BANGMOD INTERNATIONAL JOURNAL OF MATHEMATICAL & COMPUTATIONAL SCIENCE
Volume 11 (2025)
Pages 63–74
ISSN: 2408-154X (Print)
ISSN: 3057-0557 (Online)
https://bangmodjmcs.com

Guaranteed Pursuit Time for an Infinite System in l_2 with Geometric Constraints

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Received: 14 June 2024 / Accepted: 3 January 2025

Abstract In this paper, we consider a pursuit differential game problem described by an infinite system of binary differential equations in the Hilbert space l_2. The control parameters of the players are subject to geometric constraints. The pursuer's goal is to complete the game by bringing the state of the system to the origin, while the evader's goal is the contrary. The guaranteed pursuit time required to achieve the pursuer's goal is estimated. To this end, we constructed an optimal strategy for the pursuer.

MSC: 47H09

Keywords: Guaranteed pursuit time; geometric constraint; players control functions

Published online: 6 April 2025

Published by Center of Excellence in Theoretical and Computational Science (TaCS-CoE)

Please cite this article as: B.M. Umar et al., Guaranteed Pursuit Time for an Infinite System in l_2 with Geometric Constraints, Bangmod Int. J. Math. & Comp. Sci., Vol. 11 (2025), 63–74.
https://doi.org/10.58715/bangmodjmcs.2025.11.4

1. Introduction

Differential game is a mathematical method of investigating conflicts modeled in the form of differential equations, where the players' dynamics are usually described by first order, *nth* order differential equations, and also partial differential equations (PDEs). See, for instance ([1],[2],[3],[4],[5],[6],[7]). While some authors [8–10] have studied differential games described by a PDE of the form

$$\frac{\partial z}{\partial t} = Az + u - v,\ Az = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} (a_{ij}(x)) \frac{\partial z}{\partial x_i}, \tag{1.1}$$

the authors [11, 13] studied the reduced version of (1.1) described by the following infinite system of differential equations

$$\dot{z}_k + \lambda_k z_k = u_k - v_k,\ k = 1, 2, 3, \cdots, \tag{1.2}$$

where u_k and v_k are control parameters of players, z_k, u_k, $v_k \in \mathbb{R}$ and the coefficients λ_k, $k = 1, 2, ...$, satisfy the condition $0 < \lambda_1 \leq \lambda_2 \leq \cdots \to +\infty$. Ibragimov et al. [11] studied a pursuit differential game described by an infinite system of binary differential equations in Hilbert space l_2.

$$\begin{aligned} \dot{x} &= -\lambda_i x_i + y_i + u_{i1} - v_{i1},\ x_i(0) = x_{i0}, \\ \dot{y} &= -\lambda_i y_i + u_{i2} - v_{i2},\ y_i(0) = y_{i0}. \end{aligned} \tag{1.3}$$

The control parameters of the players are subject to geometric constraints. They obtained an equation for the guaranteed pursuit and evasion times.

In the same line of research, generalizing the integral and geometric constraints on players control functions has recently drawn attention of the authors ([12],[13],[14]). In [12], a pursuit and evasion differential game problems of one pursuer/one evader and many pursuers/one evader respectively, is studied in the space $\mathbb{R}^n$, with the constraints

$$\int_0^\infty ||u(t)||^p dt \leq \rho^p \ and \ \int_0^\infty ||v(t)||^p dt \leq \sigma^p.$$

The authors [12] obtained sufficient conditions that guarantee the completion of a pursuit and an evasion. They also construct the players optimal strategies in both problems, and estimated the possible distance that an evader can preserve from pursuers.

Liu Yanfang et al. [15] modelled the scenario of two missiles P and Q intercepting a single target as a two-pursuit single-evader non-zero-sum linear quadratic differential game. The intercept space is decomposed into three subspaces which are mutually disjoint and their union covers the entire intercept space. The effect of adding the second interceptor arises in the intercept space of both P and Q (PQ-intercept space). A guidance law is derived from the Nash equilibrium strategy set (NESS) of the game. Simulation studies are focused on the PQ-intercept space. It is indicated that 1) increasing the targets maneuverability will enlarge PQ-intercept space; 2) the handover conditions will be released if the initial zero-effort-miss (ZEM) of both interceptors has opposite sign; 3) overvaluation of the targets maneuverability by choosing a small weight coefficient will generate robust performance with respect to the target maneuvering command switch time and decrease the fuel requirement; and 4) cooperation between interceptors increases the interception probability.

Zhen-Yu Li [16] investigated an orbital pursuit-evasion-defense game problem with three players called the pursuer, the evader, and the defender, respectively. In this game, the pursuer aims to intercept the evader, while the evader tries to escape the pursuer. A defender accompanying the evader can protect the evader by actively intercepting the pursuer. For such a game, a linear-quadratic duration-adaptive (LQDA) strategy is first proposed as a basic strategy for the three players. Later, an advanced pursuit strategy is designed for the pursuer to evade the defender when they are chasing the evader. Meanwhile, a cooperative evasiondefense strategy is proposed for the evader and the defender to build their cooperation. Simulations determined that the proposed LQDA strategy has higher interception accuracy than the classic LQ strategy. Meanwhile, the proposed two-sided pursuit strategy can improve the interception performance of the pursuer against a non-cooperative defender. But if the evader and defender employ the proposed cooperation strategy, the pursuers interception will be much more difficult.

This present paper come up with a formula for a guaranteed pursuit time, when the players dynamic is described by (1.3) and players' control function is subject to geometric constraints.

2. Statement of the problem

Consider the Hilbert space l_2

$$l_2 = \left\{ \alpha = (\alpha_1, \alpha_2, \alpha_3, \cdots), \sum_{i=1}^{\infty} \alpha_i^2 < \infty \right\},$$

with the norm

$$\|\alpha\| = \left(\sum_{i=1}^{\infty} |\alpha_i|^p \right)^{\frac{1}{p}}.$$

A controlled object is described by the following infinite system of differential equations

$$\begin{aligned} \dot{x} &= -\lambda_i x_i + y_i + u_{i1} - v_{i1},\ x_i(0) = x_{i0}, \\ \dot{y} &= -\lambda_i y_i + u_{i2} - v_{i2},\ y_i(0) = y_{i0}, \end{aligned} \tag{2.1}$$

where $x_0 = (x_{01}, x_{02}, ...) \in l_2$, $y_0 = (y_{01}, y_{02}, ...) \in l_2$, $u = (u_{11}, u_{12}, u_{21}, u_{22}, ...)$ and $v = (v_{11}, v_{12}, v_{21}, v_{22}, ...)$ are the control parameters of pursuer and evader, respectively.

Definition 2.1. A function $u(t) = (u_1(t), u_2(t), ...)$, $t \in [0, T]$, with measurable coordinates $u(t) = (u_1(t), u_2(t),\ i = 1, 2, \cdots)$, subject to the condition

$$\sum_{i=1}^{\infty} (|u_{i1}|^p + |u_{i2}|^p) \le \rho^p,\ 0 \le t \le T \tag{2.2}$$

is referred to as the admissible control of the pursuer.

Definition 2.2. A function $v(t) = (v_1(t), v_2(t), \cdots)$, $t \in [0, T]$, with measurable coordinates $v(t) = (v_1(t), v_2(t),\ i = 1, 2, \cdots)$, subject to the condition

$$\sum_{i=1}^{\infty} (|v_{i1}|^p + |v_{i2}|^p) \le \sigma^p,\ 0 \le t \le T \tag{2.3}$$

is referred to as the admissible control of the evader.

Let $\eta_i(t)=(x_i(t),y_i(t)), U_i=(U_{i1},U_{i2}), v_i=(v_{i1},v_{i2})$.

Definition 2.3 (Guaranteed Pursuit Time, T)**.** Pursuit is said to be completed at time $T>0$, if there exist strategies of the pursuers $U(t,v(t))$, such that for any admissible control of the evader $v(t)$, $0\le t\le T$, $z(\tau)=0$ at some τ, $0\le\tau\le T$. In the sequel, the number T is called guaranteed pursuit time.

Definition 2.4. A strategy of the pursuer is defined as a function of the form

$$U(t,v)=U_0(t)+v=(U_{10}(t)+v_1,U_{20}(t)+v_2,\cdots), \tag{2.4}$$

where $U_0(t)=(U_{10}(t),U_{20}(t),\cdots),U_{i0}(t)=(U_{i01}(t),U_{i02}(t))$, has measurable coordinates $U_{i0}(t)$, $0\le t\le T$ that satisfy the condition

$$\sum_{i=1}^{\infty}\left(|U_{i1}^0(t)|^p+|U_{i2}^0(t)|^p\right)\le(\rho-\sigma)^p\,,\ 0\le t\le T. \tag{2.5}$$

The function $\eta(t)=(\eta_1(t),\eta_2(t),\cdots)$ defined by the equation

$$\begin{aligned}\eta_i(t)=&e^{A_it}\eta_{i0}+\int_0^t e^{A_i(t-s)}\left(u_i(s)-v_i(s)\right)ds,\ i=1,2,\cdots, & (2.6)\\ =&e^{A_it}\left[\eta_{i0}+\int_0^t e^{A_is}\left(u_i(s)-v_i(s)\right)ds\right] & (2.7)\end{aligned}$$

is the unique solution of (2.1). The representation

$$\eta_i(t)=e^{A_it}\gamma(t), \tag{2.8}$$

where

$$\gamma(t)=\left(\gamma_1(t),\gamma_2(t),\cdots\right),\gamma_i(t)=\eta_{i0}+\int_0^t e^{A_i(s)}\left(u_i(s)-v_i(s)\right)ds \tag{2.9}$$

shows that $\eta_i(t)=0$ if and only if $\gamma_i(t)=0$.

3. Main Result

Let for $i=1,2,3,\cdots$

$$e^{A_it}=e^{-\lambda_it}\begin{bmatrix}1 & -t\\ 0 & 1\end{bmatrix},A=\begin{bmatrix}-\lambda_i & \lambda_it\\ 0 & -\lambda_i\end{bmatrix}.$$

And let

$$\begin{aligned}\Omega_i(t)&=\int_0^t e^{-A_is}e^{-A_i^*s}ds \qquad (3.1)\\ &=\int_0^t\begin{bmatrix}e^{\lambda_is} & -e^{\lambda_is}s\\ 0 & e^{\lambda_is}\end{bmatrix}\begin{bmatrix}e^{\lambda_is} & 0\\ -e^{\lambda_is}s & e^{\lambda_is}\end{bmatrix}ds\\ &=\begin{bmatrix}\int_0^t e^{2\lambda_is}\left(1+s^2\right)ds & -\int_0^t e^{2\lambda_is}sds\\ -\int_0^t e^{2\lambda_is}sds & \int_0^t e^{2\lambda_is}ds\end{bmatrix}\\ &=\begin{bmatrix}\psi_{11}(t) & \psi_{12}(t)\\ \psi_{21}(t) & \psi_{22}(t)\end{bmatrix},\end{aligned}$$

where

$$\psi_{11}(t) = \int_0^t e^{2\lambda_i s}\left(1+s^2\right) ds,\ \psi_{12}(t) = -\int_0^t e^{2\lambda_i s} s ds \tag{3.2}$$

$$\psi_{21}(t) = -\int_0^t e^{2\lambda_i s} s ds,\ \psi_{22}(t) = \int_0^t e^{2\lambda_i s} ds.$$

A^* is a transpose of A

$$|\Omega_i(t)| = \psi_{11}(t)\psi_{22}(t) - \psi_{12}^2(t) > 0.$$

Hence the matrix $\Omega_i(t)$ is invertible

$$\Omega_i^{-1}(t) = \begin{bmatrix} \frac{\psi_{22}(t)}{|\Omega_i(t)|} & -\frac{\psi_{12}(t)}{|\Omega_i(t)|} \\ -\frac{\psi_{21}(t)}{|\Omega_i(t)|} & \frac{\psi_{11}(t)}{|\Omega_i(t)|} \end{bmatrix}.$$

This following Lemma is crucial in the prove of our main Theorem.

Lemma 3.1.

$$\lim_{t\to\infty} \sum_{i=1}^{\infty} \frac{2^p e^{p\lambda_i t} \psi_{22}^p(t) \|\eta_{i0}\|^p}{|\Omega_i(t)|^p} = 0 \tag{3.3}$$

$1 < p < \infty,\ 0 < \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots.$

Proof. We estimate $|\Omega_i(t)|^p$ as follows

$$\begin{aligned} |\Omega_i(t)| &= \int_0^t e^{2\lambda_i s}\left(1+s^2\right) \int_0^t e^{2\lambda_i s} ds - \left(\int_0^t e^{2\lambda_i s} s ds \int_0^t e^{2\lambda_i s} s ds\right) \\ &= \left(\int_0^t e^{2\lambda_i s} ds\right)^2 + \int_0^t e^{2\lambda_i s} s^2 ds \int_0^t e^{2\lambda_i s} ds - \left(\int_0^t e^{2\lambda_i s} s ds \int_0^t e^{2\lambda_i s} s ds\right) \\ &= \left(\int_0^t e^{2\lambda_i s} ds\right)^2 + \int_0^t e^{2\lambda_i s} s^2 ds \int_0^t e^{2\lambda_i s} ds - \left(\int_0^t e^{2\lambda_i s} s ds\right)^2 \end{aligned}$$

since by Cauchy Schwartz inequality

$$\left(\int_0^t e^{2\lambda_i s} s ds\right)^2 \le \int_0^t e^{2\lambda_i s} s^2 ds \int_0^t e^{2\lambda_i s} ds$$

then

$$\begin{aligned} |\Omega_i(t)| \ge& \left(\int_0^t e^{2\lambda_i s} ds\right)^2 \\ =&\psi_{22}^2(t), \end{aligned}$$

which implies

$$|\Omega_i(t)|^p \ge \psi_{22}^{2p}(t). \tag{3.4}$$

And recall that

$$\psi_{22}(t) = \int_0^t e^{2\lambda_i s} ds = \frac{e^{2\lambda_i t} - 1}{2\lambda_i} \ge \frac{e^{2\lambda_i t}}{2\lambda_i}\left(1 - \alpha_2(t)\right),\ \alpha_2(t) = e^{-2\lambda_0 t},$$

now we have

$$\psi_{22}^{p}(t) \geq \frac{e^{2p\lambda_i t}}{2^p \lambda_i^p} \left(1-\alpha_2(t)\right)^p. \tag{3.5}$$

Let

$$f_i(t) = \frac{2^p e^{p\lambda_i t} \psi_{22}^p(t) \|\eta_{i0}\|^p}{|\Omega_i(t)|^p}$$

looking at (3.4) we then have

$$\begin{aligned} f_i(t) &\leq \frac{2^p e^{p\lambda_i t} \psi_{22}^p(t) \|\eta_{i0}\|^p}{\psi_{22}^{2p}(t)} \\ &= \frac{2^p e^{p\lambda_i t} \|\eta_{i0}\|^p}{\psi_{22}^p(t)} \end{aligned}$$

from inequality (3.5), $f_i(t)$ can be express as

$$\begin{aligned} f_i(t) &\leq \frac{2^p e^{p\lambda_i t} \|\eta_{i0}\|^p}{\frac{e^{2p\lambda_i t}}{2^p \lambda_i^p} \left(1-\alpha_2(t)\right)^p} \\ &= \frac{2^{2p} \lambda_i^p \|\eta_{i0}\|^p}{e^{p\lambda_i t} \left(1-\alpha_2(t)\right)^p} \\ &= \frac{2^{2p} \lambda_i^p}{e^{p\lambda_i t}} . \frac{1}{\left(1-\alpha_2(t)\right)^p} . \|\eta_{i0}\|^p. \end{aligned}$$

In view of the inequality $e^{2p\lambda_i t} > 2^{2p} \lambda_i^p t^p$, we can establish the following relation

$$\begin{aligned} \frac{2^{2p} \lambda_i^p}{e^{p\lambda_i t}} . \frac{1}{\left(1-\alpha_2(t)\right)^p} &\leq \frac{2^{2p} \lambda_i^p}{\left(2^{2p} \lambda_i^p t^p\right)} . \frac{1}{\left(1-\alpha_2(t)\right)^p} \\ &= \frac{1}{t^p} . \frac{1}{\left(1-\alpha_2(t)\right)^p} \\ &= \frac{1}{t^p} . \frac{1}{\left(1-\alpha_2(t)\right)^p}. \end{aligned}$$

Therefore

$$\sum_{i=1}^{\infty} f_i(t) \leq \frac{1}{t^p} . \frac{1}{\left(1-\alpha_2(t)\right)^p} \sum_{i=1}^{\infty} \|\eta_{i0}\|^p. \tag{3.6}$$

The right-hand side of this inequality (3.6) approaches 0 as $t \to +\infty$ since $\lim_{t\to\infty} \alpha_2(t) = 0$, $i = 1, 2,$. This is the required conclusion. This Lemma clearly indicate that, there is indeed a number θ that satisfies inequality (3.7) in the following Theorem. ∎

Theorem 3.2. *Let $\rho > \sigma$ and let a positive number $t = \theta$ satisfy the inequality*

$$\sum_{i=1}^{\infty} \frac{2^p e^{p\lambda_i t} \psi_{22}^p(t) \|\eta_{i0}\|^p}{|\Omega_i(t)|^p} \leq (\rho - \sigma)^p, \tag{3.7}$$

then, θ is a guaranteed pursuit time in game (2.1) we construct the strategy for the pursuer as follows

$$u_i(t) = \begin{cases} -e^{-tA^*}\Omega^{-1}(\theta)\eta_{i0} + v_i(t), & 0 \le t \le \theta, \\ v_i(t), & t > \theta. \end{cases} \tag{3.8}$$

The functions U_i^0 $i = 1, 2, ...,$ in Definition 2.4 are defined as follows

$$u_i^0(t) = \begin{cases} -e^{-tA^*}\Omega^{-1}(\theta)\eta_{i0} + v_i(t), & 0 \le t \le \theta, \\ 0, & t > \theta. \end{cases} \tag{3.9}$$

To show the admissibility of the constructed strategy, first, we prove inequality (2.5), that is

$$g_i(t) = \left| -e^{-tA^*}\Omega^{-1}(\theta)\eta_{i0} \right| = \frac{e^{\lambda_i t}}{|\Omega_i(\theta)|} \left| \begin{bmatrix} 1 & 0 \\ -t & 1 \end{bmatrix} \begin{bmatrix} \psi_{22}(t) & \psi_{12}(t) \\ \psi_{21}(t) & \psi_{11}(t) \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \right|$$

$$\begin{aligned} g_i(t) =& \frac{e^{\lambda_i t}}{|\Omega_i(\theta)|} \left| \begin{bmatrix} 1 & 0 \\ -t & 1 \end{bmatrix} \begin{bmatrix} \psi_{22}(t) & \psi_{12}(t) \\ \psi_{21}(t) & \psi_{11}(t) \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \right| \\ =& \frac{e^{\lambda_i t}}{|\Omega_i(\theta)|} \left| \begin{bmatrix} \psi_{22}(t) & \psi_{12}(t) \\ -t\psi_{22}(t) + \psi_{21}(t) & -t\psi_{12}(t) + \psi_{11}(t) \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \right| \\ =& \frac{e^{\lambda_i t}}{|\Omega_i(\theta)|} \left| \begin{bmatrix} \psi_{22}(t)x_0 + \psi_{12}(t)y_0 \\ x_0\left(-t\psi_{22}(t) + \psi_{21}(t)\right) + y_0\left(-t\psi_{12}(t) + \psi_{11}(t)\right) \end{bmatrix} \right| \end{aligned}$$

$$|g_i(t)|^p = \frac{e^{p\lambda_i t}}{|\Omega_i(\theta)|^p} |\psi_{22}(t)x_0 + \psi_{12}(t)y_0|^p + |x_0\left(-t\psi_{22}(t) + \psi_{21}(t)\right) + y_0\left(-t\psi_{12}(t) + \psi_{11}(t)\right)|^p.$$

$$|g_i(t)|^p \le \frac{e^{p\lambda_i t}}{|\Omega_i(\theta)|^p} |\psi_{22}(t)x_0 + \psi_{12}(t)y_0|^p + |x_0\left(-t\psi_{22}(t) + t\psi_{21}(t)\right) + y_0\left(-t\psi_{12}(t) + t\psi_{11}(t)\right)|^p.$$

Since $|a+b|^p \le 2^p\left(a^p + b^p\right)$, then

$$\begin{aligned} |g_i(t)|^p \le& \frac{e^{p\lambda_i t}}{|\Omega_i(\theta)|^p} 2^p\left(|\psi_{22}(t)x_0|^p + |\psi_{12}(t)y_0|^p\right) + 2^p|x_0\left(-t\psi_{22}(t) + t\psi_{21}(t)\right)|^p \\ &+ 2^p|y_0\left(-t\psi_{12}(t) + t\psi_{11}(t)\right)|^p. \end{aligned}$$

From (3.2), it clear to see that $\psi_{11}(t) > \psi_{22}(t) > \psi_{12}(t) = \psi_{12}(t)$, then

$$\begin{aligned} |g_i(t)|^p \le& \frac{e^{p\lambda_i t}}{|\Omega_i(\theta)|^p} 2^p\left(|\psi_{22}(t)x_0|^p + |\psi_{22}(t)y_0|^p\right) + 2^p|x_0\left(-t\psi_{22}(t) + t\psi_{22}(t)\right)|^p \\ &+ 2^p|y_0\left(-t\psi_{12}(t) + t\psi_{11}(t)\right)|^p \\ =& \frac{e^{p\lambda_i t}}{|\Omega_i(\theta)|^p} 2^p\left(|\psi_{22}(t)x_0|^p + |\psi_{22}(t)y_0|^p\right) + 2^p\left(|y_0\left(-t\psi_{12}(t) + t\psi_{11}(t)\right)|^p\right) \\ \le& \frac{e^{p\lambda_i t}}{|\Omega_i(\theta)|^p} 2^p|\psi_{22}(t)|^p\left(|x_0|^p + |y_0|^p\right) + (2|y_0|)^p\left(|t\psi_{11}(t) - t\psi_{12}(t) + (t\psi_{12}(t) - t\psi_{11}(t))|^p\right) \\ =& \frac{e^{p\lambda_i t}}{|\Omega_i(\theta)|^p} 2^p|\psi_{22}(t)|^p\left(|x_0|^p + |y_0|^p\right) \\ =& \frac{e^{p\lambda_i t}}{|\Omega_i(\theta)|^p} 2^p|\psi_{22}(t)|^p\left(|x_0|^p + |y_0|^p\right). \end{aligned}$$

Therefore

$$\begin{aligned}|g_i(t)|^p \leq &\frac{e^{p\lambda_i t}}{|\Omega_i(\theta)|^p}.2^p|\psi_{22}|^p\|\eta_{i0}\|^p\\ =&\frac{2^p e^{p\lambda_i t}\psi_{22}^p\|\eta_{i0}\|^p}{|\Omega_i(t)|^p}.\end{aligned}$$

From this we conclude that

$$\begin{aligned}\sum_{i=1}^{\infty}\|U_i^0\|^p =&\sum_{i=1}^{\infty}\left|-e^{-tA^*}\Omega_i^{-1}(\theta)\eta_{i0}\right|^p \qquad (3.10)\\ =&\sum_{i=1}^{\infty}|g_i(t)|^p\\ \leq&\sum_{i=1}^{\infty}\frac{2^p e^{p\lambda_i t}\psi_{22}^p(t)\|\eta_{i0}\|^p}{|\Omega_i(t)|^p}, \qquad (3.11)\end{aligned}$$

and from our assertion in Theorem 3.2

$$\sum_{i=1}^{\infty}\frac{2^p e^{p\lambda_i t}\psi_{22}^p(t)\|\eta_{i0}\|^p}{|\Omega_i(t)|^p} \leq (\rho-\sigma)^p, \tag{3.12}$$

therefore

$$\sum_{i=1}^{\infty}\|U_i^0\|^p \leq (\rho-\sigma)^p.$$

Next we show the admissibility of strategy 3.9 as follows

$$\begin{aligned}\|U(t)\| =&\|U^0(t)+v(t)\| \qquad (3.13)\\ \leq&\|U^0(t)\|+\|v(t)\|\\ =&\left(\sum_{i=1}^{\infty}|U_i^0|^p|\right)^{\frac{1}{p}}+\left(\sum_{i=1}^{\infty}|v_i|^p|\right)^{\frac{1}{p}}\\ \leq&(\rho-\sigma)+\sigma\\ =&\rho.\end{aligned}$$

This show that the strategy is admissible.

What remains is to prove that the time θ is the guaranteed pursuit time, by showing that $\eta(\theta)=0$ when the pursuer use the above admissible strategy. And it suffices to show that $\gamma_i(\theta)=0$.
From equation (2.9),

$$\gamma_i(t) =\eta_{i0}+\int_0^t e^{-A_i s}\left(U_i(s)-v_i(s)\right)ds.$$

Substituting the admissible strategy and the time θ, we have

$$\begin{aligned}\gamma_i(\theta) =&\eta_{i0} + \int_0^\theta e^{-A_i s}\left(-e^{-sA^*}\Omega^{-1}(\theta)\eta_{i0} + v_i(s) - v_i(s)\right) ds\\ =&\eta_{i0} + \int_0^\theta e^{-A_i s}\left(-e^{-sA^*}\Omega^{-1}(\theta)\eta_{i0}\right) ds\\ =&\eta_{i0} - \int_0^\theta e^{-A_i s}\left(e^{-sA^*}\Omega^{-1}(\theta)\eta_{i0}\right) ds\end{aligned}$$

looking at (3.1), $\gamma_i(\theta)$ became

$$\begin{aligned}\gamma_i(\theta) =&\eta_{i0} - \Omega(\theta)\Omega^{-1}(\theta)\eta_{i0}\\ =&\eta_{i0} - \eta_{i0}.\end{aligned}$$

Therefore $\gamma_i(\theta) = 0$; hence $\eta(\theta) = 0$. Thus, the pursuit is completed exactly at the time θ. This completes the proof of the theorem.

4. Numerical Example

Consider of a differential game described by infinite system of differential equations (2.1), and the initial position of the players is given by $x_{i0} = (1, 0, ...)$ *and* $y_{i0} = (0, 0, ...)$ respectively. Let $\eta_{i0} = x_{i0} - y_{i0} = (1, 0, ...)$, $p = 2, \lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3, \cdots, \lambda_k = k, \cdots$ that is $\lambda_i = i\ \forall i \in \mathbb{Z}$. Let $\theta = \ln(\frac{1}{2})$, $\rho = 5$ and $\sigma = 2$ be the resources of pursuer and evader respectively.

According to Theorem 3.2, guaranteed pursuit time have to satisfy the following condition

$$\sum_{i=1}^{\infty} \frac{2^p e^{p\lambda_i t}\psi_{22}^p(t)\|\eta_{i0}\|^p}{|\Omega_i(t)|^p} \leq (\rho - \sigma)^p\,. \tag{4.1}$$

Now starting with the R.H.S of (4.1)

$$(\rho - \sigma)^p = (5 - 2)^2 = 9.$$

The next step is to compute the value of L.H.S of (4.1) and compare it with the R.H.S At the L.H.S of (4.1) we have

$$\frac{2^p e^{p\lambda_i t}\psi_{22}^p(t)\|\eta_{i0}\|^p}{|\Omega_i(t)|^p}.$$

We compute for the denominator as follows

$$\begin{aligned}|\Omega_i(t)| =& \left(\int_0^t e^{2\lambda_i s} ds\right)^2 + \int_0^t e^{2\lambda_i s}s^2 ds \int_0^t e^{2\lambda_i s} ds - \left(\int_0^t e^{2\lambda_i s} s ds\right)^2\\ =& \left(\frac{1}{2\lambda_i}\left[e^{2\lambda_i t} - 1\right]\right)^2 + \left(\frac{t^2 e^{2\lambda_i t}}{2\lambda_i} - \frac{t e^{2\lambda_i t}}{2\lambda_i} + \frac{e^{2\lambda_i t}}{4\lambda_i^2} - \frac{1}{2\lambda_i^2}\right)\left(\frac{1}{2\lambda_i}\left[e^{2\lambda_i t} - 1\right]\right)\\ &- \left(\frac{t e^{2\lambda_i t}}{2\lambda_i} - \frac{e^{2\lambda_i t}}{4\lambda_i^2} - \frac{1}{4\lambda_i^2}\right)^2.\end{aligned}$$

For $i = 1$, $\lambda_1 = 1$ and $t = \ln(0.5)$ we have

$$|\Omega_1(t)| = -0.779.$$

Therefore

$$\begin{aligned}\frac{2^p e^{p\lambda_1 t}\psi_{22}^p(t)\|\eta_{10}\|^p}{|\Omega_1(t)|^p} =& \frac{2^2 e^{2(1)\ln(0.5)}\left(\frac{1}{2\lambda_1}\left[e^{2\lambda_1 t}-1\right]\right)^p (1)^2}{(-0.779)^2}\\ =& \frac{4\left(\frac{1}{2(1)}\left[e^{2(1)\ln(0.5)}-1\right]\right)^2 (1)^2}{(-0.779)^2}\\ =&0.\end{aligned}$$

For $i=2$, $\lambda_2=2$ and $t=\ln(0.5)$ we have

$$|\Omega_2(t)| = -0.091$$

Therefore

$$\begin{aligned}\frac{2^p e^{p\lambda_2 t}\psi_{22}^p(t)\|\eta_{20}\|^p}{|\Omega_2(t)|^p} =& \frac{2^2 e^{2(2)\ln(0.5)}\left(\frac{1}{2\lambda_2}\left[e^{2\lambda_2 t}-1\right]\right)^p (1)^2}{(-0.091)^2}\\ =& \frac{4\left(\frac{1}{2(2)}\left[e^{2(2)\ln(0.5)}-1\right]\right)^2 (1)^2}{(-0.091)^2}\\ =&-59.52.\end{aligned}$$

For $i=3$, $\lambda_3=3$ and $t=\ln(0.5)$ we have

$$|\Omega_3(t)| = -0.243.$$

Therefore

$$\begin{aligned}\frac{2^p e^{p\lambda_3 t}\psi_{22}^p(t)\|\eta_{30}\|^p}{|\Omega_3(t)|^p} =& \frac{2^2 e^{2(3)\ln(0.5)}\left(\frac{1}{2\lambda_3}\left[e^{2\lambda_3 t}-1\right]\right)^p (1)^2}{(-0.243)^2}\\ =& \frac{4\left(\frac{1}{2(3)}\left[e^{2(3)\ln(0.5)}-1\right]\right)^2 (1)^2}{(-0.243)^2}\\ =&-5.49.\end{aligned}$$

From these computations, it is clear that all the values of

$$\frac{2^p e^{p\lambda_i t}\psi_{22}^p(t)\|\eta_{i0}\|^p}{|\Omega_i(t)|^p}$$

are running in negative side of $\mathbb{R}$, which implies that

$$\sum_{i=1}^{\infty}\frac{2^p e^{p\lambda_i t}\psi_{22}^p(t)\|\eta_{i0}\|^p}{|\Omega_i(t)|^p} = -k,\ k\in\mathbb{R}^+$$

$-k < 9\ \forall k\in\mathbb{R}^+$, hence the L.H.S of (4.1) is less than the R.H.S. That is

$$\sum_{i=1}^{\infty}\frac{2^p e^{p\lambda_i t}\psi_{22}^p(t)\|\eta_{i0}\|^p}{|\Omega_i(t)|^p} \le (\rho-\sigma)^p.$$

Now since the hypothesis of the Theorem 3.2 holds at time $t=\ln(0.5)$, then pursuit can be completed and hence $t=\ln(0.5)$ is the guaranteed pursuit time.

5. Conclusion

We have studied a pursuit differential game described by an infinite system of binary differential equations. Control functions of pursuers and evader are subject to geometric constraints. We provided a condition in the Theorem 3.2, such that when ever a certain time θ satisfies it, that time θ is the guaranteed pursuit time. We also state and proved Lemma 3.3 which helped us in the prove of our main theorem. We further gave numerical example to illustrate our results.

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