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Projection Method for Solving Large-scale System of Nonlinear Equations

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ABSTRACT

Derivative-free projection methods have proven to be highly effective and valuable in solving large-scale systems of nonlinear equations (SNE). Extensive research is continuously being conducted to enhance existing methods and develop new projection methods. In this paper, we modified the conjugate gradient parameter proposed by Zhu et al. and extend it to solve SNE with convex constrain. The advantage of the proposed method is that it does not rely on Jacobian information and does not require the storage of any matrices at each iteration. This characteristic makes it well-suited for tackling large-scale non-smooth problems. Under appropriate conditions, we show that the proposed method is globally convergent. Numerical experiments were conducted to evaluate the effectiveness of the proposed method and compare it with other approaches.

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1. Introduction

We study systems of nonlinear equations (SNE) of the form

$$\mathcal{F}(x) = 0, \;\; x \in \mathcal{C}, \tag{1}$$

where $\mathcal{C} \subseteq \mathbb{R}^n$ is a nonempty closed convex set and $\mathcal{F} : \mathbb{R}^n \to \mathbb{R}^n$ is continuous and monotone. In various modern fields, several problems can be formulated as (1). Examples of such problems include variational inequality problems [32], sub-problems in generalized proximal algorithms

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with Bregman distance [13], chemical equilibrium problems [22], power flow equations [29] and other engineering problems.

In recent decades, numerous numerical techniques have emerged to address the solution of problem (1). These include Newton-based methods [13, 27], derivative-free methods [19, 20, 14, 30, 26, 28, 3, 23, 10, 12, 8, 15, 9, 11, 2, 1] and gradient-based methods [17]. The Newton-based methods and their variations have gained significant popularity for tackling problem (1) due to their rapid local convergence [33, 5]. However, one drawback of Newton-based methods is the requirement to calculate a Jacobian matrix or an approximate Jacobian in each iteration. This process can significantly decrease their efficiency, particularly when dealing with large-scale problems.

First-order methods have garnered significant attention due to their low storage requirement. A well-known example of such a method is the spectral gradient method, which was initially introduced by Barzilai and Borwein [4] and later expanded upon by Raydan and La Cruz [24, 25]. Zhang and Zhou studied the spectral gradient projection method for unconstrained monotone equations, building upon the inexact Newton method introduced by Solodov and Svaiter [27]. Furthermore, Yu et al. applied this method to solve constrained monotone equations and reported excellent performance based on the numerical results [31].

Recently, Zhu et al. [34] proposed two modified Dai-Yuan conjugate gradient methods called the DDY1 method and DDY2 method. The authors establish the global convergence of these methods by employing the standard Wolfe line search. In the DDY1 method, the search direction is descent and is determined using the standard Wolfe line search. On the other hand, the DDY2 method generates a search direction that is guaranteed to be sufficiently descent, and this property holds independently of any line search. Preliminary numerical experiments demonstrate the effectiveness of both methods.

In this paper, the success achieved by the DDY1 method has inspired us to extend its application to constrained systems of nonlinear equation problems. The nature of constrained nonlinear equations differs significantly from unconstrained optimization, making our algorithm unique. It does not rely on a merit function and stands apart from the algorithm presented in [34]. As a result, our approach proves valuable in solving non-smooth problems as well. Furthermore, our algorithm can be viewed as a derivative-free projection method designed specifically for solving SNE with convex constraints. We have successfully demonstrated that the algorithm inherently generates a bounded iterative sequence and achieves global convergence. In other words, any limit point of the iterative sequence is guaranteed to be a solution to the original problem. The efficiency of the proposed algorithm is further supported by rigorous numerical testing.

The rest of the paper is organized as follows. In the next section, we give some preliminaries, the proposed algorithm, and the global convergence of the method. Section 3 is devoted to numerical experiments.

2. Algorithm and Convergence Result

Throughout this manuscript, the Euclidean norm is denoted as $\|\cdot\|$ and defined as the measure of the magnitude of a vector. Now, consider $\mathcal{C}$ a nonempty closed convex subset of $\mathbb{R}^n$. For any $x \in \mathbb{R}^n$, the projection of α onto $\mathcal{C}$ can be defined as follows

$$P_{\mathcal{C}}(\alpha) := \arg\min_{\beta\in\mathcal{C}} \|\alpha-\beta\|, \; \alpha \in \mathbb{R}^n.$$

This projection operator satisfies the properties

$$\|P_{\mathcal{C}}(\alpha) - P_{\mathcal{C}}(\gamma)\| \le \|\alpha-\gamma\|, \; \forall \alpha,\gamma \in \mathbb{R}^n. \tag{2}$$

In this paper, the iterative point at the next iteration is obtained through the projection procedure, and the search direction, denoted as d_k, is defined as follows

$$d_k = \begin{cases} -\mathcal{F}(x_k) & \text{if } k=0, \\ -\mathcal{F}(x_k) + \beta_k d_{k-1} & \text{if } k>0, \end{cases} \tag{3}$$

where β_k is defined as follows

$$\beta_k = \frac{\|\mathcal{F}(x_k)\|^2 - \max\left\{0, \; \frac{(\mathcal{F}(x_k)^T d_{k-1})^2}{\|\mathcal{F}(x_k)\|\|\mathcal{F}(x_{k-1})\|\|d_{k-1}\|^2} |\mathcal{F}(x_k)^T \mathcal{F}(x_{k-1})|\right\}}{\max\{\|y_{k-1}\|\|d_{k-1}\|, \; \sigma\|\mathcal{F}(x_k)\|\|d_{k-1}\|\}} \tag{4}$$

Now, we state the steps of the new method as follows.

Algorithm 1: Projected Conjugate Gradient Algorithm for SNE, (PCGA-SNE)

Input. Set an initial point $x_0 \in \mathcal{C}$, the positive constants:
$Tol \in (0,1), \; r \in (0,1), \; m \in (0,2),$
$\zeta > 0, \; \mu > 0$. Set $k = 0$.

Step 0. If $\|\mathcal{F}(x_k)\| \le tol$, stop. Otherwise, generate the search direction d_k using (3).

Step 1. Determine the step size $\gamma_k =: \max\{\mu r^w | w = 0,1,2,\cdots\}$ with w being the smallest positive integer such that

$$-\mathcal{F}(x_k+\gamma_k d_k)^T d_k \ge \zeta\gamma_k \|\mathcal{F}(x_k+\gamma_k d_k)\|\|d_k\|^2 \tag{5}$$

Step 2. Compute

$$z_k = x_k + \gamma_k d_k, \tag{6}$$

where z_k is a trial point.

Step 3. If $z_k \in \mathcal{C}$ and $\|\mathcal{F}(z_k)\| \le tol$, stop. Otherwise, compute the next iterate by

$$x_{k+1} = P_{\mathcal{C}}\left[x_k - m\frac{\mathcal{F}(z_k)^T(x_k-z_k)}{\|\mathcal{F}(z_k)\|^2}\right], \tag{7}$$

Step 4. Finally, we set $k = k+1$ and return to **Step 1.**

In order to prove the global convergence of the new method, we require the underlying mapping to satisfy the following assumption

Assumption 1.

(A1) The solution set denoted by $\mathcal{C}^*$ of $\mathcal{F}(x) = 0$ is nonempty.

(A2) The function $\mathcal{F}$ is monotone, i.e.,

$$\langle \mathcal{F}(x) - \mathcal{F}(y), x - y \rangle \geq 0, \forall x, y \in R^n. \tag{8}$$

(A3) The function $\mathcal{F}$ is Lipschitz continuous, i.e., there exists a positive constant L such that:

$$\|\mathcal{F}(x) - \mathcal{F}(y)\| \leq L\|x - y\|, \forall x, y \in R^n. \tag{9}$$

Lemma 2.1. *Let the sequences $\{\mathcal{F}(x_k)\}$ and $\{d_k\}$ be generated by Algorithm 1, then for all $k \geq 0$, the search directions $\{d_k\}$ satisfies the sufficient descent condition*

$$\mathcal{F}(x_k)^T d_k \leq -s_1\|\mathcal{F}(x_k)\|^2, \tag{10}$$

in which, $s_1 := \left(1 - \frac{1}{\sigma}\right), \sigma > 0$.

Proof. The proof does follow directly for $k = 0$, since from the formulation in (3), we have $\mathcal{F}(x_0)^T d_0 = -\mathcal{F}(x_0)^T \mathcal{F}(x_0) \leq -s_1\|\mathcal{F}(x_0)\|^2$. In the next segment, we wish to show for all $k \geq 1$, but from (4), we have

$$\beta_k \leq \frac{\|\mathcal{F}(x_k)\|^2}{\max\{\|y_{k-1}\|\|d_{k-1}\|,\ \sigma\|\mathcal{F}(x_k)\|\|d_{k-1}\|\}} \leq \frac{\|\mathcal{F}(x_k)\|}{\sigma\|d_{k-1}\|}.$$

Thus, from (3), we can deduce that

$$\begin{aligned}
\mathcal{F}(x_k)^T d_k &\leq -\|\mathcal{F}(x_k)\|^2 + \frac{\|\mathcal{F}(x_k)\|}{\sigma\|d_{k-1}\|}\mathcal{F}(x_k)^T d_{k-1} \\
&\leq -\|\mathcal{F}(x_k)\|^2 + \frac{\|\mathcal{F}(x_k)\|}{\sigma\|d_{k-1}\|}\|\mathcal{F}(x_k)\|\|d_{k-1}\| \\
&= -\left(1 - \frac{1}{\sigma}\right)\|\mathcal{F}(x_k)\|^2.
\end{aligned}$$

Hence, the inequality (10) holds for all $k \geq 0$, with $s_1 := \left(1 - \frac{1}{\sigma}\right), \sigma > 0$. ∎

Lemma 2.2. *Suppose Assumption 1 holds and let p be in the solution set $\mathcal{C}^*$. Then the sequence $\{\|x_k - p\|\}$ is decreasing and thus, convergent. Consequently, the sequences $\{x_k\}$ and $\{z_k\}$ are bounded. Moreover, we have*

$$\lim_{k\to\infty} \gamma_k\|d_k\| = 0. \tag{11}$$

Proof. Using the property of the projection operator and based on the (A2), we can deduce that

$$\begin{aligned}
\|x_{k+1}-p\|^2 &= \left\| P_{\mathcal{C}}\left[x_k - m\frac{\mathcal{F}(z_k)^T(x_k-z_k)}{\|\mathcal{F}(z_k)\|^2}\mathcal{F}(z_k)\right] - p\right\|^2 \\
&\leq \left\| x_k - m\frac{\mathcal{F}(z_k)^T(x_k-z_k)}{\|\mathcal{F}(z_k)\|^2}\mathcal{F}(z_k) - p\right\|^2 \\
&= \|x_k-p\|^2 - 2m\frac{\mathcal{F}(z_k)^T(x_k-z_k)}{\|\mathcal{F}(z_k)\|^2}(\mathcal{F}(z_k))^T(x_k-p) \\
&\quad + m^2\left(\frac{\mathcal{F}(z_k)^T(x_k-z_k)}{\|\mathcal{F}(z_k)\|^2}\right)^2\|\mathcal{F}(z_k)\|^2 \\
&\leq \|x_k-p\|^2 - 2m\frac{\mathcal{F}(z_k)^T(x_k-z_k)}{\|\mathcal{F}(z_k)\|^2}\mathcal{F}(z_k)^T(x_k-z_k) \\
&\quad + 2m\frac{\mathcal{F}(z_k)^T(x_k-z_k)}{\|\mathcal{F}(z_k)\|^2}\mathcal{F}(z_k)^T(z_k-p) \\
&\quad + m^2\left(\frac{\mathcal{F}(z_k)^T(x_k-z_k)}{\|\mathcal{F}(z_k)\|^2}\right)^2\|\mathcal{F}(z_k)\|^2 \\
&\leq \|x_k-p\|^2 - m(2-m)\frac{(\mathcal{F}(z_k)^T(x_k-z_k))^2}{\|\mathcal{F}(z_k)\|^2} \\
&\leq \|x_k-p\|^2 - m(2-m)\zeta^2\|x_k-z_k\|^4 \qquad (12) \\
&\leq \|x_k-p\|^2.
\end{aligned}$$

Obviously, the sequence $\{\|x_k-p\|\}$ is decreasing and convergent, which implies that the sequence $\{x_k\}$ is bounded. This means that there exists a constant, $\bar{m}_1 > 0$ such that $\|x_k\| \leq \bar{m}_1, \ \forall k \geq 0$. Thus, using the established boundedness of $\{x_k\}$ coupled with the continuity of $\mathcal{F}$, indicate that the existence of a constant, $\bar{m}_2 > 0$ such that $\|\mathcal{F}(x_k)\| \leq \bar{m}_2$, for all $k \geq 0$.

Next, we prove the remaining conclusion. It follows from (12) that

$$m(2-m)\zeta^2\sum_{k=0}^{\infty}\|x_k-z_k\|^4 \leq \sum_{k=0}^{\infty}(\|x_k-p\|^2 - \|x_{k+1}-p\|^2) < \infty.$$

Using the property of convergent series, it implies that

$$\lim_{k\to\infty}\gamma_k\|d_k\| = 0.$$

∎

Lemma 2.3. *Let Assumptions in 1 hold, and suppose that the sequences $\{d_k\}$, $\{z_k\}$ and $\{x_k\}$ as formulated in (3),(6) and (7) respectively, then we have for all $k \geq 0$, there exists a step-size $\gamma_k =: \mu r^w$ that fulfills condition (5) for some $w \in \mathbb{N}\cup\{0\}$. Furthermore, the step-size γ_k obtained via (5) satisfy the following inequality*

$$\gamma_k > \min\left\{\mu,\ \frac{rs_2\|\mathcal{F}(x_k)\|^2}{(L+\zeta\bar{m}_2)\|d_k\|^2}\right\}, \qquad (13)$$

where $s_2 := (1-1/\sigma)$. Thus, Algorithm 1 is well-defined.

Proof. If the algorithm ends in iteration k, then $\|\mathcal{F}(x_k)\| = 0$, indicating that x_k (or z_k) is a solution. Let's assume that $\mathcal{F}(x_k) \neq 0$ for all k, and as a result, $d_k \neq 0$ according to Lemma 2.1. Now, we will demonstrate that the line search procedure (5) always terminates within a finite number of steps. If $\gamma_k \neq \mu$, we know that $\gamma_k' = r^{-1}\gamma_k$ does not satisfy (5), i.e.,

$$-\mathcal{F}(\bar{z}_k)^T d_k < \zeta\gamma_k'\|\mathcal{F}(\bar{z}_k)\| \cdot \|d_k\|^2, \tag{14}$$

in which $\bar{z}_k = x_k + \gamma_k' d_k$. More so, we conclude from Lemma 2.2, that the sequence $\{\bar{z}_k\}$ is bounded and consequently using the continuity assumption on the operator, $\mathcal{F}$, we get the boundedness of the sequence $\{\mathcal{F}(\bar{z}_k)\}$. Thus, this indicate that there exists a constant, $\bar{m}_1$ for which $\|\mathcal{F}(\bar{z}_k)\| \leq \bar{m}_2$, for all $k \geq 0$. From (10), applying (A3), and taking note of (14) yields

$$\begin{aligned}
s_2\|\mathcal{F}(x_k)\|^2 &\leq -\mathcal{F}(x_k)^T d_k \\
&= \left(\mathcal{F}(\bar{z}_k)^T - \mathcal{F}(x_k)\right)^T d_k - \mathcal{F}(\bar{z}_k)^T d_k \\
&< \|\mathcal{F}(\bar{z}_k) - \mathcal{F}(x_k)\|\|d_k\| + \zeta\gamma_k'\|\mathcal{F}(\bar{z}_k)\|\|d_k\|^2 \\
&\leq L\|\bar{z}_k - x_k\|\|d_k\| + \zeta\gamma_k'\bar{m}_1\|d_k\|^2 \\
&= (L + \zeta\bar{m}_1)\, r^{-1}\gamma_k\|d_k\|^2
\end{aligned}$$

By the above relation,

$$\gamma_k \geq \frac{s_2 r\|\mathcal{F}(x_k)\|^2}{(L + \zeta\bar{m}_1)\|d_k\|^2}, \tag{15}$$

which concludes the proof. ∎

Lemma 2.4. *The search direction $\{d_k\}$ generated by Algorithm 1 is bounded. Moreover,*

$$s_1\|\mathcal{F}(x_k)\| \leq \|d_k\| \leq s_2, \tag{16}$$

where s_1, s_2 are positive constants.

Proof. We obtain the following by applying the Cauchy-Schwartz inequality to the sufficient descent property (10).

$$\|d_k\| \geq \left(1 - \frac{1}{\sigma}\right)\|\mathcal{F}(x_k)\|.$$

On the other hand, recall that we have established that $\{x_k\}$ is bounded. Therefore, by (A3) it is easy to see that $\{\mathcal{F}(x_k)\}$ is bounded. That is, $\|\mathcal{F}(x_k)\| \leq \bar{m}_2$. Thus, we have

$$\begin{aligned}
\|d_k\| &\leq \|\mathcal{F}(x_k)\| + |\beta_k|\|d_{k-1}\| \\
&\leq \|\mathcal{F}(x_k)\| + \frac{\|\mathcal{F}(x_k)\|}{\sigma\|d_{k-1}\|}\|d_{k-1}\| \\
&= \left(1 + \frac{1}{\sigma}\right)\|\mathcal{F}(x_k)\| \\
&\leq \left(1 + \frac{1}{\sigma}\right)\bar{m}_2.
\end{aligned}$$

Combining the above two inequalities with $s_1 = 1 - 1/\sigma$, $s_2 = (1 + 1/\sigma)\bar{m}_2$ implies that (10) holds for all $k \geq 0$. ∎

Theorem 2.5. *Suppose Assumption (1) hold and $\{x_k\}$ generated by Algorithm 1, then:*

$$\liminf_{k\to\infty} \|\mathcal{F}(x_k)\| = 0. \tag{17}$$

Furthermore, the sequence $\{x_k\}$ converges to a solution of (1).

Proof. Suppose by contradiction, $\liminf_{k\to\infty} \|\mathcal{F}(x_k)\| \neq 0$, then there exist a positive constant ϵ such that $\forall k \geq 0$:

$$\|\mathcal{F}(x_k)\| \geq \epsilon. \tag{18}$$

This together with (10) implies that

$$\|d_k\| \geq \left(1 - \frac{1}{\sigma}\right)\epsilon \triangleq \nu, \ \forall k \geq 0. \tag{19}$$

By virtue of (15), (18), (19) we have

$$\gamma_k\|d_k\| \geq \frac{r\|\mathcal{F}(x_k)\|^2}{(L+\zeta\bar{m}_1)s_2^2}\|d_k\| \geq \frac{r\epsilon^2}{(L+\zeta\bar{m}_1)s_2^2}\nu,$$

which contradicts (11) and therefore (17) holds. ∎

3. Numerical Experiments

In this section, we evaluate the computational performance of the proposed method PCGA-SNE by comparing it with the ACGD [6] and PDY [18]. The parameters employed for implementing PCGA-SNE, ACGD and PDY are set as reported in their respective papers. For implementing the proposed method PCGA-SNE, the specific parameters utilized are specified as follows: $r = 0.5, \ m = 1.8, \ \zeta = 0.0001, \ \sigma = 0.8$.

We used four different dimensions DIM $= 500, \ 10000, \ 50000, \ 100000$ with 10 different initial points as follows: $x_0^1 = (1, \cdots, 1)^T$, $x_0^2 = (0.1, \cdots, 0.1)^T$, $x_0^3 = \left(\frac{1}{2}, \cdots, \frac{1}{2^n}\right)^T$, $x_0^4 = \left(1 - \frac{1}{n}, 1 - \frac{2}{n}, \cdots, 0\right)^T$, $x_0^5 = \left(0, \frac{1}{n}, \frac{2}{n}, \cdots, \frac{n-1}{n}\right)^T$, $x_0^6 = \left(1, \frac{1}{2}, \cdots, \frac{1}{n}\right)^T$, $x_0^7 = \left(\frac{1}{3}, \cdots, \frac{1}{3^n}\right)^T$, $x_0^8 = \left(\frac{1}{n}, \frac{2}{n}, \cdots, 1\right)^T$, $x_0^9 = \left(\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n}\right)^T$ and $x_0^{10} = \text{rand}(n, 1)$. Here, rand$(n, 1)$ means the initial point is chosen randomly from the interval $(0, 1)$. All methods are implemented in MATLAB R2019b. The implementation process for each method is halted under two conditions: either when the norm of the objective function, $\|\mathcal{F}(x_k)\|$, becomes less than or equal to 10^{-6} or when the number of iterations exceeds 2000.

We tested all the algorithms on the following seven problems, where $\mathcal{F} = (\mathcal{F}_1(x), \mathcal{F}_2(x), \cdots, \mathcal{F}_n(x))^T$.

Problem 1: Modified exponential function [16]

$$\mathcal{F}_1(x) = e^{x_1} - 1$$
$$\mathcal{F}_i(x) = e^{x_i} + x_i - 1, \quad i = 1, 2, \cdots, n-1,$$
$$\mathcal{C} = \mathbb{R}^n_+.$$

Problem 2: Logarithmic function [16]

$$\mathcal{F}_i(x_i) = \log(x_i + 1) - \frac{x_i}{n}, \quad i = 1, 2, \cdots, n,$$
$$\mathcal{C} = \mathbb{R}^n_+.$$

Problem 3: Strictly convex function I [16]

$$\mathcal{F}_i(x) = e^{x_i} - 1, \quad i = 1, 2, \cdots, n,$$
$$\mathcal{C} = \mathbb{R}^n_+.$$

Problem 4: Strictly convex function II [16]

$$\mathcal{F}_i(x) = \left(\frac{i}{n}\right) e^{x_i} - 1, \quad i = 1, 2, \cdots, n,$$
$$\mathcal{C} = \mathbb{R}^n_+.$$

Problem 5: Tridiagonal exponential function [16]

$$\begin{aligned}
\mathcal{F}_1(x) &= x_1 - e^{\cos(h(x_1+x_2))}\\
\mathcal{F}_i(x) &= x_i - e^{\cos(h(x_{i-1}+x_i+x_{i+1}))}, \quad i = 2, \cdots, n-1,\\
\mathcal{F}_n(x) &= x_n - e^{\cos(h(x_{n-1}+x_n))},\\
h &= \frac{1}{n+1} \text{ and } \mathcal{C} = \mathbb{R}^n_+.
\end{aligned}$$

Problem 6: Nonsmooth function II [31]

$$\mathcal{F}_i(x) = x_i - \sin(|x_i - 1|), \text{ for } i = 1, 2, \cdots, n,$$
$$\mathcal{C} = \left\{ x \in \mathbb{R}^n : x \geq -1, \ \sum_{i=1}^{n} x_i \leq n \right\}.$$

Problem 7: Trig-Exp Function [21]

$$\begin{aligned}
\mathcal{F}_1(x) &= 3x_1^3 + 2x_2 - 5 + \sin(x_1 - x_2)\sin(x_1 + x_2),\\
\mathcal{F}_i(x) &= 3x_i^3 + 2x_{i+1} - 5 + \sin(x_i - x_{i+1})\sin(x_i + x_{i+1})\\
&\quad + 4x_i - x_{i-1}e^{(x_{i-1}-x_i)} - 3, \text{ for } 1 < i < n,\\
\mathcal{F}_n(x) &= 4x_n - x_{n-1}e^{(x_{n-1}-x_n)} - 3,\\
C &= \mathbb{R}^n_+.
\end{aligned}$$

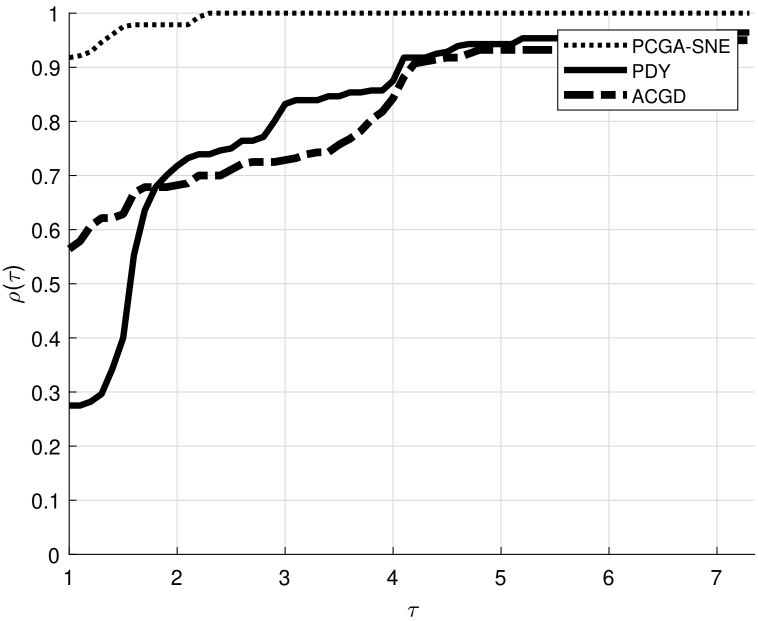

Fig. 1. Dolan and Moré performance profile with respect to number of iterations

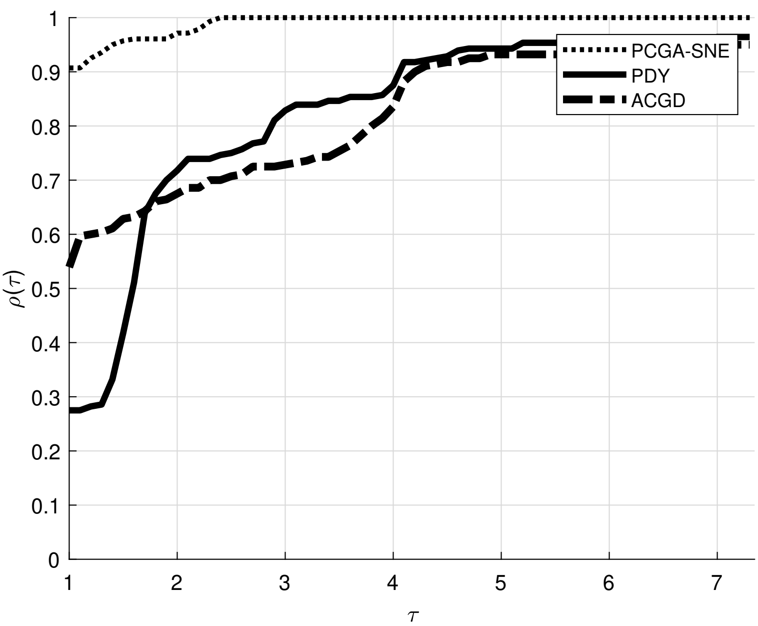

Fig. 2. Dolan and Moré performance profile with respect to number of function evaluations

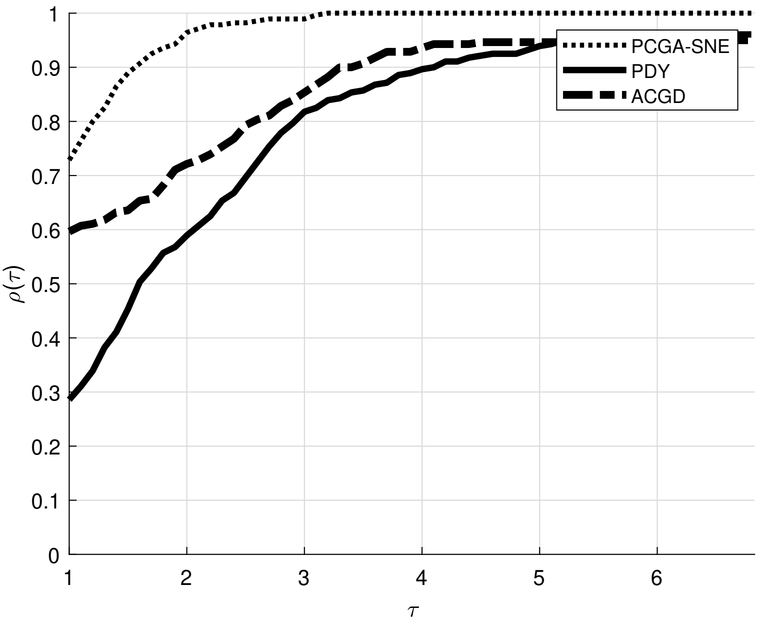

Fig. 3. Dolan and Moré performance profile with respect to time

Readers should refer to Tables 1 to 7 for comprehensive information on the numerical experiments conducted. The tables include specific details such as the number of iterations needed for convergence to an approximate solution (ITER), the count of function evaluations (FVAL), the CPU time in seconds (TIME), and the norm of the function at the approximate solution (NORM). The *empty-blank* space for example in 7 indicates failure cases obtained from a solver for those problem instances. These tables provide a thorough breakdown of the experimental results.

The performance of the tested methods in terms of iteration, function evaluation, and CPU time can be observed in Figures 1 to 3. These figures depict the efficiency comparisons using the performance profile introduced by Dolan and Moré [7]. Analyzing Figures 1 to 3, it is evident that PCGA-SNE emerges as the superior solver based on the conducted numerical experiments. It demonstrates remarkable efficiency and robustness. In terms of the number of iterations metric (Fig. 1), PCGA-SNE outperforms other methods by solving over 90% of the problems with only a few iterations. Fig. 2 further demonstrates the efficiency of PCGA-SNE, as it successfully solves more than 85% of the problems with a minimal number of function evaluations. Fig. 3 also reveals that PCGA-SNE surpasses the compared methods in terms of efficiency, solving over 70% of the problems with significantly less CPU time.

Table 1. Detailed numerical result for test problem 1

Test Problem 1													
		PCGS-SNE				**PDY**				**ACGD**			
DIMENSION	**INITIAL POINT**	**ITER**	**FVAL**	**TIME**	**NORM**	**ITER**	**FVAL**	**TIME**	**NORM**	**ITER**	**FVAL**	**TIME**	**NORM**
1000	x_1	1	3	0.0251	0.00E+00	24	71	0.0407	7.80E-06	11	32	0.0368	6.19E-06
	x_2	12	35	0.0828	8.87E-06	21	62	0.0275	8.99E-06	4	12	0.0419	5.81E-07
	x_3	17	51	0.0841	8.46E-06	27	80	0.0320	7.48E-06	16	48	0.0244	7.86E-06
	x_4	6	18	0.0265	2.89E-06	26	77	0.0319	8.98E-06	12	35	0.0163	6.46E-06
	x_5	6	18	0.0285	2.82E-06	22	65	0.0422	5.66E-06	12	35	0.0203	6.44E-06
	x_6	19	56	0.0425	8.12E-06	37	110	0.0331	8.03E-06	21	63	0.0185	9.79E-06
	x_7	6	18	0.0145	2.89E-06	26	77	0.0185	8.98E-06	12	35	0.0209	6.46E-06
	x_8	6	18	0.0155	2.84E-06	22	65	0.0183	5.67E-06	12	35	0.0228	6.45E-06
5000	x_1	1	3	0.0217	0.00E+00	24	71	0.1231	7.57E-06	11	33	0.0465	9.21E-06
	x_2	11	33	0.0744	8.34E-06	21	62	0.1137	9.56E-06	4	12	0.0224	8.02E-07
	x_3	17	51	0.0642	8.46E-06	27	80	0.0630	7.48E-06	16	48	0.0466	7.86E-06
	x_4	6	18	0.0220	6.35E-06	25	74	0.0681	7.95E-06	13	38	0.0394	4.14E-06
	x_5	6	18	0.0476	6.32E-06	23	68	0.1298	6.33E-06	13	38	0.0408	4.14E-06
	x_6	19	56	0.1082	8.12E-06	37	110	0.1723	8.02E-06	13	38	0.0444	7.36E-06
	x_7	6	18	0.0233	6.35E-06	25	74	0.1144	7.95E-06	13	38	0.0946	4.14E-06
	x_8	6	18	0.0514	6.33E-06	23	68	0.1301	6.34E-06	13	38	0.0480	4.14E-06
10000	x_1	1	3	0.0256	0.00E+00	24	71	0.1538	8.67E-06	12	35	0.0796	5.18E-06
	x_2	11	32	0.0822	9.85E-06	22	65	0.1293	6.24E-06	4	12	0.0254	1.05E-06
	x_3	17	51	0.1114	8.46E-06	27	80	0.1304	7.48E-06	16	48	0.0767	7.86E-06
	x_4	6	18	0.0705	8.96E-06	25	74	0.1783	7.11E-06	13	38	0.0528	5.90E-06
	x_5	6	18	0.0383	8.94E-06	23	68	0.1492	8.96E-06	13	38	0.0720	5.90E-06
	x_6	19	56	0.1048	8.12E-06	37	110	0.1603	8.02E-06	32	95	0.1155	9.64E-06
	x_7	6	18	0.0721	8.96E-06	25	74	0.1555	7.11E-06	13	38	0.1071	5.90E-06
	x_8	6	18	0.0522	8.95E-06	23	68	0.1250	8.96E-06	13	38	0.0676	5.90E-06
50000	x_1	1	3	0.0840	0.00E+00	73	218	0.8937	9.05E-06	12	36	0.1510	8.06E-06
	x_2	10	30	0.2960	9.48E-06	23	68	0.3185	5.67E-06	4	12	0.0769	2.18E-06
	x_3	17	51	0.3043	8.46E-06	27	80	0.3296	7.48E-06	16	48	0.2676	7.86E-06
	x_4	7	21	0.2285	2.00E-06	25	74	0.4014	7.52E-06	13	39	0.1474	9.34E-06
	x_5	7	20	0.1211	1.00E-05	25	74	0.5230	5.01E-06	13	39	0.2256	9.34E-06
	x_6	19	56	0.3561	8.12E-06	37	110	0.5474	8.02E-06	23	68	0.3110	9.69E-06
	x_7	7	21	0.1825	2.00E-06	25	74	0.4350	7.52E-06	13	39	0.1832	9.34E-06
	x_8	7	20	0.2423	1.00E-05	25	74	0.3846	5.01E-06	13	39	0.2104	9.34E-06
100000	x_1	1	3	0.0876	0.00E+00	163	488	2.6633	9.42E-06	13	38	0.3068	4.56E-06
	x_2	10	30	0.3193	8.22E-06	23	68	0.8728	7.60E-06	4	12	0.1246	3.06E-06
	x_3	17	51	0.4671	8.46E-06	27	80	0.7761	7.48E-06	16	48	0.3551	7.86E-06
	x_4	7	21	0.3743	2.83E-06	73	218	1.9385	9.73E-06	14	41	0.3824	5.30E-06
	x_5	7	21	0.3707	2.83E-06	60	179	1.4907	8.93E-06	14	41	0.3356	5.30E-06
	x_6	19	56	0.4845	8.12E-06	37	110	1.3030	8.02E-06	23	69	0.6139	9.47E-06
	x_7	7	21	0.3670	2.83E-06	73	218	1.7620	9.73E-06	14	41	0.3254	5.30E-06
	x_8	7	21	0.2567	2.83E-06	60	179	1.2472	8.93E-06	14	41	0.3388	5.30E-06

Table 2. Detailed numerical result for test problem 2

Test Problem 2													
		PCGS-SNE				PDY				ACGD			
DIMENSION	INITIAL POINT	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM
1000	x_1	7	20	0.0268	2.71E-06	5	9	0.0281	3.60E-08	8	20	0.0171	9.31E-06
	x_2	7	19	0.0156	7.67E-06	3	5	0.0077	5.17E-07	7	18	0.0223	7.37E-06
	x_3	7	18	0.0093	9.81E-06	18	50	0.0235	5.39E-06	21	61	0.0498	8.89E-06
	x_4	8	22	0.0179	5.52E-06	22	60	0.0320	7.42E-06	36	105	0.0356	7.90E-06
	x_5	8	22	0.0160	5.52E-06	22	60	0.0306	7.42E-06	36	105	0.0320	7.90E-06
	x_6	7	19	0.0085	9.85E-06	19	53	0.0225	7.68E-06	12	34	0.0127	9.10E-06
	x_7	8	22	0.0120	5.52E-06	22	60	0.0244	7.42E-06	36	105	0.0703	7.90E-06
	x_8	8	22	0.0089	5.53E-06	25	66	0.0266	6.32E-06	36	105	0.0438	7.86E-06
5000	x_1	7	20	0.0527	6.19E-06	5	9	0.0335	6.26E-09	8	21	0.0295	8.79E-06
	x_2	7	20	0.0285	3.31E-06	3	5	0.0237	1.75E-07	7	19	0.0267	6.95E-06
	x_3	7	18	0.0498	9.89E-06	18	50	0.0739	5.37E-06	21	61	0.0761	8.86E-06
	x_4	8	23	0.0778	2.44E-06	23	66	0.1200	5.26E-06	16	43	0.0603	8.44E-06
	x_5	8	23	0.0361	2.44E-06	23	66	0.1404	5.26E-06	16	43	0.0699	8.44E-06
	x_6	7	19	0.0274	9.83E-06	19	53	0.1020	7.43E-06	12	35	0.0915	8.99E-06
	x_7	8	23	0.0343	2.44E-06	23	66	0.0819	5.26E-06	16	43	0.0473	8.44E-06
	x_8	8	23	0.0886	2.44E-06	23	66	0.1228	5.26E-06	16	43	0.0849	6.90E-06
10000	x_1	7	20	0.0472	8.79E-06	5	9	0.0377	3.62E-09	9	23	0.0553	3.22E-06
	x_2	7	20	0.0924	4.65E-06	3	5	0.0315	1.21E-07	7	19	0.0855	9.80E-06
	x_3	7	18	0.1117	9.90E-06	18	50	0.1575	5.37E-06	21	61	0.1338	8.85E-06
	x_4	8	23	0.0646	3.45E-06	23	66	0.1746	7.43E-06	9	24	0.1097	5.75E-06
	x_5	8	23	0.1174	3.45E-06	23	66	0.1564	7.43E-06	9	24	0.0836	5.75E-06
	x_6	7	19	0.1210	9.82E-06	19	53	0.1184	7.40E-06	12	35	0.0885	8.91E-06
	x_7	8	23	0.0675	3.45E-06	23	66	0.1476	7.43E-06	9	24	0.0489	5.75E-06
	x_8	8	23	0.1402	3.45E-06	23	66	0.1689	7.43E-06	9	24	0.0572	5.76E-06
50000	x_1	8	22	0.2318	9.84E-06	26	77	0.4527	7.75E-06	9	23	0.2045	7.18E-06
	x_2	8	22	0.1682	5.18E-06	3	5	0.0834	6.32E-08	8	21	0.1144	5.68E-06
	x_3	7	18	0.1254	9.91E-06	18	50	0.3740	5.36E-06	21	61	0.3232	8.84E-06
	x_4	8	23	0.2058	7.72E-06	24	69	0.3298	8.30E-06	11	28	0.2536	2.66E-06
	x_5	8	23	0.2165	7.72E-06	24	69	0.4526	8.30E-06	11	28	0.2257	2.66E-06
	x_6	7	19	0.1765	9.82E-06	19	53	0.4085	7.37E-06	12	35	0.2348	8.82E-06
	x_7	8	23	0.2822	7.72E-06	24	69	0.4685	8.30E-06	11	28	0.2464	2.66E-06
	x_8	8	23	0.1972	7.72E-06	24	69	0.4948	8.30E-06	11	28	0.1780	2.66E-06
100000	x_1	8	23	0.3460	2.79E-06	63	188	1.7301	7.77E-06	9	24	0.2241	4.38E-06
	x_2	8	22	0.3170	7.31E-06	3	5	0.1848	5.40E-08	8	21	0.2632	8.03E-06
	x_3	7	18	0.2619	9.91E-06	18	50	0.6095	5.36E-06	21	61	0.6810	8.84E-06
	x_4	9	25	0.3467	5.46E-06	26	77	0.7986	5.19E-06	11	28	0.4451	3.80E-06
	x_5	9	25	0.6451	5.46E-06	26	77	0.7970	5.19E-06	11	28	0.3569	3.80E-06
	x_6	7	19	0.2983	9.82E-06	19	53	0.7436	7.36E-06	12	35	0.4135	8.81E-06
	x_7	9	25	0.3568	5.46E-06	26	77	0.6589	5.19E-06	11	28	0.3803	3.80E-06
	x_8	9	25	0.3191	5.46E-06	26	77	0.6971	5.19E-06	11	28	0.3894	3.80E-06

Table 3. Detailed numerical result for test problem 3

Test Problem 3													
		PCGS-SNE				PDY				ACGD			
DIMENSION	INITIAL POINT	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM
1000	x_1	17	50	0.0119	5.04E-06	22	65	0.0341	6.10E-06	9	26	0.019642	3.80E-06
	x_2	14	42	0.0070	8.46E-06	19	56	0.0160	6.02E-06	6	18	0.003396	6.16E-06
	x_3	12	36	0.0061	9.13E-06	16	47	0.0263	8.43E-06	181	542	0.24699	9.80E-06
	x_4	16	47	0.0076	8.23E-06	21	62	0.0411	7.67E-06	24	72	0.018594	8.89E-06
	x_5	16	47	0.0088	8.23E-06	21	62	0.0350	7.67E-06	24	72	0.03324	8.89E-06
	x_6	13	38	0.0104	8.95E-06	17	50	0.0331	8.55E-06	50	149	0.034652	9.70E-06
	x_7	16	47	0.0118	8.23E-06	21	62	0.0335	7.67E-06	24	72	0.020981	8.89E-06
	x_8	16	47	0.0068	8.24E-06	21	62	0.0181	7.68E-06	24	72	0.055374	8.92E-06
5000	x_1	17	51	0.0268	9.01E-06	23	68	0.0840	6.82E-06	9	26	0.031384	8.49E-06
	x_2	15	44	0.0263	9.46E-06	20	59	0.0607	6.73E-06	7	20	0.012363	3.58E-06
	x_3	12	36	0.0239	9.13E-06	16	47	0.0567	8.43E-06	191	572	0.40204	9.54E-06
	x_4	17	50	0.0265	7.36E-06	22	65	0.1030	8.58E-06	33	98	0.10225	6.88E-06
	x_5	17	50	0.0196	7.36E-06	22	65	0.1546	8.58E-06	33	98	0.1442	6.88E-06
	x_6	13	38	0.0230	8.95E-06	17	50	0.0419	8.56E-06	53	158	0.12934	8.57E-06
	x_7	17	50	0.0227	7.36E-06	22	65	0.0744	8.58E-06	33	98	0.1083	6.88E-06
	x_8	17	50	0.0219	7.37E-06	22	65	0.0653	8.58E-06	33	98	0.097549	6.88E-06
10000	x_1	18	53	0.0543	6.37E-06	23	68	0.1279	9.65E-06	10	29	0.15458	1.92E-06
	x_2	16	47	0.0440	5.35E-06	20	59	0.1578	9.52E-06	7	20	0.017829	5.06E-06
	x_3	12	36	0.0377	9.13E-06	16	47	0.0806	8.43E-06	195	584	0.56594	9.71E-06
	x_4	17	51	0.0467	8.33E-06	23	68	0.1620	6.06E-06	33	98	0.16357	9.98E-06
	x_5	17	51	0.0501	8.33E-06	23	68	0.1495	6.06E-06	33	98	0.18848	9.98E-06
	x_6	13	38	0.0376	8.95E-06	17	50	0.1042	8.56E-06	38	114	0.23636	9.18E-06
	x_7	17	51	0.0447	8.33E-06	23	68	0.1461	6.06E-06	33	98	0.14626	9.98E-06
	x_8	17	51	0.0479	8.33E-06	23	68	0.1675	6.07E-06	33	98	0.14059	9.98E-06
50000	x_1	19	56	0.1578	5.70E-06	59	176	0.8724	8.47E-06	10	29	0.39346	4.30E-06
	x_2	16	48	0.1565	9.58E-06	22	65	0.3881	5.32E-06	7	21	0.13122	4.88E-06
	x_3	12	36	0.1116	9.13E-06	16	47	0.3502	8.43E-06	205	614	1.8942	9.44E-06
	x_4	18	53	0.1420	9.32E-06	24	71	0.3343	6.78E-06	35	104	0.6616	6.07E-06
	x_5	18	53	0.1596	9.32E-06	24	71	0.4304	6.78E-06	35	104	0.56177	6.07E-06
	x_6	13	38	0.1064	8.95E-06	17	50	0.3017	8.56E-06	40	119	0.54235	9.13E-06
	x_7	18	53	0.1539	9.32E-06	24	71	0.4095	6.78E-06	35	104	0.52486	6.07E-06
	x_8	18	53	0.1484	9.32E-06	24	71	0.4258	6.78E-06	35	104	0.53309	6.07E-06
100000	x_1	19	56	0.2842	8.06E-06	60	179	1.5106	8.99E-06	10	29	0.3622	6.08E-06
	x_2	17	50	0.2293	6.77E-06	22	65	0.7706	7.52E-06	7	21	0.40895	6.90E-06
	x_3	12	36	0.2230	9.13E-06	16	47	0.4068	8.43E-06	209	626	4.8398	9.56E-06
	x_4	19	56	0.3061	5.27E-06	58	173	1.8373	9.64E-06	32	95	0.69973	6.56E-06
	x_5	19	56	0.2461	5.27E-06	58	173	1.6541	9.64E-06	32	95	0.80597	6.56E-06
	x_6	13	38	0.1784	8.95E-06	17	50	0.5134	8.56E-06	41	122	1.176	8.25E-06
	x_7	19	56	0.2475	5.27E-06	58	173	1.4193	9.64E-06	32	95	1.0582	6.56E-06
	x_8	19	56	0.2703	5.27E-06	58	173	1.5480	9.64E-06	32	95	1.2126	6.56E-06

Table 4. Detailed numerical result for test problem 4

Test Problem 4													
		PCGS-SNE				PDY				ACGD			
DIMENSION	**INITIAL POINT**	**ITER**	**FVAL**	**TIME**	**NORM**	**ITER**	**FVAL**	**TIME**	**NORM**	**ITER**	**FVAL**	**TIME**	**NORM**
1000	x_1	1	3	0.0044	0.00E+00	20	59	0.0318	7.37E-06	1	3	0.0023	0.00E+00
	x_2	6	17	0.0100	8.23E-06	19	56	0.0359	5.45E-06	6	17	0.0030	8.30E-06
	x_3	1	3	0.0055	2.22E-16	16	48	0.0099	5.62E-06	62	185	0.0244	9.67E-06
	x_4	7	20	0.0130	6.67E-06	20	59	0.0192	8.76E-06	13	39	0.0052	8.45E-06
	x_5	7	20	0.0080	6.67E-06	20	59	0.0301	8.76E-06	13	39	0.0063	8.45E-06
	x_6	6	18	0.0144	4.79E-06	17	50	0.0339	6.76E-06	17	50	0.0074	7.13E-06
	x_7	7	20	0.0167	6.67E-06	20	59	0.0348	8.76E-06	13	39	0.0056	8.45E-06
	x_8	7	20	0.0077	6.76E-06	20	59	0.0281	8.77E-06	13	39	0.0059	8.49E-06
5000	x_1	1	3	0.0134	0.00E+00	21	62	0.0561	8.24E-06	1	3	0.0049	0.00E+00
	x_2	6	18	0.0374	3.68E-06	20	59	0.0929	6.10E-06	6	18	0.0090	7.99E-06
	x_3	1	3	0.0067	2.22E-16	16	48	0.0919	5.62E-06	62	185	0.0744	9.67E-06
	x_4	7	21	0.0403	3.00E-06	21	62	0.0446	9.80E-06	14	41	0.0178	9.01E-06
	x_5	7	21	0.0382	3.00E-06	21	62	0.0505	9.80E-06	14	41	0.0163	9.01E-06
	x_6	6	18	0.0173	4.82E-06	17	50	0.0400	6.76E-06	17	50	0.0179	7.13E-06
	x_7	7	21	0.0499	3.00E-06	21	62	0.0793	9.80E-06	14	41	0.0125	9.01E-06
	x_8	7	21	0.0462	3.01E-06	21	62	0.0956	9.80E-06	14	41	0.0115	9.02E-06
10000	x_1	1	3	0.0193	0.00E+00	22	65	0.0942	5.83E-06	1	3	0.0052	0.00E+00
	x_2	6	18	0.0580	5.20E-06	20	59	0.1154	8.62E-06	7	20	0.0130	2.94E-06
	x_3	1	3	0.0118	2.22E-16	16	48	0.0670	5.62E-06	62	185	0.0909	9.67E-06
	x_4	7	21	0.0410	4.24E-06	22	65	0.1248	6.93E-06	14	42	0.0294	9.83E-06
	x_5	7	21	0.0501	4.24E-06	22	65	0.1351	6.93E-06	14	42	0.0240	9.83E-06
	x_6	6	18	0.0579	4.83E-06	17	50	0.1209	6.76E-06	17	50	0.0299	7.13E-06
	x_7	7	21	0.0367	4.24E-06	22	65	0.0961	6.93E-06	14	42	0.0256	9.83E-06
	x_8	7	21	0.0659	4.25E-06	22	65	0.1321	6.93E-06	14	42	0.0213	9.83E-06
50000	x_1	1	3	0.0225	0.00E+00	57	170	0.5238	8.87E-06	1	3	0.0138	0.00E+00
	x_2	7	20	0.1574	5.82E-06	21	62	0.2359	9.64E-06	7	20	0.0353	6.57E-06
	x_3	1	3	0.0267	2.22E-16	16	48	0.1831	5.62E-06	62	185	0.3034	9.67E-06
	x_4	7	21	0.1838	9.50E-06	56	167	0.5082	7.97E-06	15	45	0.0726	7.73E-06
	x_5	7	21	0.1545	9.50E-06	56	167	0.5058	7.97E-06	15	45	0.0922	7.73E-06
	x_6	6	18	0.0852	4.83E-06	17	50	0.2325	6.76E-06	17	50	0.0996	7.13E-06
	x_7	7	21	0.1367	9.50E-06	56	167	0.5103	7.97E-06	15	45	0.0766	7.73E-06
	x_8	7	21	0.1104	9.50E-06	56	167	0.5681	7.97E-06	15	45	0.0839	7.74E-06
100000	x_1	1	3	0.0695	0.00E+00	126	377	1.9557	9.15E-06	1	3	0.0282	0.00E+00
	x_2	7	20	0.1776	8.23E-06	22	65	0.4198	6.82E-06	7	20	0.0573	9.29E-06
	x_3	1	3	0.0624	2.22E-16	16	48	0.3843	5.62E-06	62	185	0.5275	9.67E-06
	x_4	8	23	0.1700	6.71E-06	57	170	1.0812	8.46E-06	16	47	0.1353	4.98E-06
	x_5	8	23	0.1384	6.71E-06	57	170	1.0702	8.46E-06	16	47	0.1575	4.98E-06
	x_6	6	18	0.2569	4.83E-06	17	50	0.3592	6.76E-06	17	50	0.1404	7.13E-06
	x_7	8	23	0.1911	6.71E-06	57	170	1.0048	8.46E-06	16	47	0.1550	4.98E-06
	x_8	8	23	0.1627	6.72E-06	57	170	1.3807	8.46E-06	16	47	0.1496	4.98E-06

Table 5. Detailed numerical result for test problem 5

Test Problem 5													
		PCGS-SNE				PDY				ACGD			
DIMENSION	INITIAL POINT	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM
1000	x_1	8	24	0.0152	4.81E-06	25	74	0.0402	5.22E-06	25	75	0.0059	8.84E-06
	x_2	9	24	0.0218	5.37E-06	25	68	0.0148	7.57E-06	22	61	0.0120	8.27E-06
	x_3	8	22	0.0241	8.16E-06	24	65	0.0209	5.51E-06	47	135	0.0155	9.75E-06
	x_4	52	153	0.0713	9.46E-06	24	65	0.0475	8.15E-06	21	59	0.0080	9.08E-06
	x_5	29	86	0.0241	8.79E-06	28	83	0.0240	5.81E-06	157	469	0.0438	9.50E-06
	x_6	8	22	0.0185	6.85E-06	25	67	0.0413	6.04E-06	45	129	0.0179	9.35E-06
	x_7	52	153	0.0422	9.46E-06	24	65	0.0392	8.15E-06	21	59	0.0082	9.08E-06
	x_8	29	86	0.0433	8.90E-06	28	83	0.0535	5.78E-06	157	469	0.0449	9.55E-06
5000	x_1	10	29	0.0288	7.05E-06	27	80	0.0752	5.74E-06	63	188	0.0815	9.11E-06
	x_2	10	27	0.0637	6.47E-06	26	70	0.1014	5.44E-06	84	247	0.1063	9.42E-06
	x_3	9	25	0.0362	7.12E-06	26	70	0.0805	8.59E-06	112	331	0.1274	9.99E-06
	x_4	114	338	0.2975	9.69E-06	24	65	0.0723	8.94E-06	31	89	0.0368	9.86E-06
	x_5	32	95	0.0700	8.85E-06	30	89	0.1230	6.71E-06	392	1175	0.4766	9.98E-06
	x_6	9	25	0.0531	3.92E-06	30	79	0.1100	6.40E-06	81	238	0.1058	8.80E-06
	x_7	114	338	0.1972	9.69E-06	24	65	0.0805	8.94E-06	31	89	0.0335	9.86E-06
	x_8	32	95	0.1372	8.87E-06	30	89	0.0796	6.70E-06	392	1175	0.4747	9.98E-06
10000	x_1	10	30	0.1010	5.14E-06	28	83	0.1394	5.41E-06	86	257	0.2122	9.17E-06
	x_2	10	28	0.0429	4.29E-06	26	70	0.1276	5.55E-06	117	346	0.2832	9.25E-06
	x_3	10	28	0.1022	2.21E-06	26	70	0.1422	5.32E-06	160	475	0.3342	9.57E-06
	x_4	109	324	0.3239	9.74E-06	25	68	0.1444	5.39E-06	145	430	0.3402	9.88E-06
	x_5	34	100	0.1376	7.88E-06	31	92	0.1735	6.42E-06	523	1567	1.2969	9.92E-06
	x_6	10	27	0.1082	5.76E-06	27	73	0.1343	6.00E-06	154	457	0.3779	9.47E-06
	x_7	109	324	0.2936	9.74E-06	25	68	0.1295	5.39E-06	145	430	0.3736	9.88E-06
	x_8	34	100	0.1558	7.89E-06	31	92	0.1786	6.42E-06	523	1567	1.2648	9.91E-06
50000	x_1	11	33	0.1699	8.06E-06	75	224	0.9077	9.99E-06	87	260	0.6046	9.49E-06
	x_2	11	31	0.1358	6.30E-06	34	101	0.5472	5.92E-06	118	349	0.9240	9.92E-06
	x_3	11	31	0.1988	3.34E-06	34	101	0.4451	5.01E-06	154	457	1.1023	9.50E-06
	x_4	117	348	1.0323	9.77E-06	27	74	0.3893	6.13E-06	709	2123	4.8375	9.99E-06
	x_5	37	109	0.4198	7.80E-06	80	239	0.7520	8.32E-06	31	91	2.1652	5.42E-06
	x_6	11	30	0.2283	8.65E-06	33	98	0.3980	7.79E-06	120	356	0.7805	9.97E-06
	x_7	117	348	0.9909	9.77E-06	27	74	0.3902	6.13E-06	709	2123	4.7744	9.99E-06
	x_8	37	109	0.5868	7.80E-06	80	239	0.8455	8.32E-06	31	91	0.8834	5.42E-06
100000	x_1	12	36	0.4568	2.52E-06	78	233	1.5349	8.43E-06	117	350	1.2015	9.46E-06
	x_2	12	33	0.2915	9.68E-06	35	104	0.9613	5.62E-06	208	620	2.2770	9.91E-06
	x_3	12	33	0.3090	5.24E-06	34	101	0.8293	9.54E-06	163	484	1.7285	9.65E-06
	x_4	56	164	0.9205	8.90E-06	33	98	0.8898	8.46E-06	821	2458	8.6939	9.98E-06
	x_5	38	112	0.8562	9.23E-06	82	245	1.6194	9.35E-06	31	92	1.2728	6.75E-06
	x_6	11	31	0.2925	5.53E-06	34	101	0.9241	7.42E-06	162	481	1.7146	9.36E-06
	x_7	56	164	1.1177	8.90E-06	33	98	0.7542	8.46E-06	821	2458	8.7567	9.98E-06
	x_8	38	112	0.7000	9.23E-06	82	245	1.7929	9.35E-06	31	92	1.3877	6.75E-06

Table 6. Detailed numerical result for test problem 6

Test Problem 6													
		PCGS-SNE				PDY				ACGD			
DIMENSION	INITIAL POINT	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM
1000	x_1	7	21	0.0143	5.41E-06	23	68	0.0371	6.47E-06	8	23	0.0061	3.11E-06
	x_2	7	21	0.0108	8.25E-06	23	68	0.0191	9.86E-06	8	23	0.0037	4.74E-06
	x_3	7	21	0.0335	8.56E-06	24	71	0.0657	5.11E-06	8	23	0.0041	4.92E-06
	x_4	7	21	0.0120	7.05E-06	23	68	0.0208	8.42E-06	8	23	0.0045	4.05E-06
	x_5	7	21	0.0133	7.05E-06	23	68	0.0299	8.42E-06	8	23	0.0047	4.05E-06
	x_6	7	21	0.0182	8.54E-06	24	71	0.0454	5.10E-06	8	23	0.0067	4.91E-06
	x_7	7	21	0.0250	7.05E-06	23	68	0.0514	8.42E-06	8	23	0.0040	4.05E-06
	x_8	7	21	0.0149	7.05E-06	23	68	0.0322	8.42E-06	8	23	0.0060	4.05E-06
5000	x_1	8	23	0.0746	6.07E-06	24	71	0.1520	7.24E-06	8	23	0.0196	6.98E-06
	x_2	8	23	0.0360	9.26E-06	25	74	0.0902	5.52E-06	8	24	0.0188	4.58E-06
	x_3	8	23	0.0307	9.61E-06	25	74	0.1453	5.73E-06	8	24	0.0192	4.76E-06
	x_4	8	23	0.0456	7.91E-06	24	71	0.1688	9.43E-06	8	23	0.0125	9.09E-06
	x_5	8	23	0.0315	7.91E-06	24	71	0.0847	9.43E-06	8	23	0.0198	9.09E-06
	x_6	8	23	0.0378	9.60E-06	25	74	0.1430	5.72E-06	8	24	0.0176	4.76E-06
	x_7	8	23	0.0387	7.91E-06	24	71	0.1395	9.43E-06	8	23	0.0197	9.09E-06
	x_8	8	23	0.0367	7.91E-06	24	71	0.0991	9.43E-06	8	23	0.0133	9.09E-06
10000	x_1	8	23	0.1063	8.59E-06	25	74	0.1767	5.12E-06	8	23	0.0283	9.88E-06
	x_2	8	24	0.0644	2.62E-06	60	179	0.2878	8.35E-06	8	24	0.0318	6.48E-06
	x_3	8	24	0.0802	2.72E-06	60	179	0.3106	8.67E-06	8	24	0.0308	6.73E-06
	x_4	8	24	0.0746	2.24E-06	59	176	0.4522	9.51E-06	8	24	0.0347	5.54E-06
	x_5	8	24	0.0712	2.24E-06	59	176	0.3535	9.51E-06	8	24	0.0352	5.54E-06
	x_6	8	24	0.1182	2.72E-06	60	179	0.3557	8.67E-06	8	24	0.0299	6.73E-06
	x_7	8	24	0.1074	2.24E-06	59	176	0.2937	9.51E-06	8	24	0.0334	5.54E-06
	x_8	8	24	0.0844	2.24E-06	59	176	0.3384	9.51E-06	8	24	0.0268	5.54E-06
50000	x_1	8	24	0.2230	3.84E-06	61	182	1.4231	9.19E-06	8	24	0.0928	9.51E-06
	x_2	8	24	0.2143	5.85E-06	134	401	2.0432	9.92E-06	9	26	0.0988	3.77E-06
	x_3	8	24	0.2656	6.08E-06	135	404	2.1475	9.01E-06	9	26	0.1038	3.91E-06
	x_4	8	24	0.2068	5.00E-06	133	398	2.8030	9.69E-06	9	26	0.0992	3.22E-06
	x_5	8	24	0.3934	5.00E-06	133	398	1.9219	9.69E-06	9	26	0.1100	3.22E-06
	x_6	8	24	0.2035	6.08E-06	135	404	2.3747	9.01E-06	9	26	0.1016	3.91E-06
	x_7	8	24	0.2618	5.00E-06	133	398	1.9778	9.69E-06	9	26	0.1087	3.22E-06
	x_8	8	24	0.1989	5.00E-06	133	398	1.8595	9.69E-06	9	26	0.0858	3.22E-06
100000	x_1	8	24	0.3954	5.43E-06	134	401	6.2068	9.21E-06	9	26	0.1882	3.50E-06
	x_2	8	24	0.3705	8.28E-06	283	848	18.5278	9.68E-06	9	26	0.1634	5.33E-06
	x_3	8	24	0.3534	8.60E-06	284	851	16.7759	9.42E-06	9	26	0.2438	5.53E-06
	x_4	8	24	0.5067	7.07E-06	136	407	5.5720	9.18E-06	9	26	0.1895	4.55E-06
	x_5	8	24	0.4281	7.07E-06	136	407	6.1390	9.18E-06	9	26	0.1678	4.55E-06
	x_6	8	24	0.3753	8.60E-06	284	851	18.8131	9.42E-06	9	26	0.1656	5.53E-06
	x_7	8	24	0.3735	7.07E-06	136	407	6.6805	9.18E-06	9	26	0.2138	4.55E-06
	x_8	8	24	0.3845	7.07E-06	136	407	6.2586	9.18E-06	9	26	0.1715	4.55E-06

Table 7. Detailed numerical result for test problem 7

Test Problem 7													
		PCGS-SNE				PDY				ACGD			
DIMENSION	INITIAL POINT	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM	ITER	FVAL	TIME	NORM
1000	x_1	29	87	0.029436	9.54E-06	36	107	0.077023	8.43E-06				
	x_2	33	99	0.040312	8.78E-06	96	287	0.083037	8.45E-06	25	74	0.032175	7.36E-06
	x_3	27	81	0.050235	8.82E-06								
	x_4	28	83	0.07407	9.08E-06	325	974	0.12589	5.64E-06	523	1568	0.26082	4.49E-06
	x_5	24	71	0.034917	7.69E-06	32	95	0.066748	8.89E-06	519	1556	0.22013	8.95E-06
	x_6	28	84	0.037449	9.19E-06					470	1410	0.19079	8.37E-06
	x_7	28	83	0.035963	9.08E-06	325	974	0.12293	5.64E-06	464	1392	0.1815	8.60E-06
	x_8	24	71	0.033757	7.73E-06	32	95	0.043311	8.26E-06	490	1469	0.24578	8.17E-06
5000	x_1	29	86	0.1142	7.74E-06	31	92	0.13469	8.81E-06				
	x_2	32	95	0.15466	7.97E-06	444	1331	0.9011	4.70E-06				
	x_3	29	87	0.17967	7.57E-06								
	x_4	28	83	0.1319	9.72E-06					354	1062	0.7242	9.94E-06
	x_5	24	72	0.14644	8.87E-06	33	98	0.14179	7.06E-06	402	1205	0.782	8.50E-06
	x_6	31	93	0.17888	9.07E-06					385	1154	0.88027	9.91E-06
	x_7	28	83	0.14363	9.72E-06					403	1208	0.80547	9.84E-06
	x_8	24	72	0.1655	8.87E-06	33	98	0.055278	7.11E-06	399	1196	0.85232	8.64E-06
10000	x_1	38	114	0.22998	9.22E-06	32	95	0.26336	5.83E-06				
	x_2	32	95	0.19153	8.70E-06	788	2363	2.993	7.43E-06				
	x_3	30	89	0.20675	8.20E-06								
	x_4	28	84	0.16387	9.51E-06	63	188	0.21545	7.77E-06	388	1163	1.4832	9.95E-06
	x_5	25	75	0.19492	7.85E-06	63	188	0.41833	7.72E-06	433	1298	1.7328	8.84E-06
	x_6	30	89	0.18587	9.46E-06					375	1124	1.4414	8.99E-06
	x_7	28	84	0.19174	9.51E-06	63	188	0.36421	7.77E-06	370	1109	1.5212	8.96E-06
	x_8	25	75	0.18039	7.84E-06	63	188	0.32567	7.72E-06	412	1236	1.6443	9.70E-06
50000	x_1	29	86	0.72278	8.39E-06	30	89	1.0592	9.84E-06				
	x_2	27	81	0.40974	9.52E-06								
	x_3	29	86	0.60097	8.58E-06	24	71	0.58916	6.04E-06				
	x_4	27	80	0.48751	8.84E-06	66	197	1.029	7.76E-06	435	1304	5.4546	8.27E-06
	x_5	25	75	0.80655	7.80E-06	66	197	1.4308	7.75E-06	432	1296	5.6062	9.31E-06
	x_6	30	89	0.71092	8.54E-06	30	89	1.0585	7.45E-06	546	1637	6.9536	9.64E-06
	x_7	27	80	0.56227	8.84E-06	66	197	1.1381	7.76E-06	414	1242	5.2427	9.13E-06
	x_8	25	75	0.42647	7.78E-06	66	197	1.2139	7.75E-06	420	1259	5.4757	6.30E-06
100000	x_1	32	96	1.273	7.80E-06	26	77	1.3177	6.50E-06				
	x_2	32	95	1.1809	8.64E-06	26	77	1.3031	8.99E-06				
	x_3	32	96	1.4997	8.59E-06	25	74	0.91989	4.80E-06				
	x_4	28	84	1.3567	9.70E-06	32	95	1.9077	6.80E-06	435	1304	10.0887	8.80E-06
	x_5	26	77	0.87055	8.21E-06	25	74	1.0497	3.56E-06	443	1328	10.4205	8.67E-06
	x_6	28	83	1.0334	9.57E-06	25	74	1.1845	6.23E-06	392	1175	9.2345	4.66E-06
	x_7	28	84	1.2074	9.70E-06	32	95	1.6595	6.80E-06	417	1250	10.1438	8.48E-06
	x_8	26	77	1.0651	8.21E-06	25	74	1.163	3.56E-06	438	1313	10.4184	9.70E-06

4. Conclusion

This paper introduces a derivative-free projection method based on conjugate gradients method to solve SNE. The proposed method ensures that the search direction always exhibits sufficient descent, regardless of any line search. By making appropriate assumptions, we establish the global convergence of the proposed method. Numerical experiments demonstrate the robustness and efficiency of the proposed method when solving large-scale monotone SNE with convex constraints.

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Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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