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RESULTS ON IMPULSIVE ψ-CAPUTO FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS WITH BOUNDARY CONDITIONS

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Abstract In this research article, we investigate the existence and uniqueness of solutions for a type of boundary value problems involving fractional integro-differential equations with the ψ-Caputo fractional derivative via the Banach contraction principle and Krasnoselskii's fixed point theorem. Results regarding the existence and uniqueness of solutions are obtained. We establish our main results through theoretical analysis and present a specific case study to illustrate the practical significance of our findings.

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1. Introduction

Fractional calculus broadens the familiar concepts of integration and differentiation to arbitrary (real or complex) orders, going beyond integer-order derivatives and integrals. This field has received a lot of attention in recent years, particularly for modeling various phenomena in biology, physics, mathematical engineering, and other disciplines. Compared to ordinary differential operators, which are local in nature, fractional differential operators provide a more nuanced and accurate representation of many real-world processes. Readers interested in the theory of fractional differential equations should consult the works of Abbas et al. [1], Hilfer [21], Kilbas et al. [26], Miller and Ross [29], Oldham [30], Pudlubny [31], Sabatier et al. [32], Tarasov [35], and their references. These monographs and studies provide a deeper understanding and advances in the field of fractional calculus. The theory is a beautiful mixture of pure and applied analysis. Over the years, the theory of fixed points has been revealed as a very powerful and important tool in the study of nonlinear phenomena.

Integro-differential equations combine differential and integral equations and are widely used to model physical phenomena that involve memory effects, hereditary processes, or systems where the current state depends on the cumulative history of prior states such as Electromagnetism, Population Dynamics, Fluid Mechanics, Heat Conduction with Memory and Control Theory; see, for example [3–10, 16, 22, 23].

Impulsive conditions are used to model systems that experience sudden, abrupt changes due to external forces or internal dynamics. These conditions are essential for describing scenarios where the state of a system shifts instantaneously such as impact and collision dynamics, gravitational slingshot, explosions, shock waves, control systems, electrical, structural and aerospace engineering; see, for example [11, 27]. In particular, fixed-point techniques have been applied in many areas of mathematics, sciences, and engineering. Various fixed-point theorems have been utilized to establish sufficient conditions for the existence and uniqueness of solutions for different types of fractional differential problems; see, for example, [14, 17, 28, 33, 34, 36, 37].

The study of the existence and uniqueness of solutions to fractional differential equations has garnered considerable attention in recent research. Interested readers can find more details in earlier works [15, 19, 26] and the references therein. However, due to the often daunting task of finding exact solutions, particularly in nonlinear analysis and optimization, approximate solutions are also considered. It is crucial to emphasize that only stable approximations are deemed acceptable. Therefore, various stability analysis techniques are employed. One widely addressed approach is the concept of HU-type stability, which is straightforward and has been extensively discussed in the literature. This type of stability was initially proposed by Ulam and further developed by Hyers in the subsequent year. Originally, this concept was applied to ordinary differential equations and later extended to fractional differential equations (FDEs). For further reading, we refer the readers to sources [24, 25].

In [12] M. S. Abdo et al. discussed the Ψ-Caputo fractional differential equation fractional boundary value problem

$$\begin{aligned}
&{}^{\mathfrak{c}}\mathcal{D}^{\mathfrak{p};\Psi}\wp(\mathfrak{t}) = \mathscr{F}(\mathfrak{t}, \wp(\mathfrak{t})),\ \mathfrak{t} \in [\mathfrak{a}, \mathfrak{b}],\\
&\wp_{\Psi}^{[\mathfrak{k}]}(\mathfrak{a}) = \wp_{\mathfrak{a}}^{\mathfrak{k}},\ \mathfrak{k} = 0, 1, ..., \mathfrak{n}-2,\\
&\wp_{\Psi}^{[\mathfrak{n}-1]}(\mathfrak{b}) = \wp_{\mathfrak{b}},\ \mathfrak{k} = 0, 1, ..., \mathfrak{n}-2,
\end{aligned}$$

where ${}^{\mathfrak{c}}\mathcal{D}^{\mathfrak{p};\Psi}$-$\Psi$-Caputo derivative and $\mathscr{F}$ is continuous function.

In [20], D. B. Dhaigude et al. established the solution of nonlinear Ψ-Caputo fractional differential equations involving boundary conditions

$$\begin{aligned} &{}^{\mathfrak{c}}\mathcal{D}_{\mathfrak{t}}^{\mathfrak{p};\Psi}\wp(\mathfrak{t}) = \mathscr{F}(\mathfrak{t},\wp(\mathfrak{t})),\ 0<\mathfrak{t}\le\mathscr{T},\\ &\mathscr{G}(\wp(0),\wp(\mathscr{T}))=0,\end{aligned}$$

where ${}^{\mathfrak{c}}\mathcal{D}_{\mathfrak{t}}^{\mathfrak{p};\Psi}$-$\Psi$-Caputo derivative and $\mathscr{F}$ is continuous function.

R. Arul et al. [13] studied the Ψ-Caputo fractional integro-differential equations with non instantaneous impulsive boundary conditions of the form

$$\begin{aligned} &{}^{\mathfrak{c}}\mathcal{D}^{\mathfrak{p};\Psi}\wp(\mathfrak{t}) = \mathscr{F}(\mathfrak{t},\wp(\mathfrak{t}),\mathscr{B}\wp(\mathfrak{t})), \mathfrak{t}\in(\mathfrak{s}_{\mathfrak{i}},\mathfrak{t}_{\mathfrak{i}+1}], 0<\mathfrak{p}<1,\\ &\wp(\mathfrak{t}) = \mathscr{H}_{\mathfrak{i}}(\mathfrak{t},\wp(\mathfrak{t})),\quad \mathfrak{t}\in(\mathfrak{t}_{\mathfrak{i}},\mathfrak{s}_{\mathfrak{i}}],\quad \mathfrak{i}=1,...,\mathfrak{m},\\ &\mathfrak{a}\wp(0)+\mathfrak{b}\wp(\mathscr{T})=\mathfrak{c},\end{aligned}$$

where ${}^{\mathfrak{c}}\mathcal{D}^{\mathfrak{p};\Psi}$ is the Ψ- Caputo fractional derivatives of order $\mathfrak{p}$,$\mathfrak{a},\mathfrak{b},\mathfrak{c}$ are real constants with $\mathfrak{a}+\mathfrak{b}\neq 0$ and $0=\mathfrak{s}_0<\mathfrak{t}_1\le\mathfrak{t}_2<...<\mathfrak{t}_{\mathfrak{m}}\le\mathfrak{s}_{\mathfrak{m}}\le\mathfrak{s}_{\mathfrak{m}+1}=\mathscr{T}$,- pre-fixed, $\mathscr{F}:[0,\mathscr{T}]\times\mathbb{R}\times\mathbb{R}\longrightarrow\mathbb{R}$ and $\mathscr{H}_{\mathfrak{i}}:[\mathfrak{t}_{\mathfrak{i}},\mathfrak{s}_{\mathfrak{i}}]\times\mathbb{R}\longrightarrow\mathbb{R}$ is continuous. Moreover, $\mathscr{B}\wp(\mathfrak{t}) = \int_0^{\mathfrak{t}}\mathfrak{k}(\mathfrak{t},\mathfrak{s})\wp(\mathfrak{s})\partial\mathfrak{s}$ and $\mathfrak{k}\in\mathscr{C}(\mathtt{D},\mathbb{R}^+)$ with domain $\mathtt{D}=\{(\mathfrak{t},\mathfrak{s})\in\mathbb{R}^2: 0\le\mathfrak{s}\le\mathfrak{t}\le\mathscr{T}\}$.

We examine the existence and uniqueness of solutions for boundary value problems involving nonlinear ψ-Caputo FDEs.

$$ {}^{C}\mathcal{D}_{a^+}^{\alpha,\psi}x(\iota) = f(\iota,x(\iota),Bx(\iota)),\quad \iota\in J:=[a,T], \tag{1.1}$$

$$ x(\iota_k^+) = x(\iota_k^-)+y_k,\quad y_k\in\mathbb{R},\quad k=1,...,m, \tag{1.2}$$

$$ x(T)=\Xi x(\eta), \tag{1.3}$$

where ${}^{C}\mathcal{D}_{a^+}^{\alpha;\psi}$ is the ψ-Caputo fractional derivative of order $\alpha\in(0,1]$, $f:[a,T]\times\mathbb{R}\longrightarrow\mathbb{R}$ is a given continuous function. Ξ is real constant and $\eta\in(a,T)$, where $Bx(\iota)=\int_0^{\iota}k(\iota,\varrho,x(\varrho))d\varrho$ and $k:\Delta\times[a,T]\to\mathbb{R}$, $\Delta=\{(\iota,\varrho): a\le\varrho\le\iota\le T\}$, $a=\iota_0<\iota_1<\iota_2<...<\iota_m=T$, $\Delta x|_{\iota=\iota_k}=x(\iota_k^+)-x(\iota_k^-)$, and $x(\iota_k^+)=\lim_{h\to0^+}x(\iota_k+h)$ and $x(\iota_k^-)=\lim_{h\to0^-}x(\iota_k+h)$ represent the right and left hand limits of $x(\iota)$ at $\iota=\iota_k$.

The paper is structured as follows: Section 2 lays the groundwork by providing essential definitions and preliminary results necessary for the development of our main findings. It also introduces an auxiliary lemma, which offers a solution representation for the solutions to Problem (1.1)-(1.3). This foundational content sets the stage for the more complex discussions that follow. In Section 3, we delve into the core of our research by establishing the existence and uniqueness of solutions for FDEs that incorporate the ψ-Caputo fractional differential operator. Through rigorous proofs and detailed analysis, we demonstrate the conditions under which these solutions are guaranteed to exist and be unique. In Section 4, to bring our theoretical results to life, this section presents a concrete example. This example serves to illustrate the practical applicability and relevance of the theoretical results obtained in the previous sections, providing a clear and tangible understanding of the concepts discussed.

2. Preliminaries and Lemmas

In this section, we introduce some fractional calculus notations and terminology, as well as offer early results that will be used in our later proofs.
Consider $\mathcal{PC}(J,\mathbb{R})$, the space of real and continuous functions with the norm

$$\|x\|_{\infty} = \sup\{\|x(\iota)\| : \quad \iota \in J\}.$$

Let $L^1(J,\mathbb{R})$ be the Banach space of Lebesgue integrable functions $x : J \to \mathbb{R}$, equipped with the norm

$$\|x\|_{L^1} = \int_J |x(\iota)| d\iota.$$

We start by introducing ψ-Riemann-Liouville fractional integrals and derivatives. Moving forward:

Definition 2.1. [5] Let $\alpha > 0$, the left-sided ψ-Riemann-Liouville fractional integral of order α for an integrable function $u : \mathrm{J} \longrightarrow \mathbb{R}$ with respect to another function $\psi : \mathrm{J} \to \mathbb{R}$, where ψ is an increasing differentiable function and $\psi'(\iota) \neq 0$, for all $\iota \in \mathrm{J}$ is defined as follows

$$\mathcal{I}_{a^+}^{\alpha;\psi} u(\iota) = \frac{1}{\Gamma(\alpha)} \int_a^{\iota} \psi'(\varrho)(\psi(\iota) - \psi(\varrho))^{\alpha-1} u(\varrho) \mathrm{d}\varrho,$$

where Γ is the classical Euler Gamma function.

Definition 2.2. [5] Let $n \in \mathbb{N}$ and let $\psi \in C^n(\mathrm{J},\mathbb{R})$ and $u \in C^n(\mathrm{J},\mathbb{R})$ such that $\psi'(\iota) \neq 0$, for all $\iota \in \mathrm{J}$. The left-sided ψ Riemann-Liouville fractional derivative of a function u of order α is defined by

$$\begin{aligned} \mathcal{D}_{a^+}^{\alpha;\psi} u(\iota) &= \left(\frac{1}{\psi'(\iota)} \frac{d}{d\iota} \right)^n \mathcal{I}_{a^+}^{n-\alpha;\psi} u(\iota) \\ &= \frac{1}{\Gamma(n-\alpha)} \left(\frac{1}{\psi'(\iota)} \frac{d}{d\iota} \right)^n \int_a^{\iota} \psi'(\varrho)(\psi(\iota) - \psi(\varrho))^{n-\alpha-1} u(\varrho) \mathrm{d}\varrho, \end{aligned}$$

where $n = [\alpha] + 1$.

Definition 2.3. [5] Let $n \in \mathbb{N}$ and let $\psi \in C^n(\mathrm{J},\mathbb{R})$ and $u \in C^n(\mathrm{J},\mathbb{R})$ such that $\psi'(\iota) \neq 0$, for all $\iota \in \mathrm{J}$. The left-sided ψ-Caputo fractional derivative of u of order α is defined by

$${}^{C}\mathcal{D}_{a^+}^{\alpha;\psi} u(\iota) = \mathcal{I}_{a^+}^{n-\alpha;\psi} \left(\frac{1}{\psi'(\iota)} \frac{d}{d\iota} \right)^n u(\iota),$$

where $n = [\alpha] + 1$ for $\alpha \notin \mathbb{N}, n = \alpha$ for $\alpha \in \mathbb{N}$.

Lemma 2.4. [2] *Let $\alpha, \beta > 0$, and $u \in L^1(\mathrm{J},\mathbb{R})$. Then*

$$\mathcal{I}_{a^+}^{\alpha;\psi} \mathcal{I}_{a^+}^{\beta;\psi} u(\iota) = \mathcal{I}_{a^+}^{\alpha+\beta;\psi} u(\iota), \ \ a.e. \ \iota \in \mathrm{J}.$$

In particular, if $u \in \mathcal{C}(\mathrm{J},\mathbb{R})$, then

$$\mathcal{I}_{a^+}^{\alpha;\psi} \mathcal{I}_{a^+}^{\beta;\psi} u(\iota) = \mathcal{I}_{a^+}^{\alpha+\beta;\psi} u(\iota), \iota \in \mathrm{J}.$$

Lemma 2.5. [2] *Let $\alpha > 0$, if $u \in \mathcal{PC}(\mathrm{J},\mathbb{R})$ then*

$${}^{C}\mathcal{D}_{a^+}^{\alpha;\psi} \mathcal{I}_{a^+}^{\alpha;\psi} u(\iota) = u(\iota), \iota \in \mathrm{J}.$$

If $u \in C^n(\mathrm{J}, \mathbb{R}), n-1<\alpha<n$. *Then*

$$\mathcal{I}_{a^+}^{\alpha;\psi}\left({}^C\mathcal{D}_{a^+}^{\alpha;\psi} u\right)(\iota) = u(\iota) - \sum_{k=0}^{n-1} \frac{u_\psi^{[k]}(a)}{k!}[\psi(\iota)-\psi(a)]^k, \quad \iota \in \mathrm{J}.$$

Lemma 2.6. [2] *Let* $\iota > a, \alpha \geq 0$, *and* $\beta > 0$. *Then*

(i) $\mathcal{I}_{a^+}^{\alpha;\psi}(\psi(\iota)-\psi(a))^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(\psi(\iota)-\psi(a))^{\beta+\alpha-1}$;

(ii) ${}^C\mathcal{D}_{a^+}^{\alpha;\psi}(\psi(\iota)-\psi(a))^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(\psi(\iota)-\psi(a))^{\beta-\alpha-1}$;

(iii) ${}^C\mathcal{D}_{a^+}^{\alpha;\psi}(\psi(\iota)-\psi(a))^{k} = 0$, *for all* $k \in \{0,\ldots,n-1\}$, $n \in \mathbb{N}$.

Lemma 2.7. *Let* $0 < \alpha < 1$, $\rho > 0$ *and* $x \in \mathcal{PC}(J,\mathbb{R})$. *Then the linear antiperiodic boundary value problem*

$${}^C\mathcal{D}^{\alpha,\psi} x(\iota) = \sigma(\iota), \quad \iota \in J := [a,T], \tag{2.1}$$

$$x(\iota_k^+) = x(\iota_k^-) + y_k, \quad y_k \in \mathbb{R} \quad k = 1,...,m, \tag{2.2}$$

$$x(T) = \Xi x(\eta). \tag{2.3}$$

$$x(\iota) = \begin{cases} \frac{1}{\Gamma(\alpha)}\int_a^\iota \psi'(\varrho)(\psi(\iota)-\psi(\varrho))^{\alpha-1}\sigma(\iota)d\varrho \\ +\frac{1}{\nu}\{\frac{\Xi}{\Gamma(\alpha)}\int_a^\eta \psi'(\varrho)(\psi(\eta)-\psi(\varrho))^{\alpha-1}\sigma(\iota)d\varrho \\ +\frac{1}{\Gamma(\alpha)}\int_a^T \psi'(\varrho)(\psi(T)-\psi(\varrho))^{\alpha-1}\sigma(\iota)d\varrho\}, \ \mathtt{for}\ \iota \in [1,\iota_1), \\ \vartheta_1 + \frac{1}{\Gamma(\alpha)}\int_a^\iota \psi'(\varrho)(\psi(\iota)-\psi(\varrho))^{\alpha-1}\sigma(\iota)d\varrho \\ +\frac{1}{\nu}\{\frac{\Xi}{\Gamma(\alpha)}\int_a^\eta \psi'(\varrho)(\psi(\eta)-\psi(\varrho))^{\alpha-1}\sigma(\iota)d\varrho \\ +\frac{1}{\Gamma(\alpha)}\int_a^T \psi'(\varrho)(\psi(T)-\psi(\varrho))^{\alpha-1}\sigma(\iota)d\varrho\}, \ \mathtt{for}\ \iota \in (\iota_1,\iota_2), \\ \vartheta_1 + \vartheta_2 + \frac{1}{\Gamma(\alpha)}\int_a^\iota \psi'(\varrho)(\psi(\iota)-\psi(\varrho))^{\alpha-1}\sigma(\iota)d\varrho \\ +\frac{1}{\nu}\{\frac{\Xi}{\Gamma(\alpha)}\int_a^\eta \psi'(\varrho)(\psi(\eta)-\psi(\varrho))^{\alpha-1}\sigma(\iota)d\varrho \\ +\frac{1}{\Gamma(\alpha)}\int_a^T \psi'(\varrho)(\psi(T)-\psi(\varrho))^{\alpha-1}\sigma(\iota)d\varrho\}, \ \mathtt{for}\ \iota \in (\iota_2,\iota_3), \\ . \\ . \\ . \\ \sum_{\kappa=1}^m \vartheta_i + \frac{1}{\Gamma(\alpha)}\int_a^\iota \psi'(\varrho)(\psi(\iota)-\psi(\varrho))^{\alpha-1}\sigma(\iota)d\varrho \\ +\frac{1}{\nu}\{\frac{\Xi}{\Gamma(\alpha)}\int_a^\eta \psi'(\varrho)(\psi(\eta)-\psi(\varrho))^{\alpha-1}\sigma(\iota)d\varrho \\ +\frac{1}{\Gamma(\alpha)}\int_a^T \psi'(\varrho)(\psi(T)-\psi(\varrho))^{\alpha-1}\sigma(\iota)d\varrho\}, \ \mathtt{for}\ \iota \in (\iota_k,\iota_{k+1}), \end{cases} \tag{2.4}$$

where

$$\nu = (1-\Xi).$$

Proof. Assume that $\aleph$ satisfies (2.1) and (2.3). If $\iota \in [1,\iota_1)$ then

$${}^C\mathcal{D}^{\alpha,\psi} x(\iota) = \sigma(\iota), \quad \iota \in J$$

$$x(T) = \Xi x(\eta).$$

We can obtain

$$\begin{aligned}x(\iota) =&\frac{1}{\Gamma(\alpha)}\int_a^{\iota}\psi'(\varrho)(\psi(\iota)-\psi(\varrho))^{\alpha-1}\sigma(\iota)d\varrho\\&+\frac{1}{\nu}\left\{\frac{\Xi}{\Gamma(\alpha)}\int_a^{\eta}\psi'(\varrho)(\psi(\eta)-\psi(\varrho))^{\alpha-1}\sigma(\iota)d\varrho\right.\\&\left.+\frac{1}{\Gamma(\alpha)}\int_a^{T}\psi'(\varrho)(\psi(T)-\psi(\varrho))^{\alpha-1}\sigma(\iota)d\varrho\right\}.\end{aligned}$$

If $\iota \in (\iota_1, \iota_2)$ then

$$\begin{aligned}x(\iota) =&y(\iota_1^+)-\frac{1}{\Gamma(\alpha)}\int_a^{\iota_1}\psi'(\varrho)(\psi(\iota)-\psi(\varrho))^{\alpha-1}\sigma(\iota)d\varrho\\&+\frac{1}{\Gamma(\alpha)}\int_a^{\iota}\psi'(\varrho)(\psi(\iota)-\psi(\varrho))^{\alpha-1}\sigma(\iota)d\varrho\\&+\frac{1}{\nu}\left\{\frac{\Xi}{\Gamma(\alpha)}\int_a^{\eta}\psi'(\varrho)(\psi(\eta)-\psi(\varrho))^{\alpha-1}\sigma(\iota)d\varrho\right.\\&\left.+\frac{1}{\Gamma(\alpha)}\int_a^{T}\psi'(\varrho)(\psi(T)-\psi(\varrho))^{\alpha-1}\sigma(\iota)d\varrho\right\}\\=&y(\iota_1^+)+y_1-\frac{1}{\Gamma(\alpha)}\int_a^{\iota}\psi'(\varrho)(\psi(\iota)-\psi(\varrho))^{\alpha-1}\sigma(\iota)d\varrho\\&+\frac{1}{\nu}\left\{\frac{\Xi}{\Gamma(\alpha)}\int_a^{\eta}\psi'(\varrho)(\psi(\eta)-\psi(\varrho))^{\alpha-1}\sigma(\iota)d\varrho\right.\\&\left.+\frac{1}{\Gamma(\alpha)}\int_a^{T}\psi'(\varrho)(\psi(T)-\psi(\varrho))^{\alpha-1}\sigma(\iota)d\varrho\right\}\\=&y_1+\frac{1}{\Gamma(\alpha)}\int_a^{\iota}\psi'(\varrho)(\psi(\iota)-\psi(\varrho))^{\alpha-1}\sigma(\iota)d\varrho\\&+\frac{1}{\nu}\left\{\frac{\Xi}{\Gamma(\alpha)}\int_a^{\eta}\psi'(\varrho)(\psi(\eta)-\psi(\varrho))^{\alpha-1}\sigma(\iota)d\varrho\right.\\&\left.+\frac{1}{\Gamma(\alpha)}\int_a^{T}\psi'(\varrho)(\psi(T)-\psi(\varrho))^{\alpha-1}\sigma(\iota)d\varrho\right\}.\end{aligned}$$

If $\iota \in (\iota_2, \iota_3)$ then

$$\begin{aligned}x(\iota) =&y(\iota_2^+)-\frac{1}{\Gamma(\alpha)}\int_a^{\iota_2}\psi'(\varrho)(\psi(\iota)-\psi(\varrho))^{\alpha-1}\sigma(\iota)d\varrho\\&+\frac{1}{\Gamma(\alpha)}\int_a^{\iota}\psi'(\varrho)(\psi(\iota)-\psi(\varrho))^{\alpha-1}\sigma(\iota)d\varrho\\&+\frac{1}{\nu}\left\{\frac{\Xi}{\Gamma(\alpha)}\int_a^{\eta}\psi'(\varrho)(\psi(\eta)-\psi(\varrho))^{\alpha-1}\sigma(\iota)d\varrho\right.\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\alpha)}\int_a^T \psi'(\varrho)(\psi(T)-\psi(\varrho))^{\alpha-1}\sigma(\iota)d\varrho\Bigg\}\\
=&y(\iota_2^+)+y_2-\frac{1}{\Gamma(\alpha)}\int_a^\iota \psi'(\varrho)(\psi(\iota)-\psi(\varrho))^{\alpha-1}\sigma(\iota)d\varrho\\
&+\frac{1}{\nu}\Bigg\{\frac{\Xi}{\Gamma(\alpha)}\int_a^\eta \psi'(\varrho)(\psi(\eta)-\psi(\varrho))^{\alpha-1}\sigma(\iota)d\varrho\\
&+\frac{1}{\Gamma(\alpha)}\int_a^T \psi'(\varrho)(\psi(T)-\psi(\varrho))^{\alpha-1}\sigma(\iota)d\varrho\Bigg\}\\
=&y_1+y_2+\frac{1}{\Gamma(\alpha)}\int_a^\iota \psi'(\varrho)(\psi(\iota)-\psi(\varrho))^{\alpha-1}\sigma(\iota)d\varrho\\
&+\frac{1}{\nu}\Bigg\{\frac{\Xi}{\Gamma(\alpha)}\int_a^\eta \psi'(\varrho)(\psi(\eta)-\psi(\varrho))^{\alpha-1}\sigma(\iota)d\varrho\\
&+\frac{1}{\Gamma(\alpha)}\int_a^T \psi'(\varrho)(\psi(T)-\psi(\varrho))^{\alpha-1}\sigma(\iota)d\varrho\Bigg\}.
\end{aligned}
$$

If $\iota \in (\iota_{\mathtt{m}}, T)$ then

$$
\begin{aligned}
x(\iota)=&\sum_{\kappa=1}^{\mathtt{m}}\vartheta_{\mathtt{i}}+\frac{1}{\Gamma(\alpha)}\int_a^\iota \psi'(\varrho)(\psi(\iota)-\psi(\varrho))^{\alpha-1}\sigma(\iota)d\varrho\\
&+\frac{1}{\nu}\Bigg\{\frac{\Xi}{\Gamma(\alpha)}\int_a^\eta \psi'(\varrho)(\psi(\eta)-\psi(\varrho))^{\alpha-1}\sigma(\iota)d\varrho\\
&+\frac{1}{\Gamma(\alpha)}\int_a^T \psi'(\varrho)(\psi(T)-\psi(\varrho))^{\alpha-1}\sigma(\iota)d\varrho\Bigg\}.
\end{aligned}
\tag{2.5}
$$

On the other hand, suppose that x fulfills the impulsive fractional boundary condition of equation (2.5). ∎

Theorem 2.8. *(Completely Continuous) The operator $\mathbb{T}$ is said to be completely continuous if it is continuous and maps any bounded subset of D into a relatively compact subset of X.*

Theorem 2.9. *(Banach Contraction Mapping Principle) If $T : X \to X$ is a contraction mapping on a complete metric space (x, d), then there is exactly one solution of $T(x) = x$ for $x \in X$.*

Theorem 2.10. *(Krasnoselkii's fixed point theorem) Let K be a closed convex, bounded and nonempty subset of a Banach space X. Let A_1, A_2 be two operators such that*

(i) $A_1x + A_2y \in K$ *for any* $x, y \in K$.
(ii) A_1 *is completely continuous operator.*
(iii) A_1 *is contraction operator.*

Then there exists at least one fixed point $z_1 \in K$ such that

$$z_1 = A_1z_1 + A_2z_1.$$

3. Main Results

Theorem 3.1. *Suppose that the following assumption is hold.*

(Al_1): *There is a positive constant L_1, L_2 and M such that*

$$
\begin{aligned}
&|f(\iota,\omega,y)-f(\iota,\overline{\omega},\overline{y})| \le L_1|\omega-y|+L_2|\overline{\omega}-\overline{y}|, \quad for \quad \iota\in[a,T],\\
&\qquad \omega,y,\overline{\omega},\overline{y}\in\mathbb{R},\\
&|k(\iota,\varrho,\vartheta)-k(\iota,\varrho,\nu)| \le M|\vartheta-\nu|, \quad for \quad \vartheta,\nu\in\mathbb{R}.
\end{aligned}
$$

If

$$
\begin{aligned}
\kappa:(L_1+L_2M)\Bigg\{&\frac{(\psi(T)-\psi(\varrho))^\alpha}{\Gamma(\alpha+1)}\\
&+\frac{1}{\nu}\Bigg\{\frac{\Xi(\psi(\eta)-\psi(\varrho))^\alpha}{\Gamma(\alpha+1)}+\frac{(\psi(T)-\psi(\varrho))^\alpha}{\Gamma(\alpha+1)}\Bigg\}\Bigg\}<1, \qquad (3.1)
\end{aligned}
$$

then the boundary value problem (1.1)-(1.3) *has a unique solution on* $[a,T]$.

Proof. Define the operator $\mathcal{G}:\mathcal{PC}(J,\mathbb{R})\to\mathcal{PC}(J,\mathbb{R})$ defined by

$$
\begin{aligned}
\mathcal{G}x(\iota)=&\sum_{\kappa=1}^{\mathtt{m}}\vartheta_{\mathtt{i}}+\frac{1}{\Gamma(\alpha)}\int_a^\iota\psi'(\varrho)(\psi(\iota)-\psi(\varrho))^{\alpha-1}f(\varrho,x(\varrho),Bx(\varrho))\,d\varrho\\
&+\frac{1}{\nu}\Bigg\{\frac{\Xi}{\Gamma(\alpha)}\int_a^\eta\psi'(\varrho)(\psi(\eta)-\psi(\varrho))^{\alpha-1}f(\varrho,x(\varrho),Bx(\varrho))\,d\varrho \qquad (3.2)\\
&+\frac{1}{\Gamma(\alpha)}\int_a^T\psi'(\varrho)(\psi(T)-\psi(\varrho))^{\alpha-1}f(\varrho,x(\varrho),Bx(\varrho))\,d\varrho\Bigg\}.
\end{aligned}
$$

Use the Banach mapping principle to demonstrate that $\mathcal{G}$ is contraction.

Let $x,y\in\mathcal{PC}(J,\mathbb{R})$ and for $\iota\in J$. Then, we have

$$
\begin{aligned}
&|\mathcal{G}x(\iota)-\mathcal{G}y(\iota)|\\
\le&\frac{1}{\Gamma(\alpha)}\int_a^\iota\psi'(\varrho)(\psi(\iota)-\psi(\varrho))^{\alpha-1}|f(\varrho,x(\varrho),Bx(\varrho))-f(\varrho,y(\varrho),By(\varrho))|\,d\varrho\\
&+\frac{1}{\nu}\Bigg\{\frac{\Xi}{\Gamma(\alpha)}\int_a^\eta\psi'(\varrho)(\psi(\eta)-\psi(\varrho))^{\alpha-1}|f(\varrho,u(\varrho),Bx(\varrho))-f(\varrho,y(\varrho),By(\varrho))|\,d\varrho\\
&+\frac{1}{\Gamma(\alpha)}\int_a^T\psi'(\varrho)(\psi(T)-\psi(\varrho))^{\alpha-1}|f(\varrho,x(\varrho),Bx(\varrho))-f(\varrho,y(\varrho),By(\varrho))|\,d\varrho\Bigg\}
\end{aligned}
$$

$$\begin{aligned}
\leq&(L_1+L_2M)\frac{1}{\Gamma(\alpha)}\int_a^\iota \psi'(\varrho)(\psi(\iota)-\psi(\varrho))^{\alpha-1}\left|x(\varrho)-y(\varrho)\right|d\varrho\\
&+(L_1+L_2M)\frac{1}{\nu}\left\{\frac{\Xi}{\Gamma(\alpha)}\int_a^\eta \psi'(\varrho)(\psi(\eta)-\psi(\varrho))^{\alpha-1}\left|x(\varrho)-y(\varrho)\right|d\varrho\right.\\
&\left.+(L_1+L_2M)\frac{1}{\Gamma(\alpha)}\int_a^T \psi'(\varrho)(\psi(T)-\psi(\varrho))^{\alpha-1}\left|x(\varrho)-y(\varrho)\right|d\varrho\right\}\\
\leq&(L_1+L_2M)\left\|x-y\right\|\\
&\left\{\frac{(\psi(T)-\psi(\varrho))^\alpha}{\Gamma(\alpha+1)}+\frac{1}{\nu}\left\{\frac{\Xi(\psi(\eta)-\psi(\varrho))^\alpha}{\Gamma(\alpha+1)}+\frac{(\psi(T)-\psi(\varrho))^\alpha}{\Gamma(\alpha+1)}\right\}\right\}\\
\leq&\kappa\left\|x-y\right\|.
\end{aligned}$$

Thus, $\|\mathcal{G}x-\mathcal{G}y\|\leq\kappa\|x-y\|$. Given $\kappa<1$, $\mathcal{G}$ is a contraction mapping operator. Therefore, we infer from the Banach contraction mapping principle that there is only one fixed point for the operator $\mathcal{G}$, which equates to a single solution for the problem in equation (1.1)-(1.3) on J. ∎

Theorem 3.2. *Assume the following assumption is holds:*

(Al_2) *There exist a non decreasing function $\upsilon(\iota)>0$ such that for $\iota\in J$,*

$$|f(\iota,\mu,\eta)|\leq\upsilon(\iota),\ \forall(\iota,\mu,\eta)\in[0,1]\times\mathbb{R}\times\mathbb{R}\quad \textit{with}\quad \upsilon\in\mathcal{PC}([a,T],\mathbb{R}).$$

(Al_3) *There exists a constant $M^*>0$ such that $\sum_{\mathtt{i}=1}^{\mathtt{m}}|\upsilon_{\mathtt{i}}|\leq M^*$.*

Then the problems (1.1)-(1.3) has at least one solution on $[a,T]$.

Proof. Consider the operator $\mathcal{G}:\mathcal{PC}(J,\mathbb{R})\to\mathcal{PC}(J,\mathbb{R})$ defined by (3.2). Define the ball

$$\mathcal{B}_{r_0}:=\{u\in\mathcal{PC}(J,\mathbb{R}):\|u\|\leq r_0\}.$$

Now we subdivide the operator $\mathcal{G}$ into two operators $\mathcal{G}_1$ and $\mathcal{G}_2$ on $\mathcal{B}_{r_0}$ defined by

$$\begin{aligned}
\mathcal{G}_1x(\iota)=&\frac{1}{\nu}\{\frac{\Xi}{\Gamma(\alpha)}\int_a^\eta \psi'(\varrho)(\psi(\eta)-\psi(\varrho))^{\alpha-1}f\left(\varrho,x(\varrho),Bx(\varrho)\right)d\varrho\\
&+\frac{1}{\Gamma(\alpha)}\int_a^T \psi'(\varrho)(\psi(T)-\psi(\varrho))^{\alpha-1}f\left(\varrho,x(\varrho),Bx(\varrho)\right)d\varrho\}+\sum_{\kappa=1}^{\mathtt{m}}\vartheta_{\mathtt{i}} \quad (3.3)
\end{aligned}$$

and

$$\mathcal{G}_2x(\iota)=\frac{1}{\Gamma(\alpha)}\int_a^\iota \psi'(\varrho)(\psi(\iota)-\psi(\varrho))^{\alpha-1}f\left(\varrho,x(\varrho),Bx(\varrho)\right)d\varrho. \quad (3.4)$$

Taking into account that $\mathcal{G}_1$ and $\mathcal{G}_2$ are defined on $\mathcal{B}_{r_0}$, and for any $x\in\mathcal{PC}(J,\mathbb{R})$,

$$\mathcal{G}u(\iota)=\mathcal{G}_1x(\iota)+\mathcal{G}_2x(\iota),\quad \iota\in J.$$

Step 1: $\mathcal{G}_1x_1+\mathcal{G}_2x_2\in\mathcal{B}_{r_0}$.

Now, for $x_1, x_2 \in \mathcal{B}_{r_0}$ and $\iota \in J$, we have

$$
\begin{aligned}
&|\mathcal{G}_1 x_1(\iota) + \mathcal{G}_2 x_2(\iota)| \leq |\mathcal{G}_1 x_1(\iota)| + |\mathcal{G}_2 x_2(\iota)| \\
\leq & \| \frac{1}{\Gamma(\alpha)} \int_a^\iota \psi'(\varrho)(\psi(\iota) - \psi(\varrho))^{\alpha-1} f\left(\varrho, x(\varrho), Bx(\varrho)\right) d\varrho \\
& + \frac{1}{\nu} \Bigg\{ \frac{\Xi}{\Gamma(\alpha)} \int_a^\eta \psi'(\varrho)(\psi(\eta) - \psi(\varrho))^{\alpha-1} f\left(\varrho, x(\varrho), Bx(\varrho)\right) d\varrho \\
& + \frac{1}{\Gamma(\alpha)} \int_a^T \psi'(\varrho)(\psi(T) - \psi(\varrho))^{\alpha-1} f\left(\varrho, x(\varrho), Bx(\varrho)\right) d\varrho \Bigg\} + \sum_{\kappa=1}^{\mathtt{m}} \vartheta_{\mathtt{i}} \| \\
\leq & \frac{1}{\Gamma(\alpha)} \int_a^\iota \psi'(\varrho)(\psi(\iota) - \psi(\varrho))^{\alpha-1} \left| f\left(\varrho, x(\varrho), Bx(\varrho)\right)\right| d\varrho \\
& + \frac{1}{\nu} \Bigg\{ \frac{\Xi}{\Gamma(\alpha)} \int_a^\eta \psi'(\varrho)(\psi(\eta) - \psi(\varrho))^{\alpha-1} \left| f\left(\varrho, x(\varrho), Bx(\varrho)\right)\right| d\varrho \\
& + \frac{1}{\Gamma(\alpha)} \int_a^T \psi'(\varrho)(\psi(T) - \psi(\varrho))^{\alpha-1} \left| f\left(\varrho, x(\varrho), Bx(\varrho)\right)\right| d\varrho \Bigg\} + \sum_{\kappa=1}^{\mathtt{m}} \vartheta_{\mathtt{i}} \\
\leq & \frac{1}{\Gamma(\alpha)} \int_a^\iota \psi'(\varrho)(\psi(\iota) - \psi(\varrho))^{\alpha-1} \left| f\left(\varrho, x(\varrho), Bx(\varrho)\right)\right| d\varrho \\
& + \frac{1}{\nu} \Bigg\{ \frac{\Xi}{\Gamma(\alpha)} \int_a^\eta \psi'(\varrho)(\psi(\eta) - \psi(\varrho))^{\alpha-1} \left| f\left(\varrho, x(\varrho), Bx(\varrho)\right)\right| d\varrho \\
& + \frac{1}{\Gamma(\alpha)} \int_a^T \psi'(\varrho)(\psi(T) - \psi(\varrho))^{\alpha-1} \left| f\left(\varrho, x(\varrho), Bx(\varrho)\right)\right| d\varrho \Bigg\} + \sum_{\kappa=1}^{\mathtt{m}} \vartheta_{\mathtt{i}} \\
\leq & \upsilon(\iota) \left\{ \frac{(\psi(T) - \psi(\varrho))^\alpha}{\Gamma(\alpha+1)} + \frac{1}{\nu} \left\{ \frac{\Xi(\psi(\eta) - \psi(\varrho))^\alpha}{\Gamma(\alpha+1)} + \frac{(\psi(T) - \psi(\varrho))^\alpha}{\Gamma(\alpha+1)} \right\} \right\} + M^* \\
\leq & r_0.
\end{aligned}
$$

Furthermore, the contraction mapping $\mathcal{G}_1$ is clear. The operator $(\mathcal{G}_2 x)(\iota)$ is continuous based on the continuity of X, as f is continuous. Additionally, we see that

$$
\begin{aligned}
\|\mathcal{G}_2 x\| &\leq \frac{1}{\Gamma(\alpha)} \int_a^\iota \psi'(\varrho)(\psi(\iota) - \psi(\varrho))^{\alpha-1} f\left(\varrho, x(\varrho), Bx(\varrho)\right) d\varrho \\
&\leq \frac{\upsilon(\iota)}{\Gamma(\alpha+1)} (\psi(\iota) - \psi(\varrho))^\alpha.
\end{aligned}
$$

This proves that $\mathcal{G}_2$ is uniformly bounded on $\mathcal{B}_{r_0}$.
Finally, we prove that $\mathcal{G}_2$ maps bounded sets into equicontinuous sets of $\mathcal{PC}(J, \mathbb{R})$, i.e., $(\mathcal{G}\mathcal{B}_{r_0})$ is equicontinuous.

$$
\sup_{(\iota, x, y) \in [0,1] \times B_r} \left| f\left(\varrho, x(\varrho), Bx(\varrho)\right)\right| = C_0 < \infty,
$$

we will get

$$\begin{aligned}|\mathcal{G}_2 u(\iota_1)-\mathcal{G}_2 u(\iota_2)| =&\Big|\frac{1}{\Gamma(\alpha)}\int_a^{\iota_1}\psi'(\varrho)(\psi(\iota_1)-\psi(\varrho))^{\alpha-1}f(\varrho,x(\varrho),Bx(\varrho))\,d\varrho\\ &-\frac{1}{\Gamma(\alpha)}\int_a^{\iota_2}\psi'(\varrho)(\psi(\iota_2)-\psi(\varrho))^{\alpha-1}f(\varrho,x(\varrho),Bx(\varrho))\,d\varrho\Big|\\ \leq&\frac{1}{\Gamma(\alpha)}\int_a^{\iota_1}\left[\psi'(\varrho)(\psi(\iota_1)-\psi(\varrho))^{\alpha-1}-(\psi'(\varrho)(\psi(\iota_2)-\psi(\varrho))^{\alpha-1})\right]\\ &+\int_{\iota_1}^{\iota_2}(\psi'(\varrho)(\psi(\iota_2)-\psi(\varrho))^{\alpha-1}\left|f(\varrho,x(\varrho),Bx(\varrho))\right|d\varrho,\end{aligned}$$

which is independent of x and approaches to zero as $\iota_2 \to \iota_1$. Thus, $\mathcal{G}_2(B_{r_0})$ is relatively compact. This shows that $\mathcal{G}_2$ is equicontinuous in $\mathcal{B}_{r_0}$. Assuming the above causes, the Arzel-Ascoli theorem applies, implying that $\mathcal{G}_2$ is compact on $\mathcal{B}_{r_0}$. Thus, the Krasnoselskii fixed point theorem propose is fulfilled, leading to the conclusion that there is at least one solution on J. ∎

4. Example

Consider the following problem of implicit FDEs involving ψ-Caputo type:

$${}^cD^{\frac{5}{2},\Psi}x(\iota)=\frac{1}{5e^{\iota+2}(1+|x(\iota)|}+\int_0^{\iota}\frac{e^{-(\varrho-\iota)}}{10}x(\varrho)d\varrho, \tag{4.1}$$

$$x(\iota_{\mathrm{k}}^{+})=x(\iota_{\mathrm{k}}^{-})+\frac{1}{6}, \tag{4.2}$$

$$x(T)=\Xi x(\eta). \tag{4.3}$$

Set:

$$f(\iota,u,v)=\frac{1}{5e^{\iota+2}(1+|x(\iota)|}+Bx(\iota),\iota\in[0,1],u,v\in\mathbb{R}^{+},$$

$$Bx(\iota)=\int_0^{\iota}\frac{e^{-(\varrho-\iota)}}{10}x(\varrho)d\varrho.$$

Hence, the condition (Al_1) is satisfied, where $\alpha=\frac{1}{4}$, $\Xi=\frac{3}{4}$, $\eta=\frac{1}{2}$, $a=0$, $T=1$, $\psi(\iota)=\iota, L_1=L_2=\frac{1}{5e^3}$, $M=\frac{1}{10}$.

$$(L_1+L_2M)\left\{\frac{(\psi(T)-\psi(\varrho))^{\alpha}}{\Gamma(\alpha+1)}+\frac{1}{\nu}\left\{\frac{\Xi(\psi(\eta)-\psi(\varrho))^{\alpha}}{\Gamma(\alpha+1)}+\frac{(\psi(T)-\psi(\varrho))^{\alpha}}{\Gamma(\alpha+1)}\right\}\right\}<1\cong 0.03368$$

Clearly, the hypothesis of Theorem 3.1 are fulfilled and hence its conclusion implies the existence of a unique solution of the problem in Equation (4.1)-(4.3) on $[0,1]$.

5. Conclusion

In this paper, we examined the ψ-Caputo fractional differential operator in determining the uniqueness and existence of solutions to FDEs. We applied the Krasnoselskii's fixed point theorem and Banach contraction principle with some inequality technique to demonstrate the main results. Finally, examples have been provided to demonstrate the validity of our conclusions. In future works, one can extend the uniqueness and existence of solutions using Burton-Kirk fixed-point theorem and Banach contraction principle given fractional boundary-value problem to more fractional derivatives, such as the Hilfer and Caputo-Fabrizio fractional derivatives.

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