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Revisit to Suzuki's Metric Completeness

Sehie Park[a,1]

[a] The National Academy of Sciences, Republic of Korea, Seoul 06579; Department of Mathematical Sciences, Seoul National University, Seoul 08826, Korea
[1] park35@snu.ac.kr; sehiepark@gmail.com

ABSTRACT

There have appeared a large number of articles related to the metric completeness. They are mainly concerned with generalizations of the Banach contraction on metric spaces and their modified artificial spaces. A well-known fixed point theorem of Suzuki in 2008 for the so-called Suzuki type map on a complete metric space with a lengthy proof is very popular and has a large number of followers. Such results on metric spaces are consequences of our generalized forms of the Banach contraction principle for weak contractions or the Rus-Hicks-Rhoades maps on quasi-metric spaces. In this paper, we collect the works of Suzuki and his colleagues on metric completeness and give short proofs and improvements for them. Moreover, we collect some positive but incorrect comments given by many followers of Suzuki.

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1. Prologue

Since completeness is one of the most important properties of various types of extensions of metric spaces, there have been appeared a huge number of works related to the metric completeness.

In our previous work [14] in 1984, we gave some necessary and sufficient conditions for a metric space (X, d) to be complete. Such characterizations of metric completeness were mainly given by results relevant to Caristi's fixed point theorem. Works of Cantor, Kuratowski, Ekeland, Caristi, Kirk, Wong, Weston, Ćirić, Hu, Reich, Subrahmanyam, and others are combined. For the references, see [14].

The first response for the article [14] was that: "Who dare use this kind of things to check the completeness of a metric space?" Contrary to such critical opinion, there have appeared numerous articles related to the metric completeness; see [6, 21]. The use of such articles are not only to check whether the space is complete or not.

Let (X, d) be a metric space. A Banach contraction $T : X \to X$ is a map satisfying

$$d(Tx, Ty) \leq \alpha\, d(x, y) \quad \text{for all } x, y \in X$$

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with some $\alpha \in [0,1)$. There have appeared thousands of articles related to the Banach contraction; see [19].

Recently, we introduced a weak contraction or the Rus-Hicks-Rhoades (RHR) map $T : X \to X$ satisfying

$$d(Tx, T^2x) \le \alpha\, d(x, Tx) \quad \text{for all } x \in X$$

with some $\alpha \in [0,1)$. See our recent works [13, 18, 20, 22, 23].

The RHR maps are called graphic contraction, iterative contraction, weakly contraction, or Banach mapping; see Berinde et al. [1]. Moreover, it is recently known that well-known metric fixed point theorems related to the RHR maps hold for quasi-metric spaces (without assuming the symmetry); see [13, 20, 22].

Recall that Suzuki [28] in 2008 gave a very interesting result on a particular RHR map. It is a weaker version of the Banach contraction principle and also characterizes the completeness of underlying metric spaces. It was commonly regarded that Suzuki's result gave a new direction to the subject and as a result, researchers including his colleagues made many contributions in metric fixed point theory. Most of such results are variations and refinements of Suzuki's original result with lengthy complicated proofs. However, we found in [13, 15, 18, 21, 22] that most of such Suzuki type results are incorrectly stated according to our recently developed ordered fixed point theory [16].

Our aim in this article is to collect new characterizations of completeness of quasi-metric spaces and to review such results of Suzuki and his colleagues. We give certain critical comments on them.

This article is organized as follows: Section 2 is for preliminaries on quasi-metric spaces and to introduce the Rus-Hicks-Rhoades Contraction Principle (Theorem P) and a related theorem. In Section 3, we give few-line proofs of Suzuki's 2008 theorems and their extended versions. Section 4 devotes to give another proofs of Suzuki's completeness theorem and other authors. In Section 5, we collect some other theorems on metric completeness due to Suzuki and his colleagues and give shorter proofs.

Since Suzuki's works on metric completeness are very popular and many authors praised them. In Section 6, we collect thirty-six articles following Suzuki's way. Such articles are given by the publishing year with the initials of their authors. Finally, Section 7 is for Epilogue.

2. Preliminaries

It is well-known that many key-results in Metric Fixed Point Theory hold for quasi-metric spaces. For example, Banach contraction principle, Nadler or Covitz-Nadler fixed point theorem, Ekeland variational principle, Caristi fixed point theorem, Takahashi minimization principle, and many others; see [12, 17, 23].

We recall the following:

Definition 2.1. A *quasi-metric* on a nonempty set X is a function $q : X \times X \to \mathbb{R}^+ = [0,\infty)$ satisfying the following conditions for all $x, y, z \in X$:

(a) (self-distance) $q(x,y) = q(y,x) = 0 \Longleftrightarrow x = y$;

(b) (triangle inequality) $q(x,z) \le q(x,y) + q(y,z)$.

A *metric* on a set X is a quasi-metric satisfying

(c) (symmetry) $q(x, y) = q(y, x)$ for all $x, y \in X$.

Remark 2.2.

(1) For quasi-metric spaces, the convergence of a sequence, Cauchy sequences, completeness, orbits, and orbital continuity are routinely defined; see Jleli-Samet [7].

(2) Such definitions also work for a topological space X and a function $q : X \times X \to \mathbb{R}^+ = [0, \infty)$ such that $q(x, y) = 0$ implies $x = y$ for $x, y \in X$

Definition 2.3. Let (X, q) be a quasi-metric space and $T : X \to X$ a selfmap. The *orbit* of T at $x \in X$ is the set

$$O_T(x) = \{x, Tx, \cdots, T^n x, \cdots\}.$$

The space X is said to be *T-orbitally complete* if every right-Cauchy sequence in $O_T(x)$ is convergent in X. A selfmap T of X is said to be *orbitally continuous* at $x_0 \in X$ if

$$\lim_{n\to\infty} T^n x = x_0 \Longrightarrow \lim_{n\to\infty} T^{n+1} x = Tx_0$$

for any $x \in X$.

The following Rus-Hicks-Rhoades (RHR) Contraction Principle is originally given in [12] and used in [13, 15, 18, 21–23]:

Theorem P. *Let (X, q) be a quasi-metric space and let $T : X \to X$ be an RHR map; that is,*

$$q(T(x), T^2(x)) \leq \alpha\, q(x, T(x)) \text{ for every } x \in X, \tag{p}$$

where $0 < \alpha < 1$.

(i) If X is T-orbitally complete, then, for each $x \in X$, there exists a point $x_0 \in X$ such that

$$\lim_{n\to\infty} T^n(x) = x_0$$

and

$$q(T^n(x), x_0) \leq \frac{\alpha^n}{1-\alpha} q(x, T(x)), \quad n = 1, 2, \cdots,$$

$$q(T^n(x), x_0) \leq \frac{\alpha}{1-\alpha} q(T^{n-1}(x), T^n(x)), \quad n = 1, 2, \cdots.$$

(ii) x_0 is a fixed point of T, and, equivalently,

(iii) $T : X \to X$ is orbitally continuous at $x_0 \in X$.

More early in our [23], we obtained Theorem H which gives equivalent formulations of the completeness of quasi-metric spaces. The following is a part of Theorem H.

Theorem H(γ1). *[13, 23] Let (X, q) be a quasi-metric space, $0 < \alpha < 1$, and $f : X \to X$ be a map satisfying*

$$q(f(x), f^2(x)) \leq \alpha\, q(x, f(x)) \text{ for all } x \in X \backslash \{f(x)\}.$$

Then f has a fixed element $v \in X$ if X is f-orbitally complete.

This form of the RHR theorem is also a consequence of Theorems P, and useful in practice.

In our previous works [13, 15, 18, 21–23], we applied Theorems P and H(γ1) to a large number of early extensions or relatives of theorems of Rus [27] in 1973 and Hicks-Rhoades [5] in 1979.

3. New Proofs of Suzuki's 2008 Theorems

Suzuki in 2008 generalized the Banach contraction principle as follows ([28], Theorem 2):

Theorem 3.1. *[28] Let (X, d) be a complete metric space and T be a mapping on X. Define a nonincreasing function θ from $[0, 1)$ onto $(1/2, 1]$ by*

$$\theta(r) = \begin{cases} 1 & \text{if } 0 \le r \le (\sqrt{5}-1)/2, \\ (1-r)r^{-2} & \text{if } (\sqrt{5}-1)/2 \le r \le 2^{-1/2}, \\ (1+r)^{-1} & \text{if } 2^{-1/2} \le r < 1. \end{cases} \tag{3.1}$$

Assume there exists $r \in [0, 1)$ such that

$$\theta(r)d(x, Tx) \le d(x, y) \text{ implies } d(Tx, Ty) \le rd(x, y)$$

for all $x, y \in X$. Then there exists a unique fixed point z of T. Moreover $\lim_n T^n x = z$ for all $x \in X$.

Instead of Suzuki's two page proof, we have the following for a quasi-metric space:

Proof. Note that T is an RHR map and hence Theorems P or H(γ1) can be applied. Now it suffices to show the uniqueness of the fixed point z. If $w \in X$ is a fixed point, then

$$\theta(r)d(z, Tz) \le d(z, w) \text{ implies } d(Tz, Tw) \le r\, d(z, w).$$

Hence $d(z, w) = d(Tz, Tw) \le r\, d(z, w)$ and consequently $d(z, w) = 0$. ∎

There have appeared a large number of variants of Suzuki's theorem. Many of them can be improved by easy proofs as shown as above.

From now on, T is called the Suzuki type as its many followers used to do.

Suzuki noted that the following theorem [[28], Theorem 3] says that $\theta(r)$ is the best constant for every $r \in [0, 1)$:

Theorem 3.2. *[28] Define a function θ as in Theorem 3.1. Then for each $r \in [0, 1)$, there exist a complete metric space (X, d) and a mapping T on X such that T does not have a fixed point and*

$$\theta(r)d(x, Tx) < d(x, y) \text{ implies } d(Tx, Ty) \le r\, d(x, y)$$

for all $x, y \in X$.

Proof. Note that $\theta(r)d(x, Tx) < d(x, y)$ means T can not have a fixed point $x = y = Tx$. ∎

In order to show uselessness of Theorems 3.1 and 3.2 of Suzuki, let us consider any function $\theta' : [0, \infty) \to [0, 1]$:

Theorem 3.3. *'Replace the function θ in Theorem 3.1 by θ'. Then the conclusion of Theorem 3.1 holds.*

Proof. Note that, by putting $y = Tx$, T becomes an RHR map. Then by Theorems P or H(γ1), T has a fixed point and its uniqueness follows as in the proof of Theorem 3.1. ∎

Theorem 3.4. *' Replace the function θ in Theorem 3.2 by θ'. Then the conclusion of Theorem 3.2 holds.*

Proof. Note that $\theta'(r)d(x, Tx) < d(x, y)$ means T can not have a fixed point $x = y = Tx$. ∎

Recall that there are hundreds of equivalent conditions for metric completeness, e.g. Kirk [10], Park [14], Cobzaș [4] as typical examples. Comments for them are given by Park and Rhoades [24] in 1986.

4. Another Proofs of Suzuki's 2008 Theorems

In our previous work [25] in 1980, we obtained essentially the following:

Theorem 4.1. *[25] Let f be a selfmap of a quasi-metric space (X, q) satisfying:*

(1) X is f-orbitally complete and $\delta(O_f(x)) < \infty$ for each $x \in X$, where δ denotes the diameter.

(2) There exists a $u \in X$ such that $O_f(u)$ has a cluster point $p \in X$.

(3) There exists a function $\varphi : [0, \infty) \to [0, \infty)$ which is nondecreasing, continuous from the right and satisfies $\varphi(t) < t$ for each $t > 0$ and the inequality

$$d(fx, fy) \leq \varphi(\delta(O_f(x) \cup O_f(y))) \quad \text{for each} \quad x, y \in X.$$

Then p is the unique fixed point of f and $\lim_n f^n(u) = p$.

This extends works of Pal-Maiti, Park, Hegedüs, Daneš, and many others.

The following is another proof of Suzuki's 2008 theorem:

Proof. Proof of Theorem 3.1 from Theorem 4.1. Since $\theta(r) \leq 1$, $\theta(r)d(x, Tx) \leq d(x, Tx)$ holds for every $x \in X$. By hypothesis, we have

$$d(Tx, T^2x) \leq r\, d(x, Tx) \text{ for all } x \in X.$$

This clearly satisfies condition (3) of Theorem 4.1 for $y = fx = Tx$. We now fix $u \in X$ and define a sequence $\{u_n\}$ in X by $u_n = T^n u$. Then $d(u_n, u_{n+1}) \leq r^n d(u, Tu)$, so $\sum_{n=1}^{\infty} d(u_n, u_{n+1}) < \infty$, and a standard argument shows $\{u_n\}$ is Cauchy and $\delta(O(u)) < \infty$. Then the conclusion follows from Theorem 4.1. ∎

The following extends Theorem 4.1 and the main theorems of Chandra-Arya-Joshi ([2], Theorem 2.1) for metric spaces and of Özkan ([11], Theorem 1) for partial metric spaces:

Theorem 4.2. *Let (X, q) be a complete quasi-metric space, $T, S : X \to X$ be two selfmaps and a nonincreasing function $\theta : [0, 1) \to (1/2, 1]$ defined as in Theorem 3.1. If there exists $r \in [0, 1/2)$ such that*

$$\theta(r)\min\{q(x, Tx), q(x, Sx)\} \le q(x, y) \Longrightarrow$$
$$\max\{q(Sx, Sy), q(Tx, Ty), \frac{1}{2}[q(Sx, Ty) + q(Sy, Tx)] \le r\, q(x, y), \quad \text{for all} x, y \in X, \quad (4.1)$$

then T and S have a unique common fixed point.

Proof. We divide several steps to prove.

Step 1. Fixed point sets $\mathrm{Fix}(T) = \mathrm{Fix}(S)$: Let $u = Tu$. Then

$$0 \le \theta(r)\min\{q(u, Tu), q(u, Su)\} \le q(u, Tu)$$

implies

$$q(Su, u) \le \max\{q(Su, STu), q(Tu, T^2u), \frac{1}{2}[q(Su, T^2u) + q(Su, Tu)]\} \le r\, q(u, Tu).$$

Hence, we have $q(Su, u) \le r\, q(u, u)$. Since q is a quasi-metric, $q(u, u) = 0$ and hence $Su = u$. Similarly, $u = Su$ implies $Tu = u$.

Step 2. T is an RHR map: Putting $y = Tx$, we have

$$\theta(r)\min\{q(x, Tx), q(x, Sx)\} \le q(x, Tx)$$

implies

$$\max\{q(Sx, STx), q(Tx, T^2x), \frac{1}{2}[q(Sx, T^2x) + q(STx, Tx)]\} \le r\, q(x, Tx)$$

for every $x \in X$. Hence we get $q(Tx, T^2x) \le r\, q(x, Tx)$.

Step 3. Existence and uniqueness of common fixed point: Since T is an RHR map, we have $\mathrm{Fix}(T) = \mathrm{Fix}(S) \neq \emptyset$ by Theorem H(γ1).

Now, to show the uniqueness of this common fixed point, we assume that u and v are common fixed points of T and S where $u \neq v$. Taking $x = u$ and $y = v$ in inequality (4.1), we have

$$\begin{aligned}
0 = \theta(r)\min\{q(u, Tu), q(u, Su)\} &\le q(u, v)\\
&\Longrightarrow \max\{q(Su, Sv), q(Tu, Tv), \frac{1}{2}[q(Su, Tv) + q(Tu, Sv)] \le r\, q(u, v)\\
&\Longrightarrow \max\{q(u, v), q(u, v), \frac{1}{2}[2q(u, v)]\} \le r\, q(u, v)\\
&\Longrightarrow q(u, v) \le r\, q(u, v) < q(u, v).
\end{aligned}$$

This is a contradiction. Hence, $q(u, v) = 0$ and $u = v$. ∎

Remark 4.3.

(1) We can add the statement (1) of Theorem 4.1 to Theorem 3.2.

(2) For $S = T$, Theorem 4.2 reduces to Theorem 3.1 Hence, the above proof gives the third proof of Theorem 3.1.

(3) Some particular cases of the RHR maps extending Suzuki type contractive conditions were appeared already as follows:

(3.1) Ćirić [3] in 1974: $\min\{d(Tx, Ty), d(x, Tx), d(y, Ty)\} - \min\{d(x, Ty), d(y, Tx)\} \leq r\, d(x, y)$.

(3.2) Suzuki [28] in 2008: $\theta(r)d(x, Tx) \leq d(x, y)$ implies $d(Tx, Ty) \leq r\, d(x, y)$.

(3.3) Kim-Sedghi-Shobkolaei [9] in 2015: $\theta(r)\min\{d(x, Tx), d(x, Sx)\} \leq d(x, y)$ implies

$$\max\{d(Sx, Sy), d(Tx, Ty), d(Sx, Ty), d(Sy, Tx)\} \leq r\, d(x, y).$$

(3.4) Rakočević-Samet [26] in 2017:

$$\min\{||Tx - Ty||, ||x - Tx||, ||y - Ty||\} - \min\{||x - Ty||, ||y - Tx||\} \leq r||x - y||.$$

The following is motivated by Chandra-Arya-Joshi ([2], Corollary 2.3) and Özkan ([11], Corollary 2):

Corollary 4.4. *Let (X, q) be a complete quasi-metric space, $f, S, T : X \to X$ be three selfmaps and a nonincreasing function $\theta : [0, 1) \to (1/2, 1]$ be defined as usual. If there exists $r \in [0, 1/2)$ such that*

$$\theta(r)\min\{q(x, fTx), q(x, fSx)\} \leq q(x, y) \text{ implies}$$
$$\max\{q(fSx, fSy), q(fTx, fTy), \frac{1}{2}[q(fSx, fTy) + q(fSy, fTx)]\} \leq r\, q(x, y), \tag{4.2}$$

also, if f is one to one, $fS = Sf$ and $fT = Tf$, then f, T and S have a common fixed point.

Proof. If we consider fS and fT as two maps with given contractive condition of Theorem 4.2, then fS and fT have a common fixed point $u \in X$. Namely, $fSu = fTu = u$. Since f is one to one, we get

$$fSu = fTu = u \implies Su = Tu.$$

Then, putting $x = u$ and $y = Tu$ in inequality (4.2)

$$\begin{aligned}
&\theta(r)\min\{d(u, fTu), d(u, fSu)\} \leq d(u, Tu)\\
&\implies \max\{d(fSu, fSTu), d(fTu, fT^2u), \tfrac{1}{2}[d(fSu, fT^2u) + d(fSTu, fTu)]\} \leq r\, d(u, Tu)\\
&\implies \max\{d(fSu, SfTu), d(fTu, TfTu), \tfrac{1}{2}[d(fSu, TfTu) + d(SfTu, fTu)]\} \leq r\, d(u, Tu)\\
&\implies \max\{d(u, Su), d(u, Tu), \tfrac{1}{2}[d(u, Tu) + d(Su, u)]\} \leq r\, d(u, Tu)\\
&\implies d(u, Tu) \leq r\, d(u, Tu).
\end{aligned}$$

Then, $d(u, Tu) = 0$. So, we get $Tu = u$ which implies $Tu = Su = u$ and also $fu = fTu = u$. So, f, T and S have a common fixed point. ∎

5. Theorems Due to Suzuki et al.

As we have seen in the previous sections, from 2008, Suzuki found fixed point theorems for certain RHR type maps with very sophisticated proofs. He and his colleagues continued to publish more related papers within a few years. Their papers became very popular and many followers published scores of papers of the same nature.

In this section, we collect some theorems related to metric completeness due to Suzuki and his colleagues.

In Section 4 of [28], Suzuki discussed the metric completeness:

Theorem 5.1. *[28] Let (X, d) be a metric space and define a nonincreasing function θ from $[0, 1)$ onto $(1/2, 1]$ as in Theorem 3.1. For $r \in [0, 1)$ and $\eta \in (0, \theta(r)]$, let $A_{r,\eta}$ be the family of mappings T on X satisfying the following:*

(a) For $x,\ y \in X$,

$$\eta\, d(x, Tx) \leq d(x, y) \text{ implies } d(Tx, Ty) \leq r\, d(x, y).$$

Let $B_{r,\eta}$ be the family of mappings T on X satisfying (a) and the following:

(b) $T(X)$ is countably infinite.

(c) Every subset of $T(X)$ is closed.

Then the following are equivalent:

(i) X is complete.

(ii) Every mapping $T \in A_{r,\theta(r)}$ has a fixed point for all $r \in [0, 1)$.

(iii) There exist $r \in (0, 1)$ and $\eta \in (0, \theta(r)]$ such that every mapping $T \in B_{r,\eta}$ has a fixed point.

We prove this theorem for quasi-metric spaces in view of Theorem H(γ1) as follows:

Proof. Note that

$$B_{r,\eta} \subset A_{r,\eta} \subset \{\text{RHR maps on } X\}.$$

If X is complete, by Theorem H(γ1), (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).

In order to show (iii) $\Rightarrow$ (i), assume (iii) and that X is not complete, that is, there exists a Cauchy sequence $\{u_n\}$ which does not converge. Suzuki defined a map $T : X \to X$ without having fixed points and satisfying (a)–(c). By (iii), T has a fixed point which yields a contradiction. Hence we obtain that X is complete. This completes the proof. ∎

For a complete metric space, Theorem 5.1 (ii) and (iii) hold. However (ii) and (iii) seem to be inconvenient to check whether X is complete or not.

Kikkawa and Suzuki [8] in 2008 proved three fixed point theorems for generalized contractions with constants in complete metric spaces, which are generalizations of fixed point theorems due to Suzuki in the same year.

Kikkawa-Suzuki [8] stated that the following is in Suzuki [28] in 2008:

Theorem 5.2. *[8] For a metric space (X, d), the following are equivalent:*

(i) X is complete.

(ii) Every mapping T on X satisfying the following has a fixed point:

- *There exists $r \in [0, 1)$ such that $\theta(r)d(x, Tx) \leq d(x, y)$ implies $d(Tx, Ty) \leq rd(x, y)$, for all $x, y \in X$.*

(iii) There exists $r \in (0, 1)$ such that every mapping T on X satisfying the following has a fixed point:

- $\frac{1}{10000}d(x, Tx) \leq d(x, y)$ *implies* $d(Tx, Ty) \leq rd(x, y)$ *for all* $x, y \in X$.

They stated: "The authors are very attracted by $\theta(r)$ because $\theta(r)$ does not seem to be natural. We know $\theta(r)$ is the best constant because of the existence of counterexamples. To find an intuitive reason is another motivation. However, we have not found such a reason yet. On the contrary, we have to raise one problem concerning $\theta(r)$."

This is not enough to justify the existence of $\theta(r)$ as the best constant. We have shown a more simple one has the same role in Section 3. Theorem 5.2 can be extended as follows:

$$\text{(i)} \iff \text{Theorem H}(\gamma 1) \text{ for metric spaces} \iff \text{(ii)} \iff \text{(iii)}.$$

Moreover, we have shown the following in our several papers:

Theorem 5.3. *A quasi-metric space X is complete if and only if any RHR selfmap of X has a fixed point.*

Consequently, the RHR maps characterize the metric completeness, but the Banach contraction does not as is well-known.

6. Other Authors' Comments Related to Suzuki et al.

A large number of followers of Suzuki's works on metric completeness published inaccurate comments on them and others. One of them was given by a well-known group of experts [6] as follows:

"A surprising result of Tomonari Suzuki goes in a different direction [11]. He considered a selfmapping T of a metric space (X, d) which satisfies the inequality $d(T(x), T(y)) \leq \alpha\, d(x, y)$ with some $\alpha \in [0, 1)$ for pairs (x, y) from some subset of $X \times X$. The completeness of (X, d) can be characterized by the fixed point property for such mappings. More precisely, the following result holds." Then Theorem 3.1 is stated.

We are definitely against to their opinions. There are also many papers praising Suzuki's works. In this section, wee list only thirty-four papers of the followers by indicating the published year with the initials of their authors.

[2011AE] We present a fixed-point theorem for a single-valued map in a complete metric space using implicit relation, which is a generalization of several previously stated results including that of Suzuki in 2008.

One of the most interesting generalizations is that given by Suzuki in 2008. Like other generalizations mentioned above in this paper, the Banach contraction principle does not characterize the metric completeness of X. However, Suzuki's theorem does characterize the metric completeness.

[2011DD] We obtain multi-valued mapping generalizations of two recent theorems of Kikkawa and Suzuki in 2008 and the main theorem of Enjouji, Nakanishi and Suzuki in 2009.

[2011DL] In this article we obtain a Suzuki-type generalization of a fixed point theorem for generalized multivalued mappings of Ćirić in 1972. The obtained results extend furthermore the recently developed Kikkawa-Suzuki-type contractions. Applications to certain functional equations arising in dynamic programming are also considered.

[2011P] The remarkable generalization of the classical Banach contraction theorem, due to Suzuki in 2008, has lead to some important contributions in metric fixed point theory (see, for instance, 8 papers are listed).

[2012A] Inspired by the work of Suzuki in 2008, we prove a fixed point theorem for contractive mappings that generalizes a theorem of Geraghty in 1973, and characterizes metric completeness.

[2012GR] Suzuki has shown that metric completeness can be characterized by a family of functions satisfying a generalized Banach's contraction principle.

[2012PV] Recently, Suzuki in 2008 proved a fixed point theorem that is a generalization of the Banach contraction principle and characterizes the metric completeness. In this paper we prove an analogous fixed point result for a self-mapping on a partial metric space or on a partially ordered metric space. We deduce, also, common fixed point results for two self-mappings. Moreover, using our results, we obtain a characterization of partial metric 0-completeness in terms of fixed point theory. This result extends Suzuki's characterization of metric completeness.

[2014HS] In 2008, in order to characterize the completeness of underlying metric spaces, Suzuki introduced a weaker notion of contraction. As an application of our results we deduce Suzuki type results for GF-contractions.

[2015CB] Abstract. In this paper we establish that a pair of compatible mappings have unique common fixed point in metric and partial metric spaces respectively. The mappings are Suzuki type. We give examples to illustrate our results.

Recently, Suzuki proved two fixed point theorems, one of which is a new type of generalization of the Banach contraction principle and does characterize the metric completeness. The Banach contraction does not have this property.

[2015MAK] The Banach contraction cannot describe the metric completeness. To overcome this issue, Suzuki in 2008 established a fixed point theorem which generalized the Banach contraction theorem and characterized the metric completeness. In 2008, Kikkawa and Suzuki extended the main theorem of Suzuki for the case of multivalued mappings.

[2016ND] Suzuki also introduced a new type of mapping. This mapping not only generalizes the Banach Contraction Mapping Principle but also characterizes the completeness of the underlying metric space. Kikkawa and Suzuki presented Kannan version of Suzuki theorem. In this paper we state and prove a fixed point theorem that uses Suzuki mapping and Kannan type of contraction.

[2018J] Suzuki proved a fixed point theorem which is a generalization of Banach's fixed point theorem and characterizes the metric completeness. Kikkawa and Suzuki proved a generalization of Kannan's fixed point theorem. Enjouji et al. proved a generalization of Theorem of Kikkawa and Suzuki.

[2020RKM] Abstract: In this paper, we use Suzuki-type contraction to prove three fixed point theorems for generalized contractions in an ordered space equipped with two metrics; we obtain some generalizations of the Kannan fixed point theorem. Our results on partially ordered metric spaces generalize and extend some results of Ran and Reurings as well as of Nieto and Rodriguez-Lopez. To illustrate the effectiveness of our main result, we give an application to matrix equations which involves monotone mappings.

[2021AKG] In 2009, Suzuki obtained a substantial generalization of the Banach contraction principle and introduced a new contractive condition, often referred to as Suzuki contractive condition, which enunciates that a self-mapping F on a metric space (Y, d) is called a Suzuki contractive, if for all $v, w \in Y$ with $v \neq w$,

$$\frac{1}{2}d(v, Fv) < d(v, w) \Longrightarrow d(Fv, Fw) < d(v, w).$$

Later on, Kumam et. al. unified the result of Suzuki and Khojasteh et al. by presenting a notion of a Suzuki type $\mathfrak{Z}$-contraction and establishing fixed point results for such mappings. The concept of Suzuki type $\mathfrak{Z}$-contraction was further improved by Hasanuzzaman et. al. by introducing the notion of Suzuki type $\mathfrak{Z}_{\mathfrak{R}}$-contraction in a relational metric space.

[2021BB] In 2008, Suzuki proved two fixed point theorems, one of which is a new type of generalization of the Banach contraction principle.

In Section 2, we introduce Suzuki $\mathfrak{Z}$-contraction type (I) maps and Suzuki $\mathfrak{Z}$-contraction type (II) maps for a pair of selfmaps in b-metric spaces.

[2022CJJ] In this paper, we establish some fixed point theorems for single valued and multi-valued mappings on a complete metric space. Suzuki's and some other fixed point theorems are generalized by taking a more general contractive condition for single valued mappings. It is also proved that our result characterizes the completeness of the metric space. Further, taking generalized contractive condition, a fixed point theorem is also established for multi-valued mappings.

In the last decade (2008), Suzuki gave a simple but important generalization of the BCP which also preserves the metric completeness of the space.

[2022KA] Suzuki presented a fixed point theorem that generalized BCP and characterized metric completeness as well. Recently, Ali et al. obtained completeness characterizations of b-metric spaces via the fixed point of Suzuki type contractions.

[2022KAG] Another important generalization of the Banach contraction principle was obtained by Suzuki in 2008. Suzuki generalized the principle for the class of weak contractive mappings in complete metric spaces and established a new version of the Banach contraction principle. The Banach contraction principle does not characterize the metric completeness, however the result of Suzuki characterizes the metric completeness.

[2022R] In this note we show the somewhat surprising fact that the proof of the 'if part' of the distinguished characterizations of metric completeness due to Kirk, and Suzuki and Takahashi, respectively, can be deduced in a straightforward manner from Hu's theorem that a metric space is complete if and only if any Banach contraction on bounded and closed subsets thereof has a fixed point. We also take advantage of this approach to easily deduce a characterization of metric completeness via fixed point theorems for α-ψ-contractive mappings.

[2022R′] Suzuki published in 2008 his renowned article in which he presented a necessary and sufficient condition for the metric completeness by utilizing an appealing generalization of the Banach contraction principle. This new and compelling approach was successfully continued by him, and by other authors who generalized and extended the type of contractions proposed by Suzuki to obtain new fixed point theorems both in metric spaces and in b-metric spaces, partial metric spaces, G-metric spaces, quasi-metric spaces, fuzzy metric spaces, and others.

[2023A] Abstract. We introduce Suzuki type P-contractive mappings by taking into account the concepts of contractive, P-contractive, and Suzuki type contractive mappings. Then, for such mappings on compact metric spaces, we present a fixed point theorem that is more general than the well-known Edelstein fixed point theorem.

In this paper, we introduce Suzuki type P-contractive mappings, which are inspired by the concepts of contractive, P-contractive, and Suzuki type contractive mappings.

[2023LKPS] Suzuki's generalization of the BCP introduced a new class of contractive maps that satisfy contraction conditions only for specific elements of the underlying space. Numerous mathematicians have contributed to the generalization of Nadler's theorem, with Kikkawa and Suzuki achieving significant progress in the study of generalized multi-valued maps.

[2023O] In 2008, Suzuki introduced a useful generalization of Banach fixed point theorem called as Suzuki fixed point theorem. In this paper, we prove a common fixed point theorem for Suzuki type contractions on complete partial metric spaces. We also state some corollaries related to Suzuki type common fixed point theorem. We also give an example where we apply our main theorem on complete partial metric spaces. Finally, to show usability of our results, we give its an application showing existence and uniqueness of a common solution for a class of functional equations in dynamic programming.

[2023PK] In 2008, Suzuki introduced a new class of contraction mappings where the contraction condition to be hold only on certain elements of the underlying space. He presented a remarkable generalization of the BCP which also characterizes completeness of the metric space.

[2023PR] In the literature of the crisp theory of fixed points, there are a lot of generalization of the Banach contraction result, but a remarkable and most interesting generalization was given by Suzuki in 2007. Suzuki adopted this concept and discovered a contraction as a tool to obtain a fixed point for a self-map. In this article, we have introduced various independent

Suzuki-type fuzzy contractive conditions, which are generalizations of existing results in the literature. Further hope with this technique is that researchers can extend more results in terms of Suzuki-type views with applications in crisp distance spaces.

[2023RT] Abstract: In an outstanding article published in 2008, Suzuki obtained a nice generalization of the Banach contraction principle from which derived a characterization of metric completeness. Although Suzuki's theorem has been successfully generalized and extended in several directions and contexts, we here show by means of a simple example that the problem of achieving, in an obvious way, its full extension to the framework of w-distances does not have an emphatic response. Motivated by this fact we introduce the concept of presymmetric w-distance on metric spaces, we give some properties and examples of this new structure and show that it provides a reasonable setting to obtain a real and hardly forced w-distance generalization of Suzuki's theorem.

[2023ZSPTZR] Abstract: This paper aims to give an extended class of contractive mappings combining types of Suzuki contractions α-admissible mapping and Wardowski F- contractions in b-metric-like spaces. Our results cover and generalize many of the recent advanced results on the existence and uniqueness of fixed points and fulfill the Suzuki-type nonlinear hybrid contractions on various generalized metrics.

[2024AANA] Abstract: In order to generalize classical Banach contraction principle in the setup of quasi-metric spaces, we introduce Suzuki-type contractions of quasi-metric spaces and prove some fixed point results.

Suzuki and Kikkawa-Suzuki generalized BCP by introducing Suzuki-type contractions of metric spaces. Note that Suzuki-type contractions characterize the completeness of underlying metric spaces. Moreover, Suzuki-type contractions are in the form of implication which allow many discontinuous mappings of metric spaces to be Suzuki-type contractions, while Banach contractions are necessarily continuous mappings of metric spaces. Suzuki-type contractions have a wide scope of applications.

[2024DS] Suzuki introduced a condition on mappings, which is weaker than nonexpansiveness. Let K be a nonempty subset of a metric space. A mapping $T : K \to K$ is called a Suzuki-generalized nonexpansive mapping if $(1/2)d(x, Tx) \leq d(x, y)$ implies $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in K$. It is interesting to note that nonexpansive mappings are continuous on their domains, but Suzuki-generalized nonexpansive mappings need not be continuous. Moreover, Suzuki generalized the fixed point result of Browder and Gohde.

[2024J] A surprising result of Tomonari Suzuki goes in a different direction. He considered a selfmapping T of a metric space (X, d) which satisfies the inequality $d(T(x), T(y)) \leq \alpha\, d(x, y)$ with some $\alpha \in [0, 1)$ for pairs (x, y) from some subset of $X \times X$. The completeness of (X, d) can be characterized by the fixed point property for such mappings.

[2024MSA] Abstract. We investigate stationary points of multivalued Suzuki maps within the framework of 2-uniformly convex hyperbolic spaces. Initially, we present key strong and Δ-convergence results, followed by an example that demonstrates the theoretical findings. Additionally, our results hold in uniformly convex Banach spaces, CAT(0) spaces, and certain CAT(κ) spaces. Furthermore, our findings encompass cases where the map is assumed to be nonexpansive.

[2024R] From Abstract: Our motivation is due, in part, to the fact that a successful improvement of the classical Banach fixed-point theorem obtained by Suzuki does not admit a natural and full quasi-metric extension, as we have noted in a recent article. Thus, and with the help of this new structure, we obtained a fixed-point theorem in the framework of Smyth-complete quasi-metric spaces that generalizes Suzuki's theorem. Combining right completeness with partial ordering properties, we also obtained a variant of Suzuki's theorem, which was applied to discuss types of difference equations and recurrence equations.

[2024RR] Generalized versions of the core discoveries of Banach and Edelstein were recently shown by Suzuki which sparked a great deal of interest in this area. This paper is to provide unique coupled common fixed point theorems in the set-up of A_b-Metric Space for (φ, ϱ)-type contractions and (φ, ϱ)-type Suzuki contractive mapping. Additionally, we can provide appropriate examples and an application to integral equations.

[2024RT] From Abstract: In an outstanding article published in 2008, Suzuki obtained a nice generalization of the Banach contraction principle from which derived a characterization of metric completeness. Although Suzuki's theorem has been successfully generalized and extended in several directions and contexts, we here show by means of a simple example that the problem of achieving, in an obvious way, its full extension to the framework of w-distances does not have an emphatic response. Motivated by this fact we introduce the concept of presymmetric w-distance on metric spaces, we give some properties and examples of this new structure and show that it provides a reasonable setting to obtain a real and hardly forced w-distance generalization of Suzuki's theorem. This is realized in our main result, which is a fixed point theorem that involves presymmetric w-distances and certain contractions of Suzuki-type.

From Introduction: In his already classical article, Suzuki presented an elegant generalization of Banach's contraction principle that he used to characterize complete metric spaces. The last part of the paper is devoted to obtain necessary and sufficient conditions for a metric space to be complete which is made by combining our fixed point results with both Suzuki's characterization and Suzuki-Takahashi's characterization.

[2024YMS] In 2008, Suzuki published one of the most comprehensive generalizations of Banach's and Edelstein's basic results. When all of the domain's points do not meet the necessary contractive condition, this is known as Suzuki contraction. The existence and uniqueness of fixed points of maps satisfying a Suzuki type contraction has been extensively studied.

7. Epilogue

In our previous works [13, 18, 22, 23] and the present one, we showed many metric fixed point theorems hold for quasi-metric spaces from the beginning.

Even for the Banach contraction principle, certain traditional monographs or text-books on fixed point theory or general topology stated for metric spaces only. All of them stated the principle for metric spaces only, but their proofs do not use the symmetry of a metric. Of course, they do not mention the concept of quasi-metric spaces.

Beginning from our [16] in 2022, we have published nearly two dozen articles on our 2023 Metatheorem and related topics. These could add up many new results to Ordered Fixed

Point Theory in [16]. Moreover, our Metatheorem has numerous applications.

In the present article, we corrected only some of inaccurate statements in Metric Fixed Point Theory. However there are many results related to the Rus-Hicks-Rhoades theorem due to scores of other authors. Since the theorem and its extensions properly include the corresponding ones of the Banach contraction, the future study on Metric Fixed Point Theory would be concentrated to extend the theorem and its new applications.

There are thousands of artificial generalizations of metric spaces. Most of them assume the symmetry which might be eliminated as in the present article.

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