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THE PROPOSITION FOR AN CAK-GENERALIZED NONEXPANSIVE MAPPING IN HILBERT SPACES

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Abstract In this paper, we introduce a new class of nonexpansive type of mapping namely, CAK-generalized nonexpansive mapping, which is more general than an AK-generalized nonexpansive mapping and α-nonexpansive mapping. Then, we obtain the proposition of the approximation method for an CAK-generalized nonexpansive in Hilbert spaces.

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1. Introduction

In 2011, Aoyama and Kohsaka [2] introduced the class of α-nonexpansive mappings in Banach spaces as follows: Let E be a Banach space and let C be a nonempty subset of E. A mapping $\varphi : C \to E$ is said to be α-nonexpansive for some real number $0 \leq \alpha < 1$ if

$$\|\varphi\mu - \varphi\nu\| \leq \alpha||\varphi\mu - \nu|| + \alpha||\varphi\nu - \mu|| + (1 - 2\alpha)||\mu - \nu||,$$

for all $\mu, \nu \in C$. Clearly, 0-nonexpansive maps are exactly nonexpansive maps. This mapping was generalized and extanded by many authors in several directions; see for instance [3, 4] and references therein.

One of the most interesting iteration processes is the viscosity approximation method introduced by Moudafi [5]. In 2004, Xu [6] studied such method for a nonexpansive mapping in a Hilbert space and introduced an iterative scheme for finding the set of fixed points of a nonexpansive mapping in a Hilbert space. Over the past few decades, the convergence theorem was extended and improved in many directions (see [7], [8]) due to its applications are desirable and can be used in real-world applications. So, many authors have been trying to construct new iterations to prove strong convergence theorems for nonexpansive semigroups; see for instance [9–11] and references therein. Especially, in 2008, Song and Xu [12] introduced the following implicit and explicit viscosity iterative schemes Very recently, Song *et al.* [13] proved a strong convergence theorem of the Halpern iteration for an α-nonexpansive semigroup in Hilbert spaces under suitable conditions as the following schemes. Moreover, they also proved some strong convergence theorems of Halperns iteration defined by a such iterative method for a family $\{\varphi_n\}$ of α-nonexpansive mappings.

In 2021, Suanoom and Khuangsatung [16] , we introduced a new class of nonexpansive type of mapping namely, *AK-generalized nonexpansive mapping*, which is more general than an α-nonexpansive mapping in Hilbert spaces as follow.

Definition 1.1. Let C be a nonempty closed convex subset of a Hilbert space H. A mapping $\varphi : C \to C$ is said to satisfy condition (AK) (or AK-generalized nonexpansive) for some real numbers $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ with $\max\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} < 1$ if

$$\begin{aligned}||\varphi\mu - \varphi\nu|| \leq \alpha_1||\varphi\mu - \mu|| + \alpha_2||\varphi\nu - \nu|| + \alpha_3||\varphi\mu - \nu|| + \alpha_4||\varphi\nu - \mu|| \\ + (1 - 4\max\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\})||\mu - \nu||, \end{aligned} \tag{1.1}$$

for all $\mu, \nu \in C$.

Notice that the class of AK-generalized nonexpansive mappings covers several well-known mappings. For example, every α-nonexpansive mappings is an AK-generalized nonexpansive mapping and also 0-nonexpansive maps are exactly nonexpansive maps. Hence we have the following diagram.

The following example shows that the reverse implication does not hold.

Example 1.2. [14] Let $X = \{(0, 0), (2, 0)(0, 4), (4, 0), (4, 5), (5, 4)\}$ be a subset of $\mathbb{R}^2$ with dictionary order. Define a inner product $(X, \langle\cdot, \cdot\rangle = ||\cdot, \cdot||)$. by $||\mu_1, \mu_2|| = (|\mu_1| + |\mu_1|)^2$. Then $(X, \langle\cdot, \cdot\rangle)$ is a Hilbert space. Define a mapping $\varphi : X \to X$ by

$$\varphi(0, 0) = (0, 0),\ \varphi(2, 0) = (0, 0),\ \varphi(0, 4) = (0, 0),$$
$$\varphi(4, 0) = (2, 0),\ \varphi(4, 5) = (4, 0),\ \varphi(5, 4) = (0, 4).$$

Then, we have φ is not an α-nonexpansive mapping but, φ is an AK-generalized nonexpansive.

Example 1.3. [16] Let $X = [0, 2]$ be a nonempty closed convex subset of a Hilbert space $(H = \mathbb{R}, \langle \cdot, \cdot \rangle = |\cdot|)$. Suppose that $\varphi : [0, 2] \to [0, 2]$ be given by $\varphi\mu = \sin\mu + \cos\mu$, for all $\mu \in [0, 2]$. Then φ is an AK-generalized nonexpansive.

In this paper, we introduce a new class of nonexpansive type of mapping namely, CAK-generalized nonexpansive mapping, which is more general than an AK-generalized nonexpansive mapping and α-nonexpansive mapping. Then, we obtain the proposition of the approximation method for an CAK-generalized nonexpansive in Hilbert spaces.

2. Preliminaries

Throughout this article, let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H. Let $\varphi : C \to C$ be a nonlinear mapping. A point $\mu \in C$ is called a *fixed point* of φ if $\varphi\mu = \mu$. The set of fixed points of φ is the set $F(\varphi) := \{\mu \in C : \varphi\mu = \mu\}$. The mapping $\varphi : C \to C$ is said to be nonexpansive if $\|\varphi\mu - \varphi\nu\| \le ||\mu - \nu||$ for any $\mu, \nu \in C$. In 1965, Browder [1] shown that if a nonexpansive mapping $\varphi : H \to H$ of a Hilbert space H into itself is asymptotically regular and has at least one fixed point then, for any $\mu \in H$, a weak limit of a weakly convergent subsequence of the sequence of successive approximations $\varphi^n\mu$ is a fixed point of φ. And, let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. Let C be a nonempty closed convex subset of H. Recall that the (nearest point) projection P_C from H onto C assigns to each $\mu \in H$, there exists the unique point $P_C x \in C$ satisfying the property

$$\|\mu - P_C x\| = \min_{\nu \in C} \|\mu - \nu\|.$$

For any $\mu \in H$ and $\nu \in C$. Then, $P_C\mu = \nu$ if and only if there holds the inequality

$$\langle \mu - \nu, \nu - \omega \rangle \ge 0, \forall \omega \in C.$$

In a real Hilbert space H, it is well known that H satisfies *Opial's condition*, *i.e.*, for any sequence $\{\mu_n\}$ with $\mu_n \rightharpoonup \mu$, the inequality

$$\lim_{n\to\infty} \inf \|\mu_n - \mu\| < \lim_{n\to\infty} \inf \|\mu_n - \nu\|,$$

holds for every $\nu \in H$ with $\nu \ne \mu$.

Lemma 2.1. [15] Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying

$$s_{n+1} \le (1 - \alpha_n)s_n + \delta_n, \forall n \in \mathbb{N},$$

where α_n is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence such that

$$(1) \sum_{n=1}^{\infty} \alpha_n = \infty, \ (2) \limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \le 0 \text{ or } \sum_{n=1}^{\infty} |\delta_n| < \infty.$$

Then, $\lim_{n\to\infty} s_n = 0$.

Lemma 2.2. Let H be a real Hilbert space. Then

$$\|\mu + \nu\|^2 \le \|\mu\|^2 + 2\langle \nu, \mu + \nu \rangle,$$

for all $\mu, \nu \in H$.

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Now, we introduce the main definitions follow:

Definition 2.3. Let C be a nonempty closed convex subset of a Hilbert space H. A mapping $\varphi, \chi : C \to C$ is said to satisfy condition (CAK) (or common AK-generalized nonexpansive) for some real numbers $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ with $\max\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} < 1$ if

$$\begin{aligned} ||\varphi\mu - \chi\nu|| &\leq \alpha_1||\varphi\mu - \mu|| + \alpha_2||\chi\nu - \nu|| + \alpha_3||\varphi\mu - \nu|| + \alpha_4||\chi\nu - \mu|| \\ &\quad + (1 - 4\max\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\})||\mu - \nu||, \end{aligned} \tag{2.1}$$

for all $\mu, \nu \in C$.

Now, we introduce the definitions follow on the results of Song *et al.* [13]. Let E be a Banach space. An (one-parameter) CAK-generalized nonexpansive semigroup is a family $\varphi = \{\varphi(t) : t > 0\}$ of mappings $D(\varphi) = \bigcap_{t>0} D(\varphi(t))$ and range $R(\varphi)$ such that

(1): $\varphi(0)\mu = \mu$ for all $\mu \in D(\varphi)$;
(2): $\varphi(t+s)x = \varphi(t)\varphi(s)x$ for all $t, s > 0$ and $x \in D(\varphi)$;
(3): for each $t > 0$, $T(t)$ is an CAK-generalized nonexpansive mapping.

Example 2.4. Let $X = [0, 2]$ be a nonempty closed convex subset of a Hilbert space $(H = \mathbb{R}, \langle \cdot, \cdot \rangle = |\cdot|)$. Suppose that $\varphi, \chi : [0, 2] \to [0, 2]$ be given by $\varphi\mu = 3^{-\mu}$, $\chi\mu = 5^{-\mu}$, for all $\mu \in [0, 2]$. Now, for any $t, s > 0$ and $\mu \in D(\varphi)$;
(1) $\varphi(0)x = 3^0\mu = \mu$, $\chi(0)x = 5^0\mu = \mu$;
(2) $\varphi(t+s)x = 3^{-(t+s)}\mu = 3^{-(t)}3^{-(s)}\mu = \varphi(t)\varphi(s)\mu$
$\varphi(t+s)x = 5^{-(t+s)}\mu = 5^{-(t)}5^{-(s)}\mu = \varphi(t)\varphi(s)\mu$;
(3) for each $t > 0$, $\varphi(t)$ is an CAK-generalized nonexpansive mapping, that is,

$$\begin{aligned} \|\varphi\mu - \varphi\nu\| &= |3^{-x} - 5^{-y}| \\ &= \frac{1}{2}|2(3^{-x} - 5^{-y})| \\ &= \frac{1}{2}|(3^{-x} - 5^{-y}) + (3^{-x} - 5^{-y}) + x - x + y - y| \\ &= \frac{1}{2}|3^{-x} - x - 5^{-y} + y + 3^{-x} - y - 5^{-y} + x| \\ &= \frac{1}{2}|(3^{-x} - x) - (5^{-y} - y) + (3^{-x} - y) - (5^{-y} - x)| \\ &\leq \frac{1}{2}|3^{-x} - x| + \frac{1}{2}|5^{-y} - y| + \frac{1}{2}|3^{-x} - y| + \frac{1}{2}|5^{-y} - x| \\ &\leq \alpha_1|3^{-x} - x| + \alpha_2|5^{-y} - y| + \alpha_3|3^{-x} - y| \\ &\quad + \alpha_4|5^{-y} - x| + (1 - 4\max\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\})|x - y| \\ &= \alpha_1||Tx - x|| + \alpha_2||Ty - y|| + \alpha_3||Tx - y|| + \alpha_4||Tx - y|| \\ &\quad + (1 - 4\max\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\})||x - y||, \end{aligned}$$

where $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 \leq \frac{1}{2}$.

Let $\varphi = \{\varphi(t) : t > 0\}$ stants for one-parameter CAK-generalized nonexpansive semigroup and $F(\varphi) = \bigcap_{t>0} F(\varphi(t))$. We give the concept of the uniformly asymptotically regular as the following definitions.

3. Main results

In this section, we first study some properties of CAK-generalized nonexpansive mapping in Hilbert space.

Proposition 3.1. *Let C be a nonempty closed convex subset of a Hilbert space H and $\varphi, \chi : C \to C$ be an CAK-generalized nonexpansive mapping with $F(\varphi \bigcap \chi) \neq \emptyset$. Then $F(\varphi \bigcap \chi)$ is closed convex and $\|\varphi\mu - p\| \leq \|\mu - p\|$ for all $\mu \in C$ and $p \in F(\varphi \bigcap \chi)$.*

Proof. Since φ, χ are an CAK-generalized nonexpansive mapping, for all $\mu \in C$ and $p \in F(\varphi \bigcap \chi)$

$$\begin{aligned} ||\varphi\mu - p|| &= ||\varphi\mu - \chi p|| \\ &\leq \alpha_1||Tx - x|| + \alpha_2||\chi p - p|| + \alpha_3||\varphi\mu - p|| + \alpha_4||\chi p - x|| \\ &+ (1 - 4\max\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\})||\mu - p|| \\ &\leq \alpha_1(||\varphi\mu - p|| + ||p - \mu||) + \alpha_3||\varphi\mu - p|| + \alpha_4||p - \mu|| \\ &+ (1 - 4\max\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\})||\mu - p||, \end{aligned} \tag{3.1}$$

and so

$$||\varphi\mu - p|| \leq \frac{1 - 2\max\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}}{1 - \alpha_1 - \alpha_3}||\mu - p|| < ||\mu - p||. \tag{3.2}$$

Likewise,

$$\|\chi\mu - p\| = \|\chi\mu - \varphi p\| \leq \|\varphi p - \chi\mu\| < ||\mu - p||. \tag{3.3}$$

Let $p, q \in F(\varphi \bigcap \chi)$, $(0 \leq \lambda \leq 1)$ and set $\omega = \lambda p + (1 - \lambda)q$. Using the Parallelogram Law, we get

$$\begin{aligned} ||\frac{\omega - p}{2} - \frac{\varphi\omega - p}{2}||^2 + \frac{1}{4}||\omega - \varphi\omega||^2 &= \frac{1}{2}||\omega - p||^2 + \frac{1}{2}||\varphi\omega - p||^2 \\ &\leq ||\omega - p||^2, \\ ||\frac{\omega - q}{2} - \frac{\varphi\omega - q}{2}||^2 + \frac{1}{4}||\omega - \varphi\omega||^2 &= \frac{1}{2}||\omega - q||^2 + \frac{1}{2}||\varphi\omega - q||^2 \\ &\leq ||\omega - q||^2. \end{aligned}$$

By (3.2) imply that

$$\begin{aligned} ||\frac{\omega + \varphi\omega}{2} - p||^2 = ||\frac{\omega - p}{2} + \frac{\varphi\omega - p}{2}||^2 &\leq ||\omega - p||^2 - \frac{1}{4}||\omega - \varphi\omega||^2 \\ &= (1 - \lambda)^2||p - q||^2 - \frac{1}{4}||\omega - \varphi\omega||^2, \\ ||\frac{\omega + \varphi\omega}{2} - q||^2 = ||\frac{\omega - q}{2} + \frac{\varphi\omega - q}{2}||^2 &\leq ||\omega - q||^2 - \frac{1}{4}||\omega - \varphi\omega||^2 \\ &= \lambda^2||p - q||^2 - \frac{1}{4}||\omega - \varphi\omega||^2. \end{aligned}$$

Suppose that $\omega \neq \varphi\omega$. Then, we have

$$||\frac{\omega + \varphi\omega}{2} - p||^2 < (1 - \lambda)^2||p - q||^2, \quad ||\frac{\omega + \varphi\omega}{2} - q||^2 < \lambda^2||p - q||^2.$$

So, we obtain that

$$||p-q|| \leq ||\frac{\omega+\varphi\omega}{2}-p||+||\frac{\omega+\varphi\omega}{2}-q|| < (1-\lambda)||p-q||+\lambda||p-q|| = ||p-q||,$$

which is a contradiction and so $\omega=\varphi\omega$. Vice versa, $\omega=\chi\omega$. Thus $F(\varphi\bigcap\chi)$ is convex. Now, we show $F(\varphi\bigcap\chi)$ is closed. Suppose that $\{\mu_n\}\in F(\varphi\bigcap\chi)$ with $\lim_{n\to\infty}\mu_n=\mu$, it follows from (3.2) that $||\mu_n-\varphi\mu||=||\mu_n-\mu||\to 0$ as $n\to\infty$ and hence $\lim_{n\to\infty}\mu_n=\varphi\mu=\mu$, and by (3.3) $\chi\mu=\mu$. Thus $F(\varphi\bigcap\chi)$ is closed. ∎

Proposition 3.2. *Let C be a nonempty subset of a Hilbert space H and $\Phi: C\to C$ be an CAK-generalized nonexpansive mapping. Then, for all $x,y\in C$:*

$$\begin{aligned}\|\mu-\chi\nu\| &\leq \frac{(1+\alpha_1+\alpha_3)}{(1-\alpha_2-\alpha_4)}\|\mu-\varphi\mu\| \\ &\quad+\frac{(1+\alpha_2+\alpha_3-4\max\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\})}{(1-\alpha_2-\alpha_4)}||x-y|| \\ &\leq \frac{(1+\alpha_1+\alpha_3)}{(1-\alpha_2-\alpha_4)}\|\mu-\varphi\mu\|+\|\mu-\nu\| \end{aligned} \tag{3.4}$$

and

$$\begin{aligned}\|\nu-\varphi\mu\| &\leq \frac{(1+\alpha_2+\alpha_4)}{(1-\alpha_1-\alpha_3)}\|\nu-\chi\nu\| \\ &\quad+\frac{(1+\alpha_1+\alpha_4-4\max\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\})}{(1-\alpha_1-\alpha_3)}||x-y|| \\ &\leq \frac{(1+\alpha_2+\alpha_4)}{(1-\alpha_1-\alpha_3)}\|\nu-\chi\nu\|+\|\mu-\nu\|. \end{aligned} \tag{3.5}$$

Proof. In first case, we get

$$\begin{aligned}\|\mu-\chi\nu\| &\leq \|\mu-\varphi\mu\|+\|\varphi\mu-\chi\nu\| \\ &\leq \|\mu-\varphi\mu\|+\alpha_1\|\varphi\mu-\mu\|+\alpha_2\|\chi\nu-\nu\|+\alpha_3\|\varphi\mu-\nu\| \\ &\quad+\alpha_4\|\chi\nu-\mu\|+(1-4\max\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\})\|\mu-\nu\| \\ &\leq \|\mu-\varphi\mu\|+\alpha_1\|\varphi\mu-\mu\|+\alpha_2\|\chi\nu-\mu\|+\alpha_2\|\mu-\nu\|+\alpha_3\|\varphi\mu-\mu\| \\ &\quad+\alpha_3\|\mu-\nu\|+\alpha_4\|\chi\nu-\mu\|+(1-4\max\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\})\|\mu-\nu\|\end{aligned}$$

This implies that

$$\begin{aligned}(1-\alpha_2-\alpha_4)\|\mu-\chi\nu\| &\leq (1+\alpha_1+\alpha_3)\|\mu-\varphi\mu\|+\alpha_2\|\nu-\chi\nu\| \\ &\quad+(1+\alpha_2+\alpha_3-4\max\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\})\|x-y\|.\end{aligned}$$

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From $2\alpha_2+\alpha_3+\alpha_4<4\max\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\}$, we get

$$\begin{aligned}\|\mu-\chi\nu\| &\le \frac{(1+\alpha_1+\alpha_3)}{(1-\alpha_2-\alpha_4)}\|\mu-\varphi\mu\| \\ &\quad+\frac{(1+\alpha_2+\alpha_3-4\max\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\})}{(1-\alpha_2-\alpha_4)}||x-y|| \\ &\le \frac{(1+\alpha_1+\alpha_3)}{(1-\alpha_2-\alpha_4)}\|\mu-\varphi\mu\|+\|\mu-\nu\|. \end{aligned} \tag{3.6}$$

Likewise,

$$\begin{aligned}\|\nu-\varphi\mu\| &\le \|\nu-\chi\nu\|+\|\chi\nu-\varphi\mu\| \\ &\le \|\nu-\chi\nu\|+\alpha_1\|\varphi\mu-\mu\|+\alpha_2\|\chi\nu-\nu\|+\alpha_3\|\varphi\mu-\nu\| \\ &+\alpha_4\|\chi\nu-\mu\|+(1-4\max\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\})\|\mu-\nu\| \\ &\le \|\nu-\chi\nu\|+\alpha_1\|\varphi\mu-\nu\|+\alpha_1\|\nu-\mu\|+\alpha_2\|\chi\nu-\nu\|+\alpha_3\|\varphi\mu-\nu\| \\ &+\alpha_4\|\chi\nu-\nu\|+\alpha_4\|\nu-\mu\|+(1-4\max\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\})\|\mu-\nu\| \end{aligned}$$

Since $2\alpha_2+\alpha_3+\alpha_4<4\max\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\}$, we have

$$\begin{aligned}\|\nu-\varphi\mu\| &\le \frac{(1+\alpha_2+\alpha_4)}{(1-\alpha_1-\alpha_3)}\|\nu-\chi\nu\| \\ &\quad+\frac{(1+\alpha_1+\alpha_4-4\max\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\})}{(1-\alpha_1-\alpha_3)}||x-y|| \\ &\le \frac{(1+\alpha_2+\alpha_4)}{(1-\alpha_1-\alpha_3)}\|\nu-\chi\nu\|+\|\mu-\nu\|. \end{aligned} \tag{3.7}$$

The complete proof. ∎

Theorem 3.3. *Let H be a nonempty closed convex subset of a Hilbert space H and $\varphi,\chi:C\to C$ be an CAK-generalized nonexpansive mapping. If a sequence $\{\mu_n\}$ in C converges weakly to $\mu\in C$ and $\lim_{n\to\infty}||\mu_n-\varphi\mu_n||=0=\lim_{n\to\infty}||\mu_n-\chi\mu_n||$, then $\mu=\varphi\mu=\chi\mu$.*

Proof. Since $\{\mu_n\}$ is weakly convergent, we have $\{\mu_n\}$ is bounded. Since

$$\|\varphi\mu_n\|\le\|\varphi\mu_n-\mu_n\|+\|\mu_n\|,\quad \|\chi\mu_n\|\le\|\chi\mu_n-\mu_n\|+\|\mu_n\|$$

we get $\{\varphi\mu_n\}$, $\{\chi\mu_n\}$ are a bounded. This implies that

$$\begin{aligned}\|\varphi\mu_n-\chi\mu\| &\le \|\varphi\mu_n-\mu_n\|+\|\mu_n-\chi\mu\| \\ &\le \|\varphi\mu_n-\mu_n\|+\frac{(1+\alpha_1+\alpha_3)}{(1-\alpha_2-\alpha_4)}\|\mu_n-\varphi\mu_n\|+\|\mu-\nu\| \quad ;(3.6). \end{aligned} \tag{3.8}$$

Thus,

$$\limsup_{n\to\infty}\|\varphi\mu_n-\chi\mu\|\le\limsup_{n\to\infty}\|\mu_n-\mu\|, \tag{3.9}$$

$$\limsup_{n\to\infty}\|\chi\mu_n-\chi\mu\|\le\limsup_{n\to\infty}\|\mu_n-\mu\|. \tag{3.10}$$

And so,

$$\limsup_{n\to\infty} \|\varphi\mu_n - \varphi\mu\| \leq \limsup_{n\to\infty} \|\mu_n - \mu\|, \tag{3.11}$$

$$\limsup_{n\to\infty} \|\chi\mu_n - \varphi\mu\| \leq \limsup_{n\to\infty} \|\mu_n - \mu\|. \tag{3.12}$$

Thus, by the properties of a Hilbert space H, we have

$$\begin{aligned}\|\mu_n - \mu\|^2 &= \|(\mu_n - \varphi\mu) + (\varphi\mu - \mu)\|^2 \\ &= \|\mu_n - \varphi\mu\|^2 + \|\varphi\mu - \mu\|^2 + 2\langle \mu_n - \varphi\mu, \varphi\mu - \mu\rangle \\ &\leq (\|\mu_n - \varphi\mu_n\| + \|\varphi\mu_n - \varphi\mu\|)^2 + \|\varphi\mu - \mu\|^2 + 2\langle \mu_n - \varphi\mu, \varphi\mu - \mu\rangle.\end{aligned}$$

Since $\{\mu_n\}$ weakly converges to $x \in C$, it follows that

$$\begin{aligned}\limsup_{n\to\infty} \|\mu_n - x\|^2 &\leq \limsup_{n\to\infty} \|\varphi\mu_n - \varphi\mu\|^2 + \|\varphi\mu - \mu\|^2 + 2\limsup_{n\to\infty} \langle \mu_n - \varphi\mu, \varphi\mu - \mu\rangle \\ &\leq \limsup_{n\to\infty} \|\varphi\mu_n - \varphi\mu\|^2 + \|\varphi\mu - \mu\|^2 + 2\langle \mu - \varphi\mu, \varphi\mu - \mu\rangle \\ &\leq \limsup_{n\to\infty} \|\mu_n - \mu\|^2 - \|\varphi\mu - \mu\|^2\end{aligned}$$

respectively, and hence $\|\varphi\mu - \mu\|^2 \leq 0$. In the same way $\|\chi\mu - \mu\|^2 \leq 0$. The complete proof. ∎

Competing interests

The authors declare that they have no competing interests.

Authorś contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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