), (x_2, y_2)).$$

Thus, T is nonexpansive. It is also evident that $(0, 0) \in \mathrm{End}(T)$.
Next, we prove that the sequence $\{x_n\}$ defined by (3.1) converges strongly to $(0, 0) \in \mathrm{End}(T)$.
Choose $\alpha_n, \beta_n, \gamma_n = \frac{1}{\sqrt[3]{n+1}}$ for all $n \in \mathbb{N}$.
Let $x_1 = (1, 0)$; then

$$T(x_1) = \{(t \cdot 1, -t \cdot 0) : 0 \leq t \leq 1\} = \{(t, 0) : 0 \leq t \leq 1\}.$$

Choose $v_1 \in T(x_1)$ such that

$$d(x_1, v_1) = R\left(x_1, T(x_1)\right) = \sup\left\{d\left((1, 0), (t, 0)\right) : 0 \leq t \leq 1\right\}.$$

Since the supremum is achieved when $t = 0$, we have $v_1 = (0, 0)$.
Thus,

$$z_1 = (1 - \gamma_1)x_1 + \gamma_1 v_1 = \left(1 - \frac{1}{\sqrt[3]{2}}\right)(1, 0) + \frac{1}{\sqrt[3]{2}}(0, 0) = \left(1 - \frac{1}{\sqrt[3]{2}}, 0\right).$$

Next,

$$T(z_1) = T\left(1 - \frac{1}{\sqrt[3]{2}}, 0\right) = \left\{\left(t\left(1 - \frac{1}{\sqrt[3]{2}}\right), 0\right) : 0 \leq t \leq 1\right\}.$$

Choose $w_1 \in T(z_1)$ such that $d(z_1, w_1) = R\left(z_1, T(z_1)\right)$. Again, the supremum distance is attained when $t = 0$, so $w_1 = (0, 0)$.
Then,

$$y_1 = (1 - \beta_1)v_1 + \beta_1 w_1 = \left(1 - \frac{1}{\sqrt[3]{2}}\right)(0, 0) + \frac{1}{\sqrt[3]{2}}(0, 0) = (0, 0).$$

Since $T(y_1) = T(0, 0) = \{(0, 0)\}$, it follows that $u_1 = (0, 0)$, and therefore,

$$x_2 = (1 - \alpha_1)v_1 + \alpha_1 u_1 = \left(1 - \frac{1}{\sqrt[3]{2}}\right)(0, 0) + \frac{1}{\sqrt[3]{2}}(0, 0) = (0, 0).$$

By iterating this process, we conclude that $x_n = (0, 0)$, for all $n \geq 2$, ensuring that the sequence$\{x_n\}$ converges strongly to $(0, 0) \in \mathrm{End}(T)$. Furthermore, applying a similar numerical approach, we observe that if the initial point x_1 satisfies the required conditions, the sequence $\{x_n\}$consistently converges to $(0, 0) \in \mathrm{End}(T)$, as illustrated in Table 1 and Table 2.

Table 1. The iterative process of $\{x_n\}$ with the initial points $(-1, 0)$, $(-0.5, 0)$, $(0.6, 0)$ $(0, 1)$, and $(0, -1)$.

x_1	$(-1, 0)$	$(-0.5, 0)$	$(0.6, 0)$	$(0, 1)$	$(0, -1)$
x_2	$(0, 0)$	$(0, 0)$	$(0, 0)$	$(0, -0.4125989480)$	$(0, 0.4125989480)$
x_3	$(0, 0)$	$(0, 0)$	$(0, 0)$	$(0, 0.1155046486)$	$(0, -0.1155046486)$
x_4	$(0, 0)$	$(0, 0)$	$(0, 0)$	$(0, -0.0277302348)$	$(0, 0.0277302348)$
x_5	$(0, 0)$	$(0, 0)$	$(0, 0)$	$(0, 0.0063888493)$	$(0, -0.0063888493)$
x_6	$(0, 0)$	$(0, 0)$	$(0, 0)$	$(0, -0.0014866272)$	$(0, 0.0014866272)$
x_7	$(0, 0)$	$(0, 0)$	$(0, 0)$	$(0, 0.0003570854)$	$(0, -0.0003570854)$
x_8	$(0, 0)$	$(0, 0)$	$(0, 0)$	$(0, -0.0000892714)$	$(0, 0.0000892714)$
x_9	$(0, 0)$	$(0, 0)$	$(0, 0)$	$(0, 0.0000232751)$	$(0, -0.0000232751)$
x_{10}	$(0, 0)$	$(0, 0)$	$(0, 0)$	$(0, -0.0000063234)$	$(0, 0.0000063234)$

Table 2. The iterative process of $\{x_n\}$ with the initial points $(0, 0.4)$, $(0, -0.9)$, and $(-0.67, 0.73)$

x_1	$(0, 0.4)$	$(0, -0.9)$	$(-0.67, 0.73)$
x_2	$(0, -0.1650395792)$	$(0, 0.3713390532)$	$(-0.1382206476, -0.1505986160)$
x_3	$(0, 0.0462018594)$	$(0, -0.1039541837)$	$(-0.0717710911, 0.0141603825)$
x_4	$(0, -0.0110920939)$	$(0, 0.0249572113)$	$(0, 0)$
x_5	$(0, 0.0025555397)$	$(0, -0.0057499644)$	$(0, 0)$
x_6	$(0, -0.0005946509)$	$(0, 0.0013379645)$	$(0, 0)$
x_7	$(0, 0.0001428342)$	$(0, -0.0003213769)$	$(0, 0)$
x_8	$(0, -0.0000357085)$	$(0, 0.0000803442)$	$(0, 0)$
x_9	$(0, 0.0000093100)$	$(0, -0.0000209475)$	$(0, 0)$
x_{10}	$(0, -0.0000025294)$	$(0, 0.0000056911)$	$(0, 0)$

4. Conclusion

This study proposed a novel iterative process for approximating endpoints of multi-valued nonexpansive mappings in 2-uniformly convex hyperbolic spaces, extending prior results in Banach spaces. The established Δ-convergence and strong convergence theorems strengthen the theoretical foundation of endpoint approximation in nonlinear analysis. The Fejér monotonicity of the sequence was verified, ensuring convergence under suitable conditions. These findings enhance the applicability of iterative methods in hyperbolic spaces, with potential extensions to broader classes of mappings and optimization problems. Future research may focus on refining convergence rates and exploring practical applications in computational mathematics and optimization.

Competing Interests

The authors declare that they have no competing interests.

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