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Ćirić-Rhoades-type Contractive Mappings

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ABSTRACT

The available literature shows that weakly contractive operators have been thoroughly researched in relation to metric spaces and associated fixed point theorems. These efforts, which broaden the concept of symmetric and asymmetric spaces, however, have not yet fully grasped the context of metric-like spaces. Considering this gap, this manuscripts introduces the notion of generalized weakly quasi contractive operators in metric-like space and investigates the existence and uniqueness of these operators' fixed points. Non-trivial comparative examples are constructed to support the assertions forming the main ideas herein. Some corollaries indicating that the idea of this paper encompasses a few related ones in the literature are highlighted and addressed.

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1. Introduction

Throughout this paper, metric space, metric-like space, complete metric-like space, partial metric space, contractive operator and fixed point, will hereon be written as MS, MLS, CMLS, PMS, CP and FP, respectively.

The contraction mapping concept, commonly referred to as the Banach FP theorem, is a crucial technique in the study of MSs. It provides powerful tools and techniques for examining the existence and uniqueness of FPs. Several researchers have developed generalisations of the contraction mapping in an effort to investigate more FP outcomes (for example, see [19, 23] and the references therein). Bakhtin [5] established the concept of a contraction mapping principle in quasi MSs in 1989. The latter idea was expanded by Czerwik [?] to b-MSs. The concept of cone MSs was first introduced by Huang and Zhang, [13], as an improvement of MSs and the accompanying FP results. In a related advance, Mustafa and Sims [20] devised a fresh method for generalized MSs. The quasi MS, as defined by Wilson [25], is one of the early generalizations. The inquiry into the denotational semantics of data flow networks was conducted by Matthews [18], who developed the concept of PMS in a similar manner. It is established in [18] that self-distance in the PMS is not always zero. By relaxing the PMS

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axioms of non-negativity and small self-distances, Amini-Harandi [3] proposed the idea of an MLS as a refinement of PMS. In a different path, Alber et al. [1] proposed the concept of weak contraction mappings in the context of Hilbert space, by defining extra algebraic structure on the space. After then, Rhoades [22] proved that weak contraction mappings have a unique FP in MS. From then, many authors explored different generalization of the weak contraction mapping in the context of MSs. In particular, Chen and Zhu [7] generalized the weak contraction mapping by replacing the Banach contraction with Chatterjea [6] contractive mapping. After that, Choudhury et al. [10] extended the weak contraction by using altering distance function and established that the class of weakly C-contractive mappings have a unique FP in complete MS. In a similar approach, Cho [9] generalized Choudhury et al. [10] weak contraction mapping by adding a lower semi-continuous function and developed some FP results for weakly contractive mappings in MS. For other related results in the context of weak contraction mapping, see Inuwa et al [14], Kim and Han [17] and the reference therein.

In the setting of an MLS, little or no research on weakly quasi CPs has been done, as can be seen from the review of the literature that have already been published. Inspired by the notion in [9, 21], we therefore offer a novel idea of generalized weakly quasi CP in an MLS and explore the existence and uniqueness of FPs of such operators. Some well-known results in the literature are generalized by the idea put out in this manuscript. For the purpose of comparing our suggested notion with other equivalent findings, substantial examples are provided. A few corollaries that connect our ideas to other well-known concepts in the literature are presented and analyzed.

The following is how the paper is set up: Section 1 provides the reviews of the relevant literature. Section 2 summarizes the foundational concepts needed for the rest of the paper. Section 3 presents the main results of the paper.

2. Preliminaries

In this section, we record specific basic concepts that are needed in the sequel. First, some definitions and basic results in PMSs are recalled. For more details, the reader can consult [9]. Let Θ be a non-null set. A function $\wp : \Theta \times \Theta \longrightarrow \mathbb{R}^+$ is called a partial metric on Θ if, for all $\mu, \omega, z \in \Theta$, the following are satisfied:

($\wp_1$) $\wp(\mu,\mu) = \wp(\omega,\omega) \Leftrightarrow \mu = \omega$;

($\wp_2$) $\wp(\mu,\mu) \leq \wp(\mu,\omega)$;

($\wp_3$) $\wp(\mu,\omega) = \wp(\omega,\mu)$;

($\wp_4$) $\wp(\mu,z) \leq \wp(\mu,\omega) + \wp(\omega,z) - \wp(\omega,\omega)$.

The pair $(\Theta,\wp)$ is called a PMS. Note that if $\wp(\mu,\omega) = 0$, then $\mu = \omega$. An example of a partial metric defined on $\mathbb{R}^+$, is $\wp(\mu,\omega) = \max\{\mu,\omega\}, \mu,\omega \geq 0$. For more examples of partial metrics, see [9]. Let $\{\mu_\imath\}_{\imath\in\mathbb{N}}$ be a sequence in Θ. Then,

(i) $\{\mu_\imath\}_{\imath\in\mathbb{N}}$ is convergent to μ if $\lim\limits_{\imath\to\infty} \wp(\mu,\mu_\imath) = \wp(\mu,\mu)$;

(ii) $\{\mu_\imath\}_{\imath\in\mathbb{N}}$ is called a Cauchy sequence if $\lim\limits_{\imath,\jmath\to\infty} \wp(\mu_\imath,\mu_\jmath)$ exists and is finite;

(iii) if each Cauchy sequence in Θ converges to a point $\mu \in \Theta$, then Θ is complete.

Definition 2.1. [3] A mapping $\sigma : \Theta \times \Theta \longrightarrow \mathbb{R}_+$ is referred to as a metric-like on Θ if for any $\mu, \omega, z \in \Theta$, the following hold:

(σ_1) $\sigma(\mu, \omega) = 0 \Rightarrow \mu = \omega$;

(σ_2) $\sigma(\mu, \omega) = \sigma(\omega, \mu)$;

(σ_3) $\sigma(\mu, z) \leq \sigma(\mu, \omega) + \sigma(\omega, z)$.

The pair (Θ, σ) is called an MLS.

Definition 2.2. [3] A sequence $\{\mu_\imath\}_{\imath\in\mathbb{N}}$ in an MLS (Θ, σ) converges to a point $\mu \in \Theta$ if $\sigma(\mu, \mu) = \lim_{\imath\to\infty} \sigma(\mu_\imath, \mu)$.

Definition 2.3. [3] A sequence $\{\mu_\imath\}_{\imath\in\mathbb{N}}$ in an MLS (Θ, σ) is called σ-Cauchy sequence if the $\lim_{\imath,\jmath\to\infty} \sigma(\mu_\imath, \mu_\jmath)$ exists and is finite. The MLS (Θ, σ) is called complete if there is some $\mu \in \Theta$ such that

$$\lim_{\imath\to\infty} \sigma(\mu_\imath, \mu) = \lim_{\imath,\jmath\to\infty} \sigma(\mu_\imath, \mu_\jmath).$$

Remark 2.4. [3] Every PMS is an MLS, but the converse is not always true. An example given here recognizes this observation.

Example 2.5. [3] Let $\Theta = \{0, 1\}$,and let

$$\sigma(\mu, \omega) = \begin{cases} 2, & \text{if } \mu = \omega = 0; \\ 1, & \text{otherwise.} \end{cases}$$

Then (Θ, σ) is an MLS, but since $\sigma(0, 0) \nleq \sigma(0, 1)$, then (Θ, σ) is not a PMS.

Definition 2.6. [11] Let (Θ, d) be a MS. A self-mapping $\zeta : \Theta \longrightarrow \Theta$ is referred to as a quasi-contraction if there exists $\eta \in \left[0, \frac{1}{2}\right)$ such that for all $\mu, \omega \in \Theta$,

$$d(\zeta\mu, \zeta\omega) \leq \max \eta\{d(\mu, \omega), d(\mu, \zeta\mu), d(\omega, \zeta\omega), d(\mu, \zeta\omega), d(\omega, \zeta\mu)\}.$$

Definition 2.7. [22] Let (Θ, d) be a MS. A mapping $\zeta : \Theta \longrightarrow \Theta$ is referred to as weakly contractive, if for all $\mu, \omega \in \Theta$,

$$d(\zeta\mu, \zeta\omega) \leq d(\mu, \omega) - \alpha(d(\mu, \omega)),$$

where $\alpha : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is continuous and non-decreasing function such that $\alpha(0) = 0$ and $\lim_{t\to+\infty} \alpha(t) = +\infty$.

Definition 2.8. A function $\zeta : \Theta \longrightarrow [0, \infty)$, where Θ is a MS, is called lower semi-continuous if, for all $\mu \in \Theta$ and $\{\mu_\imath\}_{\imath\in\mathbb{N}} \subset \Theta$ with $\lim_{\imath\to\infty} \mu_\imath = \mu$, we have

$$\zeta(\mu) \leq \liminf_{\imath\to\infty} \mu_\imath.$$

Let $\Psi_\iota = \{\psi_\iota : \mathbb{R}_+ \to \mathbb{R}_+ | \psi_\iota$ is continuous and $\psi_\iota(t) = 0 \Leftrightarrow t = 0\}$. In addition, let $\Phi_\iota = \{\phi_\iota : \mathbb{R}_+ \longrightarrow \mathbb{R}_+ | \phi_\iota$ is lower semi-continuous and $\phi_\iota(t) = 0 \Leftrightarrow t = 0\}$.

Raj et al. [21] established the following result in the context of MS.

Definition 2.9. [21]. Let Θ be a metric, $\zeta : \Theta \longrightarrow \Theta$ a mapping, and $\vartheta : \Theta \to \mathbb{R}_+$ be a lower semi-continuous function. Then, ζ is called a generalized weakly contractive mapping if it satisfies the following inequality:

$$\begin{aligned} &\psi_\iota(d(\zeta\mu, \zeta\omega) + \vartheta(\zeta\mu) + \vartheta(\zeta\omega) \\ &\quad \leq \psi_\iota(\mathcal{M}(\mu, \omega, \vartheta)) - \phi_\iota(\mathcal{N}(\mu, \omega, \vartheta)) \end{aligned} \tag{2.1}$$

for all $\mu, \omega \in \Theta$, where $\psi_\iota \in \Psi_\iota$, $\phi_\iota \in \Phi_\iota$ and

$$\mathcal{M}(\mu, \omega, \vartheta) = \max \left\{ \begin{array}{c} d(\mu, \omega) + \vartheta(\mu) + \vartheta(\omega), d(\mu, \zeta\mu) + \vartheta(\mu) + \vartheta(\zeta\mu), \\ d(\mu, \zeta\omega) + \vartheta(\omega) + \vartheta(\zeta\omega), \dfrac{d(\mu, \omega) + \vartheta(\mu) + \vartheta(\omega)}{1 + d(\mu, \omega) + \vartheta(\mu) + \vartheta(\omega)}, \\ \frac{1}{2}[d(\mu, \zeta\omega) + \vartheta(\mu) + \vartheta(\zeta\omega) + d(\omega, \zeta\mu) + \vartheta(\omega) + \vartheta(\zeta\mu)] \end{array} \right\},$$

$$\mathcal{N}(\mu, \omega, \vartheta) = \max \left\{ \begin{array}{c} d(\mu, \omega) + \vartheta(\mu) + \vartheta(\omega), \\ d(\omega, \zeta\omega) + \vartheta(\omega) + \vartheta(\zeta\omega), \\ \dfrac{d(\mu, \omega) + \vartheta(\mu) + \vartheta(\omega)}{1 + d(\mu, \omega) + \vartheta(\mu) + \vartheta(\omega)} \end{array} \right\}.$$

The main result of [21] is as follows.

Theorem 2.10. *[21]. Let Θ be a complete MS. If ζ is a generalized weakly contractive mapping, then there exists a unique $z \in \Theta$ such that $z = \zeta z$ and $\vartheta(z) = 0$.*

Lemma 2.11. *[4]. Let (Θ, σ) be an MLS and let $\{\mu_\imath\}_{\imath\in\mathbb{N}}$ be a sequence in Θ such that if $\{\mu_\imath\}_{\imath\in\mathbb{N}}$ is not a σ-Cauchy sequence in (Θ, σ). Then, there exists $\epsilon^+ > 0$ and two subsequences $\{\mu_{\jmath(\ell)}\}_{\ell\in\mathbb{N}}$ and $\{\mu_{\imath(\ell)}\}_{\ell\in\mathbb{N}}$ of $\{\mu_\imath\}_{\imath\in\mathbb{N}}$, where n, m are positive integers with $n(\ell) > m(\ell) > k$ such that*

$$\sigma(\mu_{\jmath(\ell)}, \mu_{\imath(\ell)}) \geq \epsilon^+ \tag{2.2}$$

and

$$\sigma(\mu_{\jmath(\ell)-1}, \mu_{\imath(\ell)}) < \epsilon^+. \tag{2.3}$$

Moreover, suppose that

$$\lim_{\imath\to\infty} \sigma(\mu_\imath, \mu_{\imath+1}) = 0. \tag{2.4}$$

Then, the following hold:

(1) $\lim_{\ell\to\infty} \sigma(\mu_{\jmath(\ell)}, \mu_{\imath(\ell)}) = \epsilon^+$;

(2) $\lim_{\ell\to\infty} \sigma(\mu_{\jmath(\ell)}, \mu_{\imath(\ell+1)}) = \epsilon^+$;

(3) $\lim_{\ell\to\infty} \sigma(\mu_{\jmath(\ell-1)}, \mu_{\imath(\ell)}) = \epsilon^+$;

(4) $\lim_{\ell\to\infty} \sigma(\mu_{\jmath(\ell-1)}, \mu_{\imath(\ell+1)}) = \epsilon^+$.

3. Main Results

In this section, the concept of Ćirić-Rhoades CP in the framework of an MLS is introduced and the conditions for the existence of a FP of such operator are examined.

Definition 3.1. Let (Θ, σ) be an MLS. A mapping $\zeta : \Theta \longrightarrow \Theta$ is called a Ćirić-Rhoades-type CP, if it satisfies the following inequality:

$$\psi_\iota(\sigma(\zeta\mu, \zeta\mu) + \sigma(\zeta\mu, \zeta\omega) + \vartheta(\zeta\mu) + \vartheta(\zeta\omega)) \leq \psi_\iota(\mathcal{K}(\mu, \omega, \vartheta)) - \phi_\iota(\mathcal{L}(\mu, \omega, \vartheta)), \quad (3.1)$$

for all $\mu, \omega \in \Theta$, where $\psi_\iota \in \Psi_\iota$, $\phi_\iota, \vartheta \in \Phi_\iota$ and

$$\mathcal{K}(\mu, \omega, \vartheta) = \max \left\{ \begin{array}{c} \sigma(\mu, \mu) + \sigma(\mu, \omega) + \vartheta(\mu) + \vartheta(\omega), \sigma(\mu, \zeta\mu) + \vartheta(\mu) + \vartheta(\zeta\mu), \\ \sigma(\omega, \omega) + \sigma(\omega, \zeta\omega) + \vartheta(\omega) + \vartheta(\zeta\omega), \dfrac{\sigma(\mu, \omega) + \vartheta(\mu) + \vartheta(\omega)}{1 + \sigma(\mu, \omega) + \vartheta(\mu) + \vartheta(\omega)}, \\ \frac{1}{2}[\sigma(\mu, \zeta\omega) + \vartheta(\mu) + \vartheta(\zeta\omega) + \sigma(\omega, \zeta\mu) + \vartheta(\omega) + \vartheta(\zeta\mu)] \end{array} \right\} \quad (3.2)$$

and

$$\mathcal{L}(\mu, \omega, \vartheta) = \max \left\{ \begin{array}{c} \sigma(\mu, \mu) + \sigma(\mu, \omega) + \vartheta(\mu) + \vartheta(\omega), \\ \sigma(\omega, \omega) + \sigma(\omega, \zeta\omega) + \vartheta(\omega) + \vartheta(\zeta\omega), \\ \dfrac{\sigma(\mu, \omega) + \vartheta(\mu) + \vartheta(\omega)}{1 + \sigma(\mu, \omega) + \vartheta(\mu) + \vartheta(\omega)} \end{array} \right\}. \quad (3.3)$$

The following is the main result of this paper.

Theorem 3.2. *Let (Θ, σ) be a σ-CMLS. If ζ is a Ćirić-Rhoades-type CP, then there exists a unique $u \in \Theta$ such that $u = \zeta u$ and $\vartheta(u) = 0$.*

Proof. Let $\mu_0 \in \Theta$ be arbitrary but fixed. We will construct a recursive sequence $\{\mu_\imath\}_{\imath \in \mathbb{N}}$ in the following manner:

$$\mu_0 = \mu \textit{ and } \mu_\imath = \zeta\mu_{\imath-1}, \textit{ for all } \imath \in \mathbb{N}.$$

It is detected that if $\mu_\imath = \mu_{\imath-1} = \zeta\mu_{\imath-1}$, for some $n \in \mathbb{N}$, then $\mu_\imath$ is a FP of ζ and the proof is finished. Hence, we assume that $\mu_\imath \neq \mu_{\imath-1}$ for all $\imath \in \mathbb{N}$.

By replacing $\mu = \mu_{\imath-1}$ and $y = \mu_\imath$ in equation (3.2), we acquire

$$\mathcal{K}(\mu_{\imath-1}, \mu_\imath, \vartheta) = \max \left\{ \begin{array}{c} \sigma(\mu_{\imath-1}, \mu_{\imath-1}) + \sigma(\mu_{\imath-1}, \mu_\imath) + \vartheta(\mu_{\imath-1}) + \vartheta(\mu_\imath), \\ \sigma(\mu_{\imath-1}, \zeta\mu_{\imath-1}) + \vartheta(\mu_{\imath-1}) + \vartheta(\zeta\mu_{\imath-1}), \\ \sigma(\mu_\imath, \mu_\imath) + \sigma(\mu_\imath, \zeta\mu_\imath) + \vartheta(\mu_\imath) + \vartheta(\zeta\mu_\imath), \\ \dfrac{\sigma(\mu_{\imath-1}, \mu_\imath) + \vartheta(\mu_{\imath-1}) + \vartheta(\mu_\imath)}{1 + \sigma(\mu_{\imath-1}, \mu_\imath) + \vartheta(\mu_{\imath-1}) + \vartheta(\mu_\imath)}, \\ \frac{1}{2}[\sigma(\mu_{\imath-1}, \zeta\mu_\imath) + \vartheta(\mu_{\imath-1}) + \vartheta(\zeta\mu_\imath) \\ +\sigma(\mu_\imath, \zeta\mu_{\imath-1}) + \vartheta(\mu_\imath) + \vartheta(\zeta\mu_{\imath-1})] \end{array} \right\}$$

$$= \max \left\{ \begin{array}{c} \sigma(\mu_{\imath-1}, \mu_{\imath-1}) + \sigma(\mu_{\imath-1}, \mu_\imath) + \vartheta(\mu_{\imath-1}) + \vartheta(\mu_\imath), \\ \sigma(\mu_{\imath-1}, \mu_\imath) + \vartheta(\mu_{\imath-1}) + \vartheta(\mu_\imath), \\ \sigma(\mu_\imath, \mu_\imath) + \sigma(\mu_\imath, \mu_{\imath+1}) + \vartheta(\mu_\imath) + \vartheta(\mu_{\imath+1}), \\ \dfrac{\sigma(\mu_{\imath-1}, \mu_\imath) + \vartheta(\mu_{\imath-1}) + \vartheta(\mu_\imath)}{1 + \sigma(\mu_{\imath-1}, \mu_\imath) + \vartheta(\mu_{\imath-1}) + \vartheta(\mu_\imath)}, \\ \frac{1}{2}[\sigma(\mu_{\imath-1}, \mu_{\imath+1}) + \vartheta(\mu_{\imath-1}) + \vartheta(\mu_{\imath+1}) \\ +\sigma(\mu_\imath, \mu_\imath) + \vartheta(\mu_\imath) + \vartheta(\mu_\imath)] \end{array} \right\}. \quad (3.4)$$

We notice that

$$\begin{aligned}
&\frac{1}{2}[\sigma(\mu_{\imath-1},\mu_{\imath+1})+\vartheta(\mu_{\imath-1})+\vartheta(\mu_{\imath+1})+\sigma(\mu_{\imath},\mu_{\imath})+\vartheta(\mu_{\imath})+\vartheta(\mu_{\imath})]\\
&\leq\frac{1}{2}[\sigma(\mu_{\imath-1},\mu_{\imath})+\sigma(\mu_{\imath},\mu_{\imath+1})+\vartheta(\mu_{\imath-1})+\vartheta(\mu_{\imath+1})+\sigma(\mu_{\imath},\mu_{\imath})+\vartheta(\mu_{\imath})+\vartheta(\mu_{\imath})]\\
&\leq\max\{\sigma(\mu_{\imath-1},\mu_{\imath})+\vartheta(\mu_{\imath-1})+\vartheta(\mu_{\imath}),\sigma(\mu_{\imath},\mu_{\imath})+\sigma(\mu_{\imath},\mu_{\imath+1})+\vartheta(\mu_{\imath})+\vartheta(\mu_{\imath+1})\}.
\end{aligned}$$

Hence, (3.4) becomes

$$\mathcal{K}(\mu_{\imath-1},\mu_{\imath},\vartheta)=\max\left\{\begin{array}{c}\sigma(\mu_{\imath-1},\mu_{\imath-1})+\sigma(\mu_{\imath-1},\mu_{\imath})+\vartheta(\mu_{\imath-1})+\vartheta(\mu_{\imath}),\\ \sigma(\mu_{\imath},\mu_{\imath})+\sigma(\mu_{\imath},\mu_{\imath+1})+\vartheta(\mu_{\imath})+\vartheta(\mu_{\imath+1})\end{array}\right\}.$$

Similarly, (3.5) reduces to

$$\begin{aligned}
\mathcal{L}(\mu_{\imath-1},\mu_{\imath},\vartheta)&=\max\left\{\begin{array}{c}\sigma(\mu_{\imath-1},\mu_{\imath-1})+\sigma(\mu_{\imath-1},\mu_{\imath})+\vartheta(\mu_{\imath-1})+\vartheta(\mu_{\imath}),\\ \sigma(\mu_{\imath},\mu_{\imath})+\sigma(\mu_{\imath},\zeta\mu_{\imath})+\vartheta(\mu_{\imath})+\vartheta(\zeta\mu_{\imath}),\\ \dfrac{\sigma(\mu_{\imath-1},\mu_{\imath})+\vartheta(\mu_{\imath-1})+\vartheta(\mu_{\imath})}{1+\sigma(\mu_{\imath-1},\mu_{\imath})+\vartheta(\mu_{\imath-1})+\vartheta(\mu_{\imath})}\end{array}\right\}\\
&=\max\left\{\begin{array}{c}\sigma(\mu_{\imath-1},\mu_{\imath-1})+\sigma(\mu_{\imath-1},\mu_{\imath})+\vartheta(\mu_{\imath-1})+\vartheta(\mu_{\imath}),\\ \sigma(\mu_{\imath},\mu_{\imath})+\sigma(\mu_{\imath},\mu_{\imath+1})+\vartheta(\mu_{\imath})+\vartheta(\mu_{\imath+1}),\\ \dfrac{\sigma(\mu_{\imath-1},\mu_{\imath})+\vartheta(\mu_{\imath-1})+\vartheta(\mu_{\imath})}{1+\sigma(\mu_{\imath-1},\mu_{\imath})+\vartheta(\mu_{\imath-1})+\vartheta(\mu_{\imath})}\end{array}\right\}.
\end{aligned}$$

Consequently, (3.1) gives

$$\begin{aligned}
&\psi_\iota(\sigma(\mu_{\imath},\mu_{\imath})+\sigma(\mu_{\imath},\mu_{\imath+1})+\vartheta(\mu_{\imath})+\vartheta(\mu_{\imath+1}))\\
&\quad=\psi_\iota(\sigma(\zeta\mu_{\imath-1},\zeta\mu_{\imath-1})+\sigma(\zeta\mu_{\imath-1},\zeta\mu_{\imath})+\vartheta(\zeta\mu_{\imath-1})+\vartheta(\zeta\mu_{\imath}))\\
&\quad\leq\psi_\iota(\mathcal{K}(\mu_{\imath-1},\mu_{\imath},\vartheta))-\phi_\iota(\mathcal{L}(\mu_{\imath-1},\mu_{\imath},\vartheta)).
\end{aligned}\tag{3.5}$$

If

$$\begin{aligned}
&\sigma(\mu_{\imath-1},\mu_{\imath-1})+\sigma(\mu_{\imath-1},\mu_{\imath})+\vartheta(\mu_{\imath-1})+\vartheta(\mu_{\imath})\\
&\quad<\sigma(\mu_{\imath},\mu_{\imath})+\sigma(\mu_{\imath},\mu_{\imath+1})+\vartheta(\mu_{\imath})+\vartheta(\mu_{\imath+1}),
\end{aligned}$$

for some positive integer $\imath$, then it follows from (3.5) that

$$\begin{aligned}
&\psi_\iota(\sigma(\mu_{\imath},\mu_{\imath})+\sigma(\mu_{\imath},\mu_{\imath+1})+\vartheta(\mu_{\imath})+\vartheta(\mu_{\imath+1}))\\
&\quad\leq\psi_\iota(\sigma(\mu_{\imath},\mu_{\imath})+\sigma(\mu_{\imath},\mu_{\imath+1})+\vartheta(\mu_{\imath})+\vartheta(\mu_{\imath+1}))\\
&\qquad-\phi_\iota(\sigma(\mu_{\imath},\mu_{\imath})+\sigma(\mu_{\imath},\mu_{\imath+1})+\vartheta(\mu_{\imath})+\vartheta(\mu_{\imath+1})),
\end{aligned}$$

which shows that

$$\phi_\iota(\sigma(\mu_{\imath},\mu_{\imath})+\sigma(\mu_{\imath},\mu_{\imath+1})+\vartheta(\mu_{\imath})+\vartheta(\mu_{\imath+1}))=0.$$

Thus,

$$\sigma(\mu_{\imath},\mu_{\imath})+\sigma(\mu_{\imath},\mu_{\imath+1})+\vartheta(\mu_{\imath})+\vartheta(\mu_{\imath+1})=0,$$

from which we notice that

$$\mu_\imath = \mu_{\imath+1}, \quad \sigma(\mu_\imath, \mu_\imath) = 0 \quad and \quad \vartheta(\mu_\imath) = \vartheta(\mu_{\imath+1}) = 0$$

which is logically impossible. Therefore,

$$\begin{aligned}&\sigma(\mu_\imath, \mu_\imath) + \sigma(\mu_\imath, \mu_{\imath+1}) + \vartheta(\mu_\imath) + \vartheta(\mu_{\imath+1}) \\ &\quad \leq \sigma(\mu_{\imath-1}, \mu_{\imath-1}) + \sigma(\mu_{\imath-1}, \mu_\imath) + \vartheta(\mu_{\imath-1}) + \vartheta(\mu_\imath)\end{aligned} \tag{3.6}$$

for all $n = 1, 2, 3, \ldots$

Hence,

$$\mathcal{K}(\mu_{\imath-1}, \mu_\imath, \vartheta) = \sigma(\mu_{\imath-1}, \mu_{\imath-1}) + \sigma(\mu_{\imath-1}, \mu_\imath) + \vartheta(\mu_{\imath-1}) + \vartheta(\mu_\imath)$$

and

$$\mathcal{L}(\mu_{\imath-1}, \mu_\imath, \vartheta) = \sigma(\mu_{\imath-1}, \mu_{\imath-1}) + \sigma(\mu_{\imath-1}, \mu_\imath) + \vartheta(\mu_{\imath-1}) + \vartheta(\mu_\imath).$$

From (3.5), we have

$$\begin{aligned}&\psi_\iota(\sigma(\mu_\imath, \mu_\imath) + \sigma(\mu_\imath, \mu_{\imath+1}) + \vartheta(\mu_\imath) + \vartheta(\mu_{\imath+1})) \\ &\quad \leq \psi_\iota(\sigma(\mu_{\imath-1}, \mu_{\imath-1}) + \sigma(\mu_{\imath-1}, \mu_\imath) + \vartheta(\mu_{\imath-1}) + \vartheta(\mu_\imath)) \\ &\qquad - \phi_\iota(\sigma(\mu_{\imath-1}, \mu_{\imath-1}) + \sigma(\mu_{\imath-1}, \mu_\imath) + \vartheta(\mu_{\imath-1}) + \vartheta(\mu_\imath)).\end{aligned} \tag{3.7}$$

It follows from (3.6) that the sequence $\{\sigma(\mu_\imath, \mu_\imath) + \sigma(\mu_\imath, \mu_{\imath+1}) + \vartheta(\mu_\imath) + \vartheta(\mu_{\imath+1})\}$ is bounded below and non-increasing. Therefore, $\sigma(\mu_\imath, \mu_\imath) + \sigma(\mu_\imath, \mu_{\imath+1}) + \vartheta(\mu_\imath) + \vartheta(\mu_{\imath+1}) \to \varrho$ as $\imath \to \infty$, for some $\varrho \geq 0$.

Assume that $\varrho > 0$. Using the continuity of ψ_ι, the lower semi-continuity of ϕ_ι, and as $\imath \to \infty$ in (3.7), lead to

$$\begin{aligned}\psi_\iota(\varrho) &\leq \psi_\iota(\varrho) \ - \liminf_{\imath\to\infty} \ \phi_\iota(\sigma(\mu_{\imath-1}, \mu_{\imath-1}) + \sigma(\mu_{\imath-1}, \mu_\imath) + \vartheta(\mu_{\imath-1}) + \vartheta(\mu_\imath)) \\ &\leq \psi_\iota(\varrho) \ - \phi_\iota(\varrho) < \psi_\iota(\varrho),\end{aligned}$$

which is logically impossible. Thus, $\lim_{\imath\to\infty}(\sigma(\mu_\imath, \mu_\imath) + \sigma(\mu_\imath, \mu_{\imath+1}) + \vartheta(\mu_\imath) + \vartheta(\mu_{\imath+1})) = 0$, from which we have

$$\lim_{\imath\to\infty} \sigma(\mu_\imath, \mu_\imath) = \lim_{\imath\to\infty} \sigma(\mu_\imath, \mu_{\imath+1}) = 0 \tag{3.8}$$

and

$$\lim_{\imath\to\infty} \vartheta(\mu_\imath) = \lim_{\imath\to\infty} \vartheta(\mu_{\imath+1}) = 0. \tag{3.9}$$

Now, we prove that the sequence $\{\mu_\imath\}_{\imath\in\mathbb{N}}$ is Cauchy. Assume that $\{\mu_\imath\}_{\imath\in\mathbb{N}}$ is not Cauchy. Then by Lemma 2.11, there exist $\epsilon^+ > 0$ and subsequences $\{\mu_{\imath(\ell)}\}_{\ell\in\mathbb{N}}$ and $\{\mu_{\jmath(\ell)}\}_{\ell\in\mathbb{N}}$ of $\{\mu_\imath\}_{\imath\in\mathbb{N}}$ such that (2.2) and (2.3) hold.

From (3.2), we have

$$\mathcal{K}(\mu_{\imath(\ell)}, \mu_{\jmath(\ell)}, \vartheta) = \max\left\{\begin{array}{c}\sigma(\mu_{\imath(\ell)}, \mu_{\imath(\ell)}) + \sigma(\mu_{\imath(\ell)}, \mu_{\jmath(\ell)}) + \vartheta(\mu_{\imath(\ell)}) + \vartheta(\mu_{\jmath(\ell)}), \\ \sigma(\mu_{\imath(\ell)}, \zeta\mu_{\imath(\ell)}) + \vartheta(\mu_{\imath(\ell)}) + \vartheta(\zeta\mu_{\imath(\ell)}), \\ \sigma(\mu_{\imath(\ell)}, \mu_{\imath(\ell)}) + \sigma(\mu_{\jmath(\ell)}, \zeta\mu_{\jmath(\ell)}) + \vartheta(\mu_{\jmath(\ell)}) + \vartheta(\zeta\mu_{\jmath(\ell)}), \\ \dfrac{\sigma(\mu_{\imath(\ell)}, \mu_{\jmath(\ell)}) + \vartheta(\mu_{\imath(\ell)}) + \vartheta(\mu_{\jmath(\ell)})}{1 + \sigma(\mu_{\imath(\ell)}, \mu_{\jmath(\ell)}) + \vartheta(\mu_{\imath(\ell)}) + \vartheta(\mu_{\jmath(\ell)})}, \\ \frac{1}{2}[\sigma(\mu_{\imath(\ell)}, \zeta\mu_{\jmath(\ell)}) + \vartheta(\mu_{\imath(\ell)}) + \vartheta(\zeta\mu_{\jmath(\ell)}) \\ +\sigma(\mu_{\jmath(\ell)}, \zeta\mu_{\imath(\ell)}) + \vartheta(\mu_{\jmath(\ell)}) + \vartheta(\zeta\mu_{\imath(\ell)})]\end{array}\right\}$$

$$= \max \left\{ \begin{array}{c} \sigma(\mu_{\imath(\ell)}, \mu_{\imath(\ell)}) + \sigma(\mu_{\imath(\ell)}, \mu_{\jmath(\ell)}) + \vartheta(\mu_{\imath(\ell)}) + \vartheta(\mu_{\jmath(\ell)}), \\ \sigma(\mu_{\imath(\ell)}, \mu_{\imath(\ell+1)}) + \vartheta(\mu_{\imath(\ell)}) + \vartheta(\mu_{\imath(\ell+1)}), \\ \sigma(\mu_{\jmath(\ell)}, \mu_{\jmath(\ell)}) + \sigma(\mu_{\jmath(\ell)}, \mu_{\jmath(\ell+1)}) + \vartheta(\mu_{\jmath(\ell)}) + \vartheta(\mu_{\jmath(\ell+1)}), \\ \dfrac{\sigma(\mu_{\imath(\ell)}, \mu_{\jmath(\ell)}) + \vartheta(\mu_{\imath(\ell)}) + \vartheta(\mu_{\jmath(\ell)})}{1 + \sigma(\mu_{\imath(\ell)}, \mu_{\jmath(\ell)}) + \vartheta(\mu_{\imath(\ell)}) + \vartheta(\mu_{\jmath(\ell)})}, \\ \frac{1}{2}[\sigma(\mu_{\imath(\ell)}, \mu_{\jmath(\ell+1)}) + \vartheta(\mu_{\imath(\ell)}) + \vartheta(\mu_{\jmath(\ell+1)}) \\ +\sigma(\mu_{\jmath(\ell)}, \mu_{\imath(\ell+1)}) + \vartheta(\mu_{\jmath(\ell)}) + \vartheta(\mu_{\imath(\ell+1)})]\} \end{array} \right\}. \tag{3.10}$$

As $\ell \to \infty$ in (3.10), applying Lemma 2.11 and using equations (3.8) and (3.9) yield

$$\lim_{\ell\to\infty} \mathcal{K}(\mu_{\imath(\ell)}, \mu_{\jmath(\ell)}, \vartheta) = \epsilon^+. \tag{3.11}$$

In similar steps, it follows from (3.3) that

$$\mathcal{L}(\mu_{\imath(\ell)}, \mu_{\jmath(\ell)}, \vartheta) = \max \left\{ \begin{array}{c} \sigma(\mu_{\imath(\ell)}, \mu_{\imath(\ell)}) + \sigma(\mu_{\imath(\ell)}, \mu_{\jmath(\ell)}) + \vartheta(\mu_{\imath(\ell)}) + \vartheta(\mu_{\jmath(\ell)}), \\ \sigma(\mu_{\jmath(\ell)}, \mu_{\jmath(\ell)}) + \sigma(\mu_{\jmath(\ell)}, \zeta\mu_{\jmath(\ell)}) \\ +\vartheta(\mu_{\jmath(\ell)}) + \vartheta(\zeta\mu_{\jmath(\ell)}), \\ \dfrac{\sigma(\mu_{\imath(\ell)}, \mu_{\jmath(\ell)}) + \vartheta(\mu_{\imath(\ell)}) + \vartheta(\mu_{\jmath(\ell)})}{1 + \sigma(\mu_{\imath(\ell)}, \mu_{\jmath(\ell)}) + \vartheta(\mu_{\imath(\ell)}) + \vartheta(\mu_{\jmath(\ell)})} \end{array} \right\}$$

$$= \max \left\{ \begin{array}{c} \sigma(\mu_{\imath(\ell)}, \mu_{\imath(\ell)}) + \sigma(\mu_{\imath(\ell)}, \mu_{\jmath(\ell)}) + \vartheta(\mu_{\imath(\ell)}) + \vartheta(\mu_{\jmath(\ell)}), \\ \sigma(\mu_{\jmath(\ell)}, \mu_{\jmath(\ell)}) + \sigma(\mu_{\jmath(\ell)}, \mu_{\jmath(\ell+1)}) \\ +\vartheta(\mu_{\jmath(\ell)}) + \vartheta(\mu_{\jmath(\ell+1)}), \\ \dfrac{\sigma(\mu_{\imath(\ell)}, \mu_{\jmath(\ell)}) + \vartheta(\mu_{\imath(\ell)}) + \vartheta(\mu_{\jmath(\ell)})}{1 + \sigma(\mu_{\imath(\ell)}, \mu_{\jmath(\ell)}) + \vartheta(\mu_{\imath(\ell)}) + \vartheta(\mu_{\jmath(\ell)})} \end{array} \right\}.$$

Thus,

$$\lim_{\ell\to\infty} \mathcal{L}(\mu_{\imath(\ell)}, \mu_{\jmath(\ell)}, \vartheta) = \epsilon^+. \tag{3.12}$$

From (3.1), we have

$$\begin{aligned} &\psi_\iota(\sigma(\mu_{\imath(\ell+1)}, \mu_{\imath(\ell+1)}) + \sigma(\mu_{\imath(\ell+1)}, \mu_{\jmath(\ell+1)}) + \vartheta(\mu_{\imath(\ell+1)}) + \vartheta(\mu_{\jmath(\ell+1)})) \\ &\quad \le \psi_\iota(\mathcal{K}(\mu_{\imath(\ell)}, \mu_{\jmath(\ell)}, \vartheta)) - \phi_\iota(\mathcal{L}(\mu_{\imath(\ell)}, \mu_{\jmath(\ell)}, \vartheta)). \end{aligned} \tag{3.13}$$

Letting $\ell \to \infty$ in (3.13), and applying Lemma 2.11, the continuity of ψ_ι, the lower semi-continuity of ϕ_ι, and by using equations (3.8) (3.9), (3.11) and (3.12), we acquire $\psi_\iota(\epsilon^+) \le \psi_\iota(\epsilon^+) - \phi_\iota(\epsilon^+)$, a contradiction because $\phi_\iota(\epsilon^+) > 0$. Therefore, $\{\mu_\imath\}_{\imath\in\mathbb{N}}$ is a Cauchy sequence. The completeness of Θ implies that there exists $u \in \Theta$ such that $\lim_{\imath\to\infty} \mu_\imath = u$. Given that ϕ_ι is lower semi-continuous, $\vartheta(u) \le \liminf_{\imath\to\infty} \vartheta(\mu_\imath) \le \lim_{\imath\to\infty} \vartheta(\mu_\imath) = 0$, from which it follows that $\vartheta(u) = 0$.

Now, from (3.2), we acquire

$$\mathcal{K}(\mu_\imath, u, \vartheta) = \max \left\{ \begin{array}{c} \sigma(\mu_\imath, \mu_\imath) + \sigma(\mu_\imath, u) + \vartheta(\mu_\imath) + \vartheta(u), \\ \sigma(\mu_\imath, \zeta\mu_\imath) + \vartheta(\mu_\imath) + \vartheta(\zeta\mu_\imath), \\ \sigma(u, u) + \sigma(u, \zeta u) + \vartheta(u) + \vartheta(\zeta u), \\ \dfrac{\sigma(\mu_\imath, u) + \vartheta(\mu_\imath) + \vartheta(u)}{1 + \sigma(\mu_\imath, u) + \vartheta(\mu_\imath) + \vartheta(u)}, \\ \frac{1}{2}[\sigma(\mu_\imath, \zeta u) + \vartheta(\mu_\imath) + \vartheta(\zeta u) \\ +\sigma(u, \zeta\mu_\imath) + \vartheta(u) + \vartheta(\zeta\mu_\imath)] \end{array} \right\}$$

$$= \max \left\{ \begin{array}{c} \sigma(\mu_\imath, \mu_\imath) + \sigma(\mu_\imath, u) + \vartheta(\mu_\imath) + \vartheta(u), \\ \sigma(\mu_\imath, \mu_{\imath+1}) + \vartheta(\mu_\imath) + \vartheta(\mu_{\imath+1}), \\ \sigma(u, u) + \sigma(u, \zeta u) + \vartheta(u) + \vartheta(\zeta u), \\ \dfrac{\sigma(\mu_\imath, u) + \vartheta(\mu_\imath) + \vartheta(u)}{1 + \sigma(\mu_\imath, u) + \vartheta(\mu_\imath) + \vartheta(u)}, \\ \frac{1}{2}[\sigma(\mu_\imath, \zeta u) + \vartheta(\mu_\imath) + \vartheta(\zeta u) \\ +\sigma(u, \mu_{\imath+1}) + \vartheta(u) + \vartheta(\mu_{\imath+1})] \end{array} \right\},$$

from which we have

$$\lim_{\imath\to\infty} \mathcal{K}(\mu_\imath, u, \vartheta) = \max\left\{\sigma(u,u), \sigma(u,u) + \sigma(u,\zeta u) + \vartheta(\zeta u), \frac{1}{2}[\sigma(u,\zeta u) + \vartheta(\zeta u)]\right\}$$

$$= \sigma(u,u) + \sigma(u,\zeta u) + \vartheta(\zeta u). \tag{3.14}$$

In like manner, we have

$$\lim_{\imath\to\infty} \mathcal{L}(\mu_\imath, u, \vartheta) = \lim_{\imath\to\infty} \max \left\{ \begin{array}{c} \sigma(\mu_\imath, \mu_\imath) + \sigma(\mu_\imath, u) + \vartheta(\mu_\imath) + \vartheta(u), \\ \sigma(u, u) + \sigma(u, \zeta u) + \vartheta(u) + \vartheta(\zeta u), \\ \dfrac{\sigma(\mu_\imath, u) + \vartheta(\mu_\imath) + \vartheta(u)}{1 + \sigma(\mu_\imath, u) + \vartheta(\mu_\imath) + \vartheta(u)} \end{array} \right\}$$

$$\begin{aligned} &= \max\{\sigma(u,u), \sigma(u,u) + \sigma(u,\zeta u) + \vartheta(\zeta u)\} \\ &= \sigma(u,u) + \sigma(u,\zeta u) + \vartheta(\zeta u). \end{aligned} \tag{3.15}$$

Therefore, from (3.1), we have

$$\begin{aligned} &\psi_\iota(\sigma(\mu_{\imath+1}, \mu_{\imath+1}) + \sigma(\mu_{\imath+1}, \zeta u) + \vartheta(\mu_{\imath+1}) + \vartheta(\zeta u)) \\ &\quad = \psi_\iota(\sigma(\zeta\mu_\imath, \zeta\mu_\imath) + \sigma(\zeta\mu_\imath, \zeta u) + \vartheta(\zeta\mu_\imath) + \vartheta(\zeta u)) \\ &\quad \le \psi_\iota(\mathcal{K}(\mu_\imath, u, \vartheta)) - \phi_\iota(\mathcal{L}(\mu_\imath, u, \vartheta)). \end{aligned} \tag{3.16}$$

Letting $\imath \to \infty$ in (3.16) and utilizing the continuity of ψ_ι, the lower continuity of ϕ_ι and using equations (3.14) and (3.13), we have

$$\begin{aligned} &\psi_\iota(\sigma(u,u) + \sigma(u,\zeta u) + \vartheta(\zeta u)) \\ &\quad \le \psi_\iota(\sigma(u,u) + \sigma(u,\zeta u) + \vartheta(\zeta u)) - \phi_\iota(\sigma(u,u) + \sigma(u,\zeta u) + \vartheta(\zeta u)). \end{aligned} \tag{3.17}$$

The expression (3.17) implies that $\phi_\iota(\sigma(u,u) + \sigma(u,\zeta u) + \vartheta(\zeta u)) = 0$ and hence, $\sigma(u,u) + \sigma(u,\zeta u) + \vartheta(\zeta u) = 0$. Therefore, $\sigma(u,u) = 0$, $u = \zeta u$ and $\vartheta(\zeta u) = 0$.

To show that the FP of ζ is unique, suppose that p is another FP of ζ with $\mu = u$ and $\omega = p$. Then, $p = \zeta p$ and $\vartheta(p) = 0$. Now, using (3.1), we have

$$\begin{aligned} &\psi_\iota(\sigma(u,u) + \sigma(u,p)) + \vartheta(u) + \vartheta(p)) \\ &= \psi_\iota(\sigma(\zeta u, \zeta u) + \sigma(\zeta u, \zeta p) + \vartheta(\zeta u) + \vartheta(\zeta p)) \\ &\le \psi_\iota(\mathcal{K}(u, p, \vartheta)) - \phi_\iota(\mathcal{L}(u, p, \vartheta)) \\ &= \psi_\iota(\sigma(u,u) + \sigma(u,p)) - \phi_\iota(\sigma(u,u) + \sigma(u,p)). \end{aligned} \tag{3.18}$$

From, (3.18), it is clear that $\psi_\iota(\sigma(u,p)) \le \psi_\iota(u,p)) - \phi_\iota(\sigma(u,p))$. From which $\phi_\iota(\sigma(u,p)) \le 0$. But $0 \le \phi_\iota(\sigma(u,p)) \le 0$. That is, $\phi_\iota(\sigma(u,p)) = 0$. Hence, by the defining property of ϕ_ι, $\sigma(u,p) = 0$. Therefore, condition (σ_1) of metric-like reveals that $u = p$. ∎

We construct the following example to support the hypotheses of Theorem 3.2.

Example 3.3. Let $\Theta = \mathbb{R}_+$ and $\sigma(\mu, \omega) = \mu + \omega$ for all $\mu, \omega \in \Theta$. Then obviously σ is a metric-like on Θ and (Θ, σ) is complete. Also, σ is not a metric on Θ, since $\sigma(1, 1) = 2 \neq 0$. Define the self-mapping ζ by $\zeta\mu = \frac{\mu}{2}$ for all $\mu \in \Theta$. To see that ζ is a Ćirić-Rhoades-type CP, let $\psi_\iota(t) = t$, $\phi_\iota(t) = \frac{t}{2}$ and $\vartheta(t) = t^2$. Let $\mu, \omega \in \mathbb{R}_+$, then

$$\begin{aligned}
&\psi_\iota(\sigma(\zeta\mu, \zeta\mu) + \sigma(\zeta\mu, \zeta\omega) + \vartheta(\zeta\mu) + \vartheta(\zeta\omega)) \\
&= \frac{\mu}{2} + \frac{\mu}{2} + \frac{\mu}{2} + \frac{\omega}{2} + \left(\frac{\mu}{2}\right)^2 + \left(\frac{\omega}{2}\right)^2 \\
&= \frac{3\mu}{2} + \frac{\omega}{2} + \frac{\mu^2}{4} + \frac{\omega^2}{4} \\
&< \frac{3\mu}{2} + \frac{\omega}{2} + \frac{\mu^2}{2} + \frac{\omega^2}{2} \\
&= \frac{1}{18}\left(27\mu + 9\omega + 9\mu^2 + 9\omega^2\right) \\
&= \frac{1}{18}\left(54\mu + 18\omega + 18\mu^2 + 18\omega^2 - 27\mu - 9\omega - 9\mu^2 - 9\omega^2\right) \\
&= \frac{1}{18}\left(54\mu + 18\omega + 18\mu^2 + 18\omega^2\right) - \frac{1}{18}\left(27\mu + 9\omega + 9\mu^2 + 9\omega^2\right) \\
&= 3\mu + \omega + \mu^2 + \omega^2 - \frac{1}{2}\left(3\mu + \omega + \mu^2 + \omega^2\right) \\
&= \mu + \mu + \mu + \omega + \mu^2 + \omega^2 - \frac{1}{2}\left(\mu + \mu + \mu + \omega + \mu^2 + \omega^2\right) \\
&= \sigma(\mu, \mu) + \sigma(\mu, \omega) + \vartheta(\mu) + \vartheta(\omega) - \frac{1}{2}\left(\sigma(\mu, \mu) + \sigma(\mu, \omega) + \vartheta(\mu) + \vartheta(\omega)\right) \\
&= \psi_\iota(\max\{\mathcal{K}(\mu, \omega, \vartheta)\}) - \phi_\iota(\max\{\mathcal{L}(\mu, \omega, \vartheta)\}).
\end{aligned}$$

Hence, all the hypotheses of Theorem 3.2 are satisfied. We can see therefore that the mapping ζ has a unique fixed in Θ. However, since σ is not a metric on Θ, then the corresponding results in [21, 9] are not applicable/useful in this example to find a FP of ζ.

Take

$$\mathcal{C}(\mu, \omega, \vartheta) = \max\left\{\begin{array}{c} \sigma(\mu, \mu) + \sigma(\mu, \omega) + \vartheta(\mu) + \vartheta(\omega), \sigma(\mu, \zeta\mu) + \vartheta(\mu) + \vartheta(\zeta\mu), \\ \sigma(\omega, \omega)\sigma(\omega, \zeta\omega) + \vartheta(\omega) + \vartheta(\zeta\omega), \\ \frac{1}{2}[\sigma(\mu, \zeta\omega) + \vartheta(\mu) + \vartheta(\zeta\omega) + \sigma(\omega, \zeta\mu) + \vartheta(\omega) + \vartheta(\zeta\mu)] \end{array}\right\},$$

for all $\mu, \omega \in \Theta$, where $\psi_\iota \in \Psi_\iota$ and $\phi_\iota, \vartheta \in \Phi_\iota$. In what follows, we present some consequences of Theorem 3.2.

Corollary 3.4. *[8] Let (Θ, σ) be a σ-CMLS. Suppose that the self-mapping ζ satisfies the following inequality:*

$$\begin{aligned}
&\psi_\iota(\sigma(\zeta\mu, \zeta\mu) + \sigma(\zeta\mu, \zeta\omega) + \vartheta(\zeta\mu) + \vartheta(\zeta\omega)) \\
&\leq \psi_\iota(\mathcal{C}(\mu, \omega, \vartheta)) - \phi_\iota(\mathcal{C}(\mu, \omega, \vartheta)),
\end{aligned} \tag{3.19}$$

for all $\mu, \omega \in \Theta$, where $\psi_\iota \in \Psi_\iota$ and $\phi_\iota, \vartheta \in \Phi_\iota$. Then, there exists a unique FP $u \in \Theta$ of ζ and $\vartheta(u) = 0$.

Corollary 3.5. *Let (Θ, σ) be a σ-CMLS. Suppose that the self-mapping ζ satisfies the following inequality:*

$$\begin{aligned}&\psi_\iota(\sigma(\zeta\mu, \zeta\mu) + \sigma(\zeta\mu, \zeta\omega) + \vartheta(\zeta\mu) + \vartheta(\zeta\omega))\\ &\leq \psi_\iota(\mathcal{K}(\mu, \omega, \vartheta)) - \phi_\iota(\mathcal{K}(\mu, \omega, \vartheta)),\end{aligned} \tag{3.20}$$

for all $\mu, \omega \in \Theta$, where $\psi_\iota \in \Psi_\iota$ and $\phi_\iota, \vartheta \in \Phi_\iota$. Then, there exists a unique FP $u \in \Theta$ of ζ and $\vartheta(u) = 0$.

By taking $\phi_\iota(t) = 0$, for all $t \in \mathbb{R}^+$ in Theorem 3.2, we have the next result.

Corollary 3.6. *Let (Θ, σ) be a σ-CMLS. Suppose that the self-mapping ζ satisfies the following inequality:*

$$\psi_\iota(\sigma(\zeta\mu, \zeta\mu) + \sigma(\zeta\mu, \zeta\omega) + \vartheta(\zeta\mu) + \vartheta(\zeta\omega)) \leq \psi_\iota(\mathcal{K}(\mu, \omega, \vartheta)), \tag{3.21}$$

for all $\mu, \omega \in \Theta$, where $\psi_\iota \in \Psi_\iota$. Then, there exists a unique FP of ζ, say $u \in \Theta$ and $\vartheta(u) = 0$.

Corollary 3.7. *Let (Θ, σ) be a σ-CMLS. Suppose that the self-mapping ζ satisfies the following inequality:*

$$\begin{aligned}&\psi_\iota(\sigma(\zeta\mu, \zeta\mu) + \sigma(\zeta\mu, \zeta\omega) + \vartheta(\zeta\mu) + \vartheta(\zeta\omega))\\ &\leq \psi_\iota(\sigma(\mu, \mu) + \sigma(\mu, \omega) + \vartheta(\mu) + \vartheta(\omega)) - \phi_\iota(\sigma(\mu, \mu) + \sigma(\mu, \omega) + \vartheta(\mu) + \vartheta(\omega)),\end{aligned}$$

for all $\mu, \omega \in \Theta$, where $\psi_\iota \in \Psi_\iota$ and $\phi_\iota, \vartheta \in \Phi_\iota$. Then, there exists a unique FP of ζ, say $u \in \Theta$ and $\vartheta(u) = 0$.

Corollary 3.8. *Let (Θ, σ) be a σ-CMLS. Suppose that the self-mapping ζ satisfies the following inequality:*

$$\begin{aligned}&\psi_\iota(\sigma(\zeta^j\mu, \zeta^j\mu) + \sigma(\zeta^j\mu, \zeta^j\omega) + \vartheta(\zeta^j\mu) + \vartheta(\zeta^j\omega))\\ &\leq \psi_\iota(\mathcal{K}(\mu, \omega, \vartheta)) - \phi_\iota(\mathcal{L}(\mu, \omega, \vartheta)),\end{aligned}$$

for all $\mu, \omega \in \Theta$, where $\psi_\iota \in \Psi_\iota$, $\phi_\iota, \vartheta \in \Phi_\iota$ and j is a positive integer. Then, there exists a unique FP of ζ, say $u \in \Theta$ and $\vartheta(u) = 0$.

Proof. Let $S = \zeta^j$. Then by Theorem 3.2, S has a unique FP, say $u \in \Theta$. Then $\zeta^j u = Su = u$ and $\vartheta(u) = \vartheta(Su) = \vartheta(\zeta^j u) = 0$. Since $\zeta^{j+1}u = \zeta u$, $S\zeta u = \zeta^j(\zeta u) = \zeta^{j+1}u = \zeta u$ and so ζu is a FP of S. By the uniqueness of a FP of S, $\zeta u = u$. ∎

4. Conclusion

In this manuscript, the idea of Ćirić-Rhoades CPs in an MLS is introduced, and conditions for the existence of fixed points for such mappings are investigated. Non-trivial comparative examples have been presented to illustrate the proposed ideas and to show that they are indeed generalizations of a few concepts in the literature. By extending the concepts from metric domains to the framework of dislocated metrics, the result produced in this paper contributes to the development of fixed point theory and serves as a foundation for further study.

The scope of the main ideas in this work is constrained by the fact that the problem formulation, analysis, and conclusion described here are all abstract.

Competing Interests

The authors declare that they have no competing interests.

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