Bangmod Int. **J.** **M**ath. **&** **C**omp. **S**ci.
Vol. 3, No. 1-2, 2017; Pages 113 - 122
http://bangmod-jmcs.kmutt.ac.th/
ISSN : 2408-154X

Ricci solitons in generalized Sasakian space form

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Abstract The objective of the present paper is to study the Ricci solitons in a generalized Sasakian space forms with Killing and conformal Killing vector field.

MSC: 53C15, 53C25, 53D15.
Keywords: Generalized Sasakian space form, Ricci tensor, Ricci operator, scalar curvature.

Submission date: 24 September 2017 / Acceptance date: 14 November 2017 /Available online 31 December 2017

1. Introduction

A Ricci soliton (g, V, λ) is a natural generalization of an Einstein metric and is defined on a Riemannian manifold (M, g) by

$$(\mathcal{L}_V g + 2S + 2\lambda g)(X, Y) = 0, \tag{1.1}$$

where S is the Ricci tensor of M, $\mathcal{L}_V$ denote the Lie derivative operator along the vector field V and λ is a real scalar. The Ricci soliton is said to be shrinking, steady or expanding according as λ is negative, zero and positive respectively.

Ricci soliton is a special solution of the Ricci flow introduced by Hamilton [12] in the year 1982. In [19], R. Sharma studied Ricci solitons in contact geometry. Thereafter Ricci solitons in contact metric manifolds have been studied by various authors such as M. M. Tripathi [20], A. Ghosh and R. Sharma [11], U. C. De and et. al. [10], H. G. Nagaraja and et. al. [16], C. S. Bagewadi and et. al. [4, 5], A. A. Shaikh and et. al. [8], S. K. Hui et. al. [13, 15] and many others.

The nature of a Riemannian manifold mostly depends on the curvature tensor R of the manifold and further it is known that the sectional curvature of a manifold determines curvature tensor completely. A Riemannian manifold with constant sectional curvature c is known as real space form and its curvature tensor is given by

$$R(X, Y)Z = c\{g(Y, Z)X - g(X, Z)Y\}. \tag{1.2}$$

Published by **T**heoretical **and** **C**omputational **S**cience Center **(TaCS)**,
King Mongkut's University of Technology Thonburi **(KMUTT)**

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A Sasakian manifold with constant ϕ-sectional curvature is called Sasakian space form and the curvature tensor of such manifold is given by

$$\begin{aligned} R(X,Y)Z &= \frac{c+3}{4}\{g(Y,Z)X - g(X,Z)Y\} + \frac{c-1}{4}\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X \\ &+ 2g(X,\phi Y)\phi Z\} + \frac{c-1}{4}\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi \\ &- g(Y,Z)\eta(X)\xi\}. \end{aligned} \tag{1.3}$$

In 2004, P. Alegre, D. E. Blair and A. Carriazo [1] introduced the concept of generalized Sasakian space forms. The generalized Sasakian space form is defined as follows:

A generalized Sasakian space form in an almost contact metric manifold (M, ϕ, ξ, η, g) whose curvature tensor is given by

$$\begin{aligned} R(X,Y)Z &= f_1\{g(Y,Z)X - g(X,Z)Y\} + f_2\{g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X \\ &+ 2g(X,\phi Y)\phi Z\} + f_3\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi \\ &- g(Y,Z)\eta(X)\xi\}, \end{aligned} \tag{1.4}$$

where f_1, f_2, f_3 are differentiable functions on M and X, Y, Z are vector fields on M. This type of manifold appears as a natural generalization of the well known Sasakian space form $M(c)$, which can be obtained as a particular case of generalized Sasakian space form by taking $f_1 = \frac{c+3}{4}$, $f_2 = \frac{c-1}{4}$ and $f_3 = \frac{c-1}{4}$, where c denotes constant ϕ-sectional curvature.

The generalized Sasakian space forms have been studied by several authors such as P. Alegre and A. Carriazo [2, 3], M. Belkhelfa, R. Deszcz and L. Verstraelen [6], U. C. De and et. al. [9], A. A. Shaikh and et. al. [17] and many others.

Motivated by the above work, in this paper we study Ricci solitons in generalized Sasakian space forms.

2. Preliminaries

A differentiable manifold M is said to be an almost contact metric manifold if there exist a $(1,1)$ tensor field ϕ, a vector field ξ, a 1-form η and Riemannian metric g, which satisfy

$$\phi^2 X = -X + \eta(X)\xi, \quad \eta(\xi) = 1, \quad \phi \cdot \xi = 0, \quad \eta(\phi X) = 0, \tag{2.1}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \quad g(X,\xi) = \eta(X), \tag{2.2}$$

$$g(\phi X, Y) = -g(X, \phi Y) \tag{2.3}$$

for all vector fields X, Y on M. An almost contact metric manifold $M(\phi, \xi, \eta, g)$ is said to be a Sasakian manifold [7] if

$$(\nabla_X \phi)Y = g(X,Y)\xi - \eta(Y)X. \tag{2.4}$$

From (2.4), it follows that

$$\nabla_X \xi = -\phi X, \tag{2.5}$$

for any vector field X on M, where ∇ is the covariant derivative of M.

By virtue of (2.2) in (1.4), we have

$$R(X,Y)\xi = (f_1 - f_3)\{\eta(Y)X - \eta(X)Y\}. \tag{2.6}$$

Again from (1.4) and by taking an account of $S(X,Y)=\sum_{i=1}^{(2n+1)} g(R(e_i,X)Y,e_i)$, we get

$$S(X,Y) = [2nf_1+3f_2-f_3]g(X,Y)+[-3f_2-(2n-1)f_3]\eta(X)\eta(Y). \quad (2.7)$$

From (2.7), we have

$$QX = [2nf_1+3f_2-f_3]X+[-3f_2-(2n-1)f_3]\eta(X)\xi, \quad (2.8)$$

$$r = 2n(2n+1)f_1+6nf_2-4nf_3, \quad (2.9)$$

where Q is the Ricci operator and r is the scalar curvature of M. Putting $Y=\xi$ in (2.7), we get

$$S(X,\xi) = 2n(f_1-f_3)\eta(X). \quad (2.10)$$

On Riemannian manifold (M,g), we have

$$(\mathcal{L}_V g)(X,Y)=g(\nabla_X V,Y)+g(X,\nabla_Y V), \quad (2.11)$$

where ∇ denotes the Levi-Civita connection of M. If (M,g) is a Ricci soliton with potential vector field V, then by using (2.11) in (1.1), we obtain

$$g(\nabla_X V,Y)+g(X,\nabla_Y V)+2S(X,Y)+2\lambda g(X,Y)=0. \quad (2.12)$$

By taking $X=Y=e_i$, where $\{e_i : i=1,2,\dots,2n+1\}$ is an orthonormal basis, we get

$$divV+r+(2n+1)\lambda=0. \quad (2.13)$$

By taking volume integral in (2.13), we obtain

$$\int divV\mu_g+\int r\mu_g+\int(2n+1)\lambda\mu_g=0. \quad (2.14)$$

We know by Green's theorem $\int divV\mu_g=0$, we have

$$\int r\mu_g=-(2n+1)\lambda vol(M). \quad (2.15)$$

Hence, we state the following theorem:

Theorem 2.1. *Let (M,g) be a Ricci soliton with respect to potential vector field V on M. Then*

- $\int r\mu_g=-(2n+1)\lambda vol(M)$.
- *$divV=0$ if and only if either $r\le-(2n+1)\lambda$ or $r\ge-(2n+1)\lambda$ on M, then $r=-(2n+1)\lambda$.*

3. Parallel symmetric second order tensors and Ricci solitons in generalized Sasakian space form

Fix h a symmetric tensor field of $(0,2)$-type which is parallel with respect to Levi-Civita connection ∇ that is $\nabla h=0$. Applying the Ricci identity [18]

$$\nabla^2h(X,Y;Z,W)-\nabla^2h(X,Y;W,Z)=0, \quad (3.1)$$

we obtain the relation

$$h(R(X,Y)Z,W)+h(Z,R(X,Y)W)=0. \quad (3.2)$$

Replacing $Z=W=\xi$ in (3.2) and by virtue of (2.6) and by the symmetry of h, we have

$$2(f_1-f_3)[\eta(Y)h(X,\xi)-\eta(X)h(Y,\xi)]=0. \quad (3.3)$$

Putting $X = \xi$ in (3.3), we get

$$2(f_1 - f_3)[\eta(Y)h(\xi,\xi) - h(Y,\xi)] = 0. \tag{3.4}$$

And supposing $f_1 - f_3 \neq 0$, it results

$$h(Y,\xi) = \eta(Y)h(\xi,\xi). \tag{3.5}$$

We call a regular generalized Sasakian space form with $f_1 - f_3 \neq 0$.

Differentiating (3.5) covariantly with respect to X, we have

$$\nabla_X(h(Y,\xi)) = \nabla_X(g(Y,\xi)h(\xi,\xi)), \tag{3.6}$$

on expanding the above equation and by virtue of (3.5), $\nabla h = 0$, $\eta(\nabla_X \xi) = 0$, we get

$$h(Y,\phi X) = g(Y,\phi X)h(\xi,\xi). \tag{3.7}$$

Replace $X = \phi X$ in (3.7), we get

$$h(X,Y) = g(X,Y)h(\xi,\xi). \tag{3.8}$$

This implies that $h(\xi,\xi)$ is a constant, via (3.5). Hence we state the following theorem:

Theorem 3.1. *A symmetric parallel second order covariant tensor in a regular generalized Sasakian space form is a constant multiple of the metric tensor.*

Suppose that the $(0,2)$-type symmetric tensor field $\mathcal{L}_V g + 2S$ is parallel for any vector field V on a generalized Sasakian space form. Then Theorem (3.1) yields $\mathcal{L}_V g + 2S$ is a constant multiple of the metric tensor g, i.e. $(\mathcal{L}_V g)(X,Y) + 2S(X,Y) = -2\lambda g(X,Y)$ for all X, Y on M, where λ is a constant. Hence the relation (1.1) holds. This implies that (g, V, λ) yields a Ricci soliton. Hence we can state the following:

Theorem 3.2. *If the tensor field $\mathcal{L}_V g + 2S$ on a generalized Sasakian space form is parallel for any vector field V, then (g, V, λ) is a Ricci soliton.*

Again for a $(0,2)$-type symmetric parallel tensor field h in a generalized Sasakian space form such that

$$h(X,Y) = (\mathcal{L}_\xi g)(X,Y) + 2S(X,Y). \tag{3.9}$$

Putting $X = Y = \xi$ in (3.9) and by virtue of (2.7), we obtain

$$h(\xi,\xi) = 4n(f_1 - f_3). \tag{3.10}$$

If (g, V, λ) is a Ricci soliton on a generalized Sasakian space form, then from (1.1) we have

$$h(X,Y) = -2\lambda g(X,Y). \tag{3.11}$$

Putting $X = Y = \xi$ in (3.11), we get

$$h(\xi,\xi) = -2\lambda. \tag{3.12}$$

From (3.10) and (3.12) we get $\lambda = -2n(f_1 - f_3)$ and consequently the Ricci soliton (g, ξ, λ) is shrinking if $f_1 > f_3$ or expanding if $f_1 < f_3$. Thus we can state the following:

Theorem 3.3. *If the tensor field $\mathcal{L}_V g + 2S$ on a generalized Sasakian space form is parallel, then the Ricci soliton (g, ξ, λ) is shrinking or expanding.*

4. Ricci solitons in generalized Sasakian space form

In this section, we study Ricci solitons in generalized Sasakian space form:

Theorem 4.1. *If a metric g in a generalized Sasakian space form is a Ricci soliton with $V = \xi$, then it is Einstein.*

Proof. Putting $V = \xi$ in (1.1), then we have

$$(\mathcal{L}_\xi g)(X, Y) + 2S(X, Y) + 2\lambda g(X, Y) = 0. \tag{4.1}$$

where

$$(\mathcal{L}_\xi g)(X, Y) = g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) = 0. \tag{4.2}$$

Substituting (4.2) in (4.1), then we get the result. ∎

Proposition 4.2. *Ricci soliton in generalized Sasakian space form with V point-wise collinear with ξ, then V is a constant multiple of ξ and the manifold is Einstein.*

Proof. Putting $V = b\xi$ in (4.1), we get

$$g(\nabla_X(b\xi), Y) + g(X, \nabla_Y(b\xi)) + 2S(X, Y) + 2\lambda g(X, Y) = 0. \tag{4.3}$$

The above equation (4.3) can be written in the form

$$(Xb)\eta(Y) + (Yb)\eta(X) + 2S(X, Y) + 2\lambda g(X, Y) = 0. \tag{4.4}$$

Putting $Y = \xi$ in (4.4), we have

$$(Xb) + (\xi b)\eta(X) + 4n(f_1 - f_3)\eta(X) + 2\lambda g(X, Y) = 0. \tag{4.5}$$

Again putting $X = \xi$ in (4.5), we obtain

$$(\xi b) = -[2n(f_1 - f_3) + \lambda]. \tag{4.6}$$

Substituting (4.6) in (4.5), we get

$$(Xb) = -[2n(f_1 - f_3) + \lambda]\eta(X). \tag{4.7}$$

which implies

$$db = -[2n(f_1 - f_3) + \lambda]\eta. \tag{4.8}$$

Applying d on both sides,

$$d^2 b = -[2n(f_1 - f_3) + \lambda]d\eta. \tag{4.9}$$

Equation (4.9) implies that $d^2 b = 0$, but $d\eta$ is nowhere vanishing. Therefore, $-2n(f_1 - f_3) - \lambda = 0$ which implies $db = 0$; that is, b is constant. As ξ is Killing, we conclude that the manifold is Einstein which completes the proof. ∎

Theorem 4.3. *A generalized Sasakian space form admitting a Ricci soliton (g, V), where the potential vector field V is orthogonal to ξ is shrinking if $f_1 > f_3$, expanding if $f_1 < f_3$ or steady if $f_1 = f_3$.*

Proof. Suppose that a generalized Sasakian space form admits a Ricci soliton (g, V), then from (2.11) in (1.1), we have

$$g(\nabla_X V, Y) + g(X, \nabla_Y V) + 2S(X, Y) + 2\lambda g(X, Y) = 0. \tag{4.10}$$

Putting $X = Y = \xi$ in (4.10), we get

$$2g(\nabla_\xi V, \xi) + 2S(\xi, \xi) + 2\lambda g(\xi, \xi) = 0. \tag{4.11}$$

For a vector field V orthogonal to ξ and by virtue of (2.10), we obtain

$$\lambda = -2n(f_1 - f_3). \tag{4.12}$$

According as $f_1 > f_3$, $f_1 < f_3$, $f_1 = f_3$, then $\lambda < 0$, $\lambda > 0$, $\lambda = 0$ that is, generalized Sasakian space form admitting a Ricci soliton is shrinking, expanding or steady. This completes the proof of the theorem. ∎

Theorem 4.4. *A Ricci soliton in generalized Sasakian space form with Killing vector field ξ is shrinking if $2n[(2n+1)f_1 + 3f_2 - 2f_3] > 0$, expanding if $2n[(2n+1)f_1 + 3f_2 - 2f_3] < 0$ or steady if $2n[(2n+1)f_1 + 3f_2 - 2f_3] = 0$.*

Proof. By using (2.11) and (2.7) in (1.1) with $V = \xi$, we have

$$\begin{aligned} &g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) + 2\{[2nf_1 + 3f_2 - f_3]g(X,Y) \\ &+[-3f_2 - (2n-1)f_3]\eta(X)\eta(Y)\} + 2\lambda g(X,Y) = 0. \end{aligned} \tag{4.13}$$

Putting $X = Y = e_i$, where $\{e_i : i = 1, 2, \ldots, (2n+1)\}$ is an orthonormal basis, we obtain

$$div\xi + 2n[(2n+1)f_1 + 3f_2 - 2f_3] + (2n+1)\lambda = 0, \tag{4.14}$$

the above equation implies that

$$\lambda = -\frac{div\xi}{(2n+1)} - \frac{2n[(2n+1)f_1 + 3f_2 - 2f_3]}{(2n+1)}. \tag{4.15}$$

If ξ is a Killing vector field then $div\xi = 0$, the above equation reduces to

$$\lambda = -\frac{2n[(2n+1)f_1 + 3f_2 - 2f_3]}{(2n+1)}. \tag{4.16}$$

That is Ricci soliton in generalized Sasakian space form with Killing vector field ξ is shrinking, expanding or steady as $\lambda < 0$, $\lambda > 0$ or $\lambda = 0$. This completes the proof. ∎

Definition 4.5. A vector field V is said to be conformal Killing vector field if it satisfies

$$\mathcal{L}_V g = 2\rho g. \tag{4.17}$$

for some scalar function ρ.

Theorem 4.6. *Let (g, V) be a Ricci soliton in a generalized Sasakian space form. If V is conformal Killing vector field then the followings are equivalent:*

(1) Einstein
(2) locally Ricci symmetric
(3) Ricci semisymmetric; that is, $R \cdot S = 0$.

Proof. Suppose that V is a conformal Killing vector field and from (1.1), we have

$$2\rho g(X,Y) + 2S(X,Y) + 2\lambda g(X,Y) = 0. \tag{4.18}$$

Equation (4.18) can be written in the form

$$S(X,Y) = -(\rho + \lambda)g(X,Y). \tag{4.19}$$

This shows that the Ricci soliton in a generalized Sasakian space form under consideration is Einstein, that is (1) holds.

The implication (1) → (2) → (3) is trivial. Now, we prove the implication (3) → (1). Now,

$$(R(X,Y) \cdot S)(U,V) = -S(R(X,Y)U,V) - S(U,R(X,Y)V). \tag{4.20}$$

Using (4.19) in (4.20), we obtain

$$(R(X,Y)\cdot S)(U,V) = (\rho+\lambda)[g(R(X,Y)U,V) + g(U,R(X,Y)V)] = 0, \tag{4.21}$$

which implies that the Ricci soliton in a generalized Sasakian space form is Ricci semisymmetric.
Consider $R(X,Y)\cdot S = 0$ and putting $X=\xi$ in (4.20) and by virtue of (2.7), we get

$$(f_1-f_3)[g(Y,U)S(\xi,V) - \eta(U)S(Y,V) + g(Y,V)S(U,\xi) - \eta(V)S(U,Y)] = 0. \tag{4.22}$$

If $f_1 \neq f_3$, then

$$g(Y,U)S(\xi,V) - \eta(U)S(Y,V) + g(Y,V)S(U,\xi) - \eta(V)S(U,Y) = 0. \tag{4.23}$$

Putting $U=\xi$ in (4.23) and by using (2.10), we obtain

$$S(Y,V) = 2n(f_1-f_3)g(Y,V) \qquad or \qquad S = 2n(f_1-f_3)g. \tag{4.24}$$

Generalized Sasakian space form with $f_1 \neq f_3$ is Einstein, that is (3) → (1). ∎

From (4.18) and (4.24), we get

$$\lambda = -[\rho + 2n(f_1-f_3)]. \tag{4.25}$$

This leads the following:

Theorem 4.7. *A Ricci soliton in a generalized Sasakian space form with conformal Killing vector field V is shrinking if $[\rho+2n(f_1-f_3)]>0$, expanding if $[\rho+2n(f_1-f_3)]<0$ or steady if $[\rho+2n(f_1-f_3)]=0$.*

5. Ricci solitons in Submanifolds of generalized Sasakian space form

Let M be a submanifold of a generalized Sasakian space form $M(c)$ then the Gauss and Weingarten formula [22] is given by

$$\bar\nabla_X Y = \nabla_X Y + B(X,Y), \qquad \bar\nabla_X V = -A_V X + D_X V, \tag{5.1}$$

for tangent vector fields X and Y, where B is the second fundamental form, A is the shape operator and $g(B(X,Y),V) = g(A_V X,Y)$.
By virtue of Gauss and Weingarten formula, we have

$$\begin{aligned} R(X,Y,Z,W) &= f_1[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] + f_2[g(X,\phi Z)g(\phi Y,W) \\ &- g(Y,\phi Z)g(\phi X,W) + 2g(X,\phi Y)g(\phi Z,W)] + f_3[\eta(X)\eta(Z)g(Y,W) \\ &- \eta(Y)\eta(Z)g(X,W) + g(X,Z)\eta(Y)\eta(W) - g(Y,Z)\eta(X)\eta(W)] \\ &+ g(B(Y,Z),B(X,W)) - g(B(X,Z),B(Y,W)). \end{aligned} \tag{5.2}$$

and

$$(\nabla_X B)(Y,Z) = (\nabla_Y B)(X,Z). \tag{5.3}$$

Putting $X=W=e_i$ in (5.2), where $\{e_i : i = 1,2,\dots,(2n+1)\}$ is an orthonormal basis, we obtain

$$\begin{aligned} S(Y,Z) &= [2nf_1 + 3f_2 - f_3]g(Y,Z) + [-3f_2 - (2n-1)f_3]\eta(Y)\eta(Z) \\ &+ \sum_{e_i}(trA_{e_i})g(A_{e_i}Y,Z) - \sum_{e_i} g(A_{e_i}Y, A_{e_i}Z). \end{aligned} \tag{5.4}$$

Theorem 5.1. *Ricci soliton in submanifold of generalized Sasakian space form with Killing vector field ξ is*

- *shrinking if* $[2n[(2n+1)f_1 + 3f_2 - 2f_3] + \sum_{e_i}(trA_{e_i})^2 - \sum_{e_i}(trA^2_{e_i})] > 0$
- *expanding if* $[2n[(2n+1)f_1 + 3f_2 - 2f_3] + \sum_{e_i}(trA_{e_i})^2 - \sum_{e_i}(trA^2_{e_i})] < 0$
- *steady if* $[2n[(2n+1)f_1 + 3f_2 - 2f_3] + \sum_{e_i}(trA_{e_i})^2 - \sum_{e_i}(trA^2_{e_i})] = 0.$

Proof. By using (2.12) and (5.4) in (1.1), we have

$$g(\nabla_X V, Y) + g(X, \nabla_Y V) + 2[[2nf_1 + 3f_2 - f_3]g(X,Y) + [-3f_2 - (2n-1)f_3]\eta(X)\eta(Y) \\ + \sum_{e_i}(trA_{e_i})g(A_{e_i}X, Y) - \sum_{e_i} g(A_{e_i}X, A_{e_i}Y)] + 2\lambda g(X,Y) = 0. \tag{5.5}$$

Putting $X = Y = e_i$ in (5.5), where $\{e_i : i = 1, 2, \ldots, (2n+1)\}$ is an orthonormal basis, we get

$$divV + 2n[(2n+1)f_1 + 3f_2 - 2f_3] + \sum_{e_i}(trA_{e_i})^2 - \sum_{e_i}(trA^2_{e_i}) + (2n+1)\lambda = 0. \tag{5.6}$$

The above equation (5.6) can be written in the form

$$\lambda = -\left[\frac{divV}{(2n+1)} + \frac{2n[(2n+1)f_1 + 3f_2 - 2f_3] + \sum_{e_i}(trA_{e_i})^2 - \sum_{e_i}(trA^2_{e_i})}{(2n+1)}\right]. \tag{5.7}$$

If $V = \xi$ is a Killing vector field then $div\xi = 0$, then the above equation reduces in the form

$$\lambda = -\left[\frac{2n[(2n+1)f_1 + 3f_2 - 2f_3] + \sum_{e_i}(trA_{e_i})^2 - \sum_{e_i}(trA^2_{e_i})}{(2n+1)}\right]. \tag{5.8}$$

That is Ricci soliton in submanifold of generalized Sasakian space form with Killing vector field ξ is shrinking, expanding or steady as $\lambda < 0$, $\lambda > 0$ or $\lambda = 0$. ∎

Conflict of Interests

The author declare that there is no conflict of interests regarding the publication of this paper.

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