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α-ADMISSIBLE MAPPING IN RECTANGULAR METRIC SPACE INVOLVING THE EXISTENCE SOLUTION OF THE CAPUTO FRACTIONAL DIFFERENTIAL EQUATION WITH NON-LOCAL INTEGRAL CONDITIONS

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Abstract In this article, we investigate and demonstrate the existence of a solution to a non-linear fractional differential equation with a non-local integral condition by finding new conditions of a fixed point in rectangular metric space for certain α-admissible condition mapping.

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Keywords: α-admissible mappings, rectangular metric space, Caputo fractional derivative, non-local integral conditions, fixed point theorem.

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1. Introduction

Fixed point theorem is a branch of function analysis. Today, mathematicians continually study and research in this field to devise new knowledge of the theorem. It is very important in other fields as an important basis for the development of basic science. Fixed point theorem is one of the most widely applied fields, especially to the study of Existence and Uniqueness of solution of various equations, especially fractional differential equations and boundary value problem.

Let (E, d) be a complete metric space and $\mathcal{F} : E \to E$ be a mapping. If there exists $\kappa \in (0, 1)$ such that for all $\mu, \nu \in E$, $d(\mathcal{F}\mu, \mathcal{F}\nu) \leq \kappa d(\mu, \nu)$, then $\mathcal{F}$ is said to be a contractive mapping.

In 1922, Polish mathematician Banach [1] demonstrated a crucial finding relating to a contraction mapping, known as the Banach contraction principle. It is one of the fundamental results in metric fixed point theory.Because of its significance and ease of use, various authors have attained. The Banach contraction principle has numerous intriguing extensions (see [2–10] and the references therein). The result was extended by S.B. Nadler [11] to the context of set valued contraction.

In 2013,[12] N. Hussain, E. Kanwal, P. Salimi and F. Akbar show the existence and uniqueness of a fixed point for certain α-admissible contraction mapping investigated the existence and uniqueness.

In 2018, [13] A. Hussain, T. Kanwal, M.Adeel, S.Radenovic, In order to demonstrate the presence of the optimal proximity point outcomes in the context of b-metric spaces, it is important to integrate the aforementioned notions in a more general approach for set valued and single valued mappings. Supplying, the authors demonstrate some of the best proximity point results using the concept of a graph with b-metric space. Some cement Examples are provided to show how the outcomes were achieved. Moreover,the authors investigated the existence of the solution of non-linear fractional differential equation with Caputo derivative.

In 2020, [14] P. Borisut, P. Kumam, I. Ahmed, K. Sitthithakerngkiet study and consider the positive solution of fractional differential equation with nonlocal multi-point conditions of the form:

$$\begin{cases} {}_{RL}\mathcal{D}^{q}_{0+}\mu(t) + g(t)f(t, \mu(t)) = 0, t \in (0, 1), \\ \mu^{(k)}(0) = 0, \ \mu(1) = \sum_{i=1}^{m} \beta_i \ {}_{RL}\mathcal{D}^{p_i}_{0+}\mu(\eta_i), \end{cases}$$

where $q \in (n-1, n)$, $n \geq 2$, $p_i \in (n-1, n)$, $q > p_i$, $m, n \in \mathbb{N}$, $k = 0, 1, \cdots, n-2$, $0 < \eta_1 < \eta_2 < \cdots < \kappa$, $\beta_i \leq 0$, $0 < \kappa \leq 1$, ${}_{RL}\mathcal{D}^{q}_{0+}$, ${}_{RL}\mathcal{D}^{p_i}_{0+}$ are the Riemann-Liouville fractional derivative of order q, p_i, $g : (0, 1) \to \mathbb{R}^+$ is continuous functions and $f : [0, 1] \times C([0, 1], E) \to E$, E be Banach space. The main tools for finding positive solutions of the above problem are the Guo-Krasnoselskii fixed point theorem and Boyd-Wong fixed point theorem.

Inspired by the above paper in [12–14], the objection of this paper is to improve the situation. We consider the existence and uniqueness of a fixed point in rectangular metric space for

$$(d(f\mu, f\nu) + l)^{\alpha(\mu, f\mu)\alpha(\nu, f\nu)} \leq \theta(d(\mu, \nu))d(\mu, \nu) + l$$

Our results generalize and existend on the topic. We will consider the existence of the solution of non-linear Caputo fractional differential equation with non-local integral conditions.

$$\begin{cases} {}^C\mathcal{D}^q\mu(t) = f(t,\mu(t)), \ t \in J = [0,T] \\ \mu^{(k)}(0) = 0, \ \mu(T) = \tau \int_0^T \mu(s)ds, \end{cases} \tag{1.1}$$

where $q \in (n-1,n)$, $n \in \mathbb{N}$, $k = 0,1,...,n-2$ and ${}^C\mathcal{D}^q$ is the Caputo fractional derivatives, $f : J \times C(J,\mathcal{B}) \to \mathcal{B}$, $\mathcal{B}$ be rectangular metric space and $\tau < \frac{n}{T}$. In Section 2, we present some known results and introduce some conditions to be used in the next section. The main results formulated and proved in Section 3, also a Caputo derivative of fractional order is presented to demonstrate the applications for guarantee of the main results.

2. Background materials

We present notations, definitions, and background information in this part that will be utilized throughout the paper.

Definition 2.1. [15] Let $\mathcal{B}$ be a nonempty set, and let $d : \mathcal{B} \times \mathcal{B} \to [0,\infty)$ be a mapping such that for all $\mu, \nu \in \mathcal{B}$ and all distinct points $r, s \in \mathcal{B}$, each distinct from μ and ν
1) $d(\mu,\nu) = 0$ if and only if $\mu = \nu$
2) $d(\mu,\nu) = d(\nu,\mu)$
3) $d(\mu,\nu) \leq d(\mu,s) + d(s,r) + d(r,\nu)$ (rectangular inequality).

Then $(\mathcal{B},d)$ is called a rectangular or generalized metric space(g.m.s).

Definition 2.2. [16] Let $f : \mathcal{B} \to \mathcal{B}$ and $\alpha : \mathcal{B}\times\mathcal{B} \to \mathbb{R}^+$. We say that f is an α-admissible mapping if $\alpha(\mu,\nu) \leq 1$ implies $\alpha(f\mu, f\nu) \leq 1, \mu,\nu \in \mathcal{B}$.

Definition 2.3. [17]. Let $f : (0,\infty) \to \mathbb{R}$ be a function and $q > 0$. The Riemann-Liouville fractional integral of orders q is defined by

$${}_{RL}\mathcal{I}^q_{0+} f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{\alpha-1} f(s)ds$$

provided that the integral exists. The Caputo fractional derivative of order q is defined by

$${}^C\mathcal{D}^q_{0+} f(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} f^{(n)}(s)ds$$

provided that the right side is point wise defined on $(0,\infty)$, where $n = [q]+1$, $q \in (n-1,n)$, and Γ denotes the gamma function. If $q = n$, then ${}^C\mathcal{D}^q_{0+} f(t) = f^{(n)}(t)$.

Lemma 2.4. [18]. *Let $q \in (n-1,n)$. If $f \in C^n([a,b])$, then*

$${}_{RL}\mathcal{I}^q_{0+}({}^C\mathcal{D}^q f)(t) = f(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1}t^{n-1}$$

for some $c_i \in \mathbb{R}$, $i = 0,1,2,\cdots,n-1$, where n is the smallest integer greater than or equal to q.

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3. Main results

In this section we study the existence of solutions for the integral boundary value problem of non-linear fractional differential equation. (1.1).By the fixed point theorem from the theorem to prove the following.

Theorem 3.1. *Let $(\mathcal{B}, d)$ be a complete rectangular metric space and $\mathcal{F} : \mathcal{B} \to \mathcal{B}$ be an α-admissible mapping. Suppose that there exists a function $\theta : [0, \infty) \to (0, 1]$ such that, $\theta(t_n) \to 1$ implies $t_n \to 0$, for any bounded sequence t_n of $\mathbb{R}^+$ and*

$$\Big(d(\mathcal{F}\mu, \mathcal{F}\nu) + \kappa\Big)^{\alpha(\mu,\mathcal{F}\mu)\alpha(\nu,\mathcal{F}\nu)} \leq \theta(d(\mu, \nu))d(\mu, \nu) + \kappa. \tag{3.1}$$

for all $\mu, \nu \in \mathcal{B}$ where $\kappa \geq 1$. Suppose that either
(a)$\mathcal{F}$ is continuous, or
(b) if μ_n is a sequence in $\mathcal{B}$ such that $\mu_n \to \mu$, $\alpha(\mu_n, \mu_{n+1}) \geq 1$ for all n, then $\alpha(\mu, \mathcal{F}\mu) \geq 1$, then $\mathcal{F}$ has a fixed point.

Proof. Let $\mu_0 \in \mathcal{B}$ such that $\alpha(\mu_0, \mathcal{F}\mu_0) \geq 1$. Define a sequence $\mu_n \in \mathcal{B}$ by $\mu_n = \mathcal{F}\mu_{n-1} = \mathcal{F}^2\mu_{n-2} = \cdots = \mathcal{F}^n\mu_0$ for all $n \in \mathbb{N}$. If $\mu_{n+1} = \mu_n$ for some $n \in \mathbb{N}$, then $\mu = \mu_n$ is a fixed point for $\mathcal{F}$ and the result is proved. Hence, we suppose that $\mu_{n+1} \neq \mu_n$ for all $n \in \mathbb{N}$. Since $\mathcal{F}$ is an α-admissible mapping and $\alpha(\mu_0, \mathcal{F}\mu_0) \geq 1$, we deduce that

$$\begin{aligned} \alpha(\mu_1, \mu_2) &= \alpha(\mathcal{F}\mu_0, \mathcal{F}\mu_1) > 1 \\ &= \alpha(\mathcal{F}\mu_0, \mathcal{F}^2\mu_0) > 1. \end{aligned}$$

By continuing this process, we get $\alpha(\mu_n, \mathcal{F}\mu_n) \geq 1$. By the inequality (3.1), we have

$$\begin{aligned} d(\mathcal{F}\mu_{n-1}, \mathcal{F}\mu_n) + \kappa &\leq \Big(d(\mathcal{F}\mu_{n-1}, \mathcal{F}\mu_n) + \kappa\Big)^{\alpha(\mu_{n-1},\mathcal{F}\mu_{n-1})\alpha(\mu_n,\mathcal{F}\mu_n)} \\ &\leq \theta\Big(d(\mu_{n-1}, \mu_n)\Big) \cdot d(\mu_{n-1}, \mu_n) + \kappa, \end{aligned}$$

then

$$d(\mu_n, \mu_{n+1}) \leq \theta\Big(d(\mu_{n-1}, \mu_n)\Big) \cdot d(\mu_{n-1}, \mu_n) \tag{3.2}$$

which implies $d(\mu_n, \mu_{n+1}) \leq d(\mu_{n-1}, \mu_n)$. It follows that the sequence $\{d(\mu_n, \mu_{n+1})\}$ is decreasing. Thus, there exists $c^* \in \mathbb{R}^+$ such that $\lim_{n\to\infty} d(\mu_n, \mu_{n+1}) = c^*$. We will prove $c^* = 0$. From (3.2), we have

$$\begin{aligned} d(\mu_n, \mu_{n+1}) &\leq \theta\Big(d(\mu_{n+1}, \mu_n)\Big) \cdot d(\mu_{n-1}, \mu_n) \\ \frac{d(\mu_n, \mu_{n+1})}{d(\mu_{n-1}, \mu_n)} &\leq \theta\Big(d(\mu_{n+1}, \mu_n)\Big) \leq 1 \end{aligned}$$

which implies $\lim_{n\to\infty} \theta\Big(d(\mu_{n-1}, \mu_n)\Big) = 1$. Using the function's attribute θ, we conclude that $\lim_{n\to\infty} d(\mu_n, \mu_{n+1}) = 0$. and we have,

$$\begin{aligned} d(\mu_{n-1}, \mathcal{F}\mu_{n+1}) + \kappa &\leq \Big(d(\mathcal{F}\mu_{n-1}, \mathcal{F}\mu_{n+1}) + \kappa\Big)^{\alpha(\mu_{n-1},\mathcal{F}\mu_{n-1})\alpha(\mu_{n+1},\mathcal{F}\mu_{n+1})} \\ &\leq \theta\Big(d(\mu_{n-1}, \mu_{n+1})\Big) \cdot d(\mu_{n-1}, \mu_{n+1}) + \kappa, \end{aligned}$$

then

$$d(\mu_n, \mu_{n+2}) \leq \theta\Big(d(\mu_{n-1}, \mu_{n+1})\Big) \cdot d(\mu_{n-1}, \mu_{n+1}) \tag{3.3}$$

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which implies $d(\mu_n, \mu_{n+2}) \leq d(\mu_{n-1}, \mu_{n-2})$. It follows that the sequence $\{d(\mu_n, \mu_{n+2})\}$ is decreasing. Thus, there exists $d^* \in \mathbb{R}^+$ such that $\lim_{n\to\infty} d(\mu_n, \mu_{n+1}) = d^*$. We will prove $d^* = 0$. From (3.3), we have

$$\begin{aligned} d(\mu_n, \mu_{n+2}) &\leq \theta\Big(d(\mu_{n-1}, \mu_{n+1})\Big) \cdot d(\mu_{n-1}, \mu_{n+1}) \\ \frac{d(\mu_n, \mu_{n+2})}{d(\mu_{n-1}, \mu_{n+1})} &\leq \theta\Big(d(\mu_{n-1}, \mu_{n+1})\Big) \leq 1 \end{aligned}$$

which implies $\lim_{n\to\infty} \theta\Big(d(\mu_{n-1}, \mu_{n+1})\Big) = 1$. Using the function's attribute θ, As a result, we say $\lim_{n\to\infty} d(\mu_n, \mu_{n+1}) = 0$. Next, we will prove that μ_n is a Cauchy sequence. Contrarily, assume that μ_n is not a Cauchy sequence. Then there is $\varepsilon > 0$ and sequences $m(k)$ and $n(k)$ such that, $\forall k \in \mathbb{I}^+$, we have $n(k) > m(k) > k$, $d(\mu_{n(k)}, \mu_{m(k)}) > \varepsilon$ and $d(\mu_{n(k)}, \mu_{m(k)-1}) < \varepsilon$. By the rectangular inequality, we derive that.

$$\begin{aligned} \varepsilon &\leq d(\mu_{n(k)}, \mu_{m(k)}) \\ &\leq d(\mu_{n(k)}, \mu_{m(k)-1}) + d(\mu_{m(k)-1}, \mu_{m(k)-2}) + d(\mu_{m(k)-2}, \mu_{m(k)}) \\ &\leq \varepsilon. \end{aligned}$$

Taking lim as $k \to +\infty$ in the above inequality, we get

$$\lim_{k\to\infty} d(\mu_{n(k)}, \mu_{m(k)}) = \varepsilon. \tag{3.4}$$

Again, by the rectangular inequality, we find that

$$d(\mu_{n(k)+1}, \mu_{m(k)+1}) \leq d(\mu_{n(k)+1}, \mu_{n(k)}) + d(\mu_{n(k)}, \mu_{m(k)}) + d(\mu_{m(k)}, \mu_{m(k)+1})$$

and

$$d(\mu_{n(k)}, \mu_{m(k)}) \leq d(\mu_{n(k)}, \mu_{n(k)+1}) + d(\mu_{n(k)+1}, \mu_{m(k)+1}) + d(\mu_{m(k)+1}, \mu_{m(k)}).$$

Taking the lim as $k \to +\infty$, together with above inequality, we deduce that

$$\lim_{k\to\infty} d(\mu_{n(k)+1}, \mu_{m(k)+1}) = \varepsilon. \tag{3.5}$$

From equation (3.1), (3.4) and (3.5), we have

$$\begin{aligned} d(\mu_{n(k)+1}, \mu_{m(k)+1}) + \kappa &\leq \Big(d(\mathcal{F}\mu_{n(k)}, \mathcal{F}_{m(k)}) + \kappa\Big)^{\alpha(\mu_{n(k)}, \mathcal{F}\mu_{n(k)}) \cdot \alpha(\mu_{m(k)}, \mathcal{F}\mu_{m(k)})} \\ &\leq \theta\Big(d(\mu_{n(k)}, \mu_{m(k)})\Big) \cdot d(\mu_{n(k)}, \mu_{m(k)}) + \kappa. \end{aligned}$$

Hence,

$$\frac{d(\mu_{n(k)+1}, \mu_{m(k)+1})}{d(\mu_{n(k)}, \mu_{m(k)})} \leq \theta\Big(d(\mu_{n(k)}, \mu_{m(k)})\Big) \leq 1.$$

Letting $t \to +\infty$ in the above inequality, we get

$$\lim_{k\to\infty} \theta\Big(d(\mu_{n(k)}, \mu_{m(k)})\Big) = 1.$$

That is, $\lim_{n\to\infty} d(\mu_{n(k)}, \mu_m(k)) = 0 < \varepsilon$, which is a contradiction. Hence μ_n is a Cauchy sequence. Since $\mathcal{B}$ is complete, then there is $\omega \in \mathcal{B}$ such that $\mu_n \to \omega$.

From (a) if $\mathcal{F}$ is continuous, then $\mathcal{F}\omega = \lim_{n\to\infty} \mathcal{F}\mu_n = \lim_{n\to\infty} \mu_{n+1} = \omega$. So, ω is a fixed point of $\mathcal{F}$.

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Next, suppose that (b) of theorem 3.1, them $\alpha(\omega, \mathcal{F}\omega) \geq 1$. and inequality (3.1), we have

$$\begin{aligned} d(\mathcal{F}\omega, \mu_{n+1}) + \kappa &\leq \Big(d(\mathcal{F}\omega, \mathcal{F}\mu_{n+1}) + \kappa\Big)^{\alpha(\omega,\mathcal{F}\omega)\cdot\alpha(\mu_{n+1},\mathcal{F}\mu_{n+1})} \\ &\leq \theta\Big(d(\omega, \mu_{n+1})\Big) \cdot d(\omega, \mu_{n+1}) + \kappa. \end{aligned}$$

That is, $d(\mathcal{F}\omega, \mu_{n+1}) \leq \theta\Big(d(\omega, \mu_{n+1})\Big) \cdot d(\omega, \mu_{n+1})$. In summary $\lim_{n\to\infty} d(\omega, \mu_{n+1}) = 0$. Similarly, we get

$$\begin{aligned} d(\mathcal{F}^2\omega, \mu_{n+2}) + \kappa &\leq \Big(d(\mathcal{F}^2\omega, \mu_{n+2}) + \kappa\Big)^{\alpha(\mathcal{F}\omega,\mathcal{F}^2\omega)\cdot\alpha(\mu_{n+2},\mathcal{F}\mu_{n+2})} \\ &\leq \theta\Big(d(\mathcal{F}\omega, \mu_{n+2})\Big) \cdot d(\mathcal{F}\omega, \mu_{n+2}) + \kappa. \end{aligned}$$

So, $\lim_{n\to\infty} d(\mathcal{F}\omega, \mu_{n+2}) = 0$, and we have

$$d(\mathcal{F}\omega, \omega) = d(\mathcal{F}\omega, \mu_{n+2}) + d(\mu_{n+2}, \mu_{n+1}) + d(\mu_{n+1}, \omega).$$

Letting $n \to \infty$ in the above inequality, we get $d(\mathcal{F}\omega, \omega) = 0$, that is, $\omega = \mathcal{F}\omega$. ∎

To prove the existence of solution to (1.1), we need the following auxiliary lemma.

Lemma 3.2. *Let $\mu \in C([0, T])$ the linear fractional boundary value problem.*

$$\begin{cases} {}^C\mathcal{D}^q\mu(t) = y(t), \ t \in [0, T] \\ \mu^{(k)}(0) = 0, \ \mu(T) = \tau \int_0^T \mu(s)ds, \end{cases} \tag{3.6}$$

where $q \in (n-1, n)$, $n \in \mathbb{N}$, $k = 0, 1, ..., n-2$
has a unique solution

$$\mu(t) = \int_0^T G(t, s)y(s)ds,$$

where

$$G(t, s) = \begin{cases} \frac{(t-s)^{q-1}(n-T\tau)T^{n+1}\Gamma(q+1)+nt^{n-1}(T\tau-s\tau-q)(T-s)^{q-1}}{(n-T\tau)T^{n-1}\Gamma(q+1)} & \text{if } 0 < s < t < T \\ \frac{nt^{n-1}(T\tau-s\tau-q)(T-s)^{q-1}}{(n-T\tau)T^{n-1}\Gamma(q+1)} & \text{if } 0 < t < s < T. \end{cases}$$

Proof. From Lemma (2.4), we have

$$\mu(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1}y(s)ds + c_0 + c_1t + c_2t^2 + \cdots + c_{n-1}t^{n-1}.$$

By the first condition in (1.1), we have

$$c_0 = c_1 = c_2 = \cdots c_{n-2} = 0.$$

So,

$$\mu(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1}y(s)ds + c_{n-1}t^{n-1},$$

and

$$c_{n-1} = \frac{1}{T^{n-1}}\Big(\tau \int_0^T \mu(s)ds - \frac{1}{\Gamma(q)} \int_0^T (T-s)^{q-1}y(s)ds\Big).$$

Thus,

$$\begin{aligned}\mu(t) &= \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}y(s)ds \\ &+\frac{t^{n-1}}{T^{n-1}}\Big(\tau\int_0^T \mu(s)ds - \frac{1}{\Gamma(q)}\int_0^T (T-s)^{q-1}y(s)ds\Big) \\ \int_0^T \mu(t)dt &= \frac{1}{\Gamma(q)}\int_0^T\int_0^t (t-s)^{q-1}y(s)dsdt \\ &+\Big(\tau\int_0^T \mu(s)ds - \frac{1}{\Gamma(q)}\int_0^T (T-s)^{q-1}y(s)ds\Big)\int_0^T \frac{t^{n-1}}{T^{n-1}}dt\end{aligned}$$

$$\int_0^T \mu(s)ds = \frac{n}{(n-T\tau)}\Big(\frac{1}{\Gamma(q+1)}\int_0^T (T-s)^q y(s)ds - \frac{T}{n\Gamma(q)}\int_0^T (T-s)^{q-1}y(s)ds\Big).$$

So,

$$\begin{aligned}\mu(t) &= \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}y(s)ds \\ &+\frac{t^{n-1}}{T^{n-1}}\Bigg(\frac{n\tau}{(n-T\tau)}\Big(\frac{1}{\Gamma(q+1)}\int_0^T (T-s)^q y(s)ds - \frac{T}{n\Gamma(q)}\int_0^T (T-s)^{q-1}y(s)ds\Big) \\ &-\frac{1}{\Gamma(q)}\int_0^T (T-s)^{q-1}y(s)ds\Bigg).\end{aligned}$$

Hence,

$$\mu(t) = \int_0^T G(t,s)y(s)ds.$$

∎

Let $\mathcal{B} = C(J,\mathbb{R})$, so that $\mathcal{B}$ is a Banach space and define the operator $\mathcal{F}:\mathcal{B}\to\mathcal{B}$ by

$$\mathcal{F}\mu(t) := \int_0^T G(t,s)f(s,\mu(s))ds, \tag{3.7}$$

then the problem (1.1) has solution if and only if the operator $\mathcal{F}$ has solution. In the next step, we show the existence of solutions with Theorem 3.3.

Theorem 3.3. *Suppose that*

(i) There exists $\mu_0 \in C(J,\mathbb{R})$ and function $u : \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ such that $u(\mu_0(t),\mathcal{F}\mu_0)\geq 0$ for all $t\in J=[0,T]$ where $\mathcal{F}: C(J,\mathbb{R})\to C(J,\mathbb{R})$ is defined by the operator (3.7).

(ii) For each $t\in J$ and $\mu,\nu\in C(J,\mathbb{R})$, $u(\mu(t),\nu(t))\geq 0$ implies $u(\mathcal{F}\mu(t),\mathcal{F}\nu(t))\geq 0$.

(iii) For each $t\in J$, if $\{\mu_n\}$ is a sequence in $C(J,\mathbb{R})$ such that $\mu_n\to\mu$ in $C(J,\mathbb{R})$ and $u(\mu_n(t),\mu_{n+1}(t))\geq 0$ for all $n\in\mathbb{N}$, then $u(\mu_n(t),\mu(t))\geq 0$ for all $n\in\mathbb{N}$.

Then, problem (1.1) has at least one solution.

Proof. Let $\mathcal{B} = C(J,\mathbb{R})$ is a rectangular metric space endowed with norm

$$d(\mu,\nu) = \Big|\Big|\mu-\nu\Big|\Big| = \sup_{t\in[0,T]}\Big|\mu(t)-\nu(t)\Big|,$$

for all $\mu, \nu \in \mathcal{B}$, we present the operator $\mathcal{F}$ has a fixed point. We divide the proof into two steps as follows:

step 1. show that $\mathcal{F}$ is $\alpha-$admissible. Also define

$$\alpha(\mu,\nu) \geq 1 \quad \text{if} \quad u(\mu(t),\nu(t)) \geq 0, \ t \in J. \tag{3.8}$$

From Theorem 3.3 (i) and (ii) there exists $\mu_0 \in C(J,\mathbb{R})$ such that $u(\mu_0, \mathcal{F}\mu_0)$, for all $\mu,\nu \in C(J,\mathbb{R})$ get that

$$\begin{aligned} \alpha(\mu,\nu) \geq 1 \quad &\Rightarrow \quad u(\mu(t),\nu(t)) \geq 0 \\ &\Rightarrow \quad u(\mathcal{F}\mu(t),\mathcal{F}\nu(t)) \geq 0 \\ &\Rightarrow \quad \alpha(\mathcal{F}\mu,\mathcal{F}\nu) \geq 1. \end{aligned}$$

Hence $\mathcal{F}$ is α-admissible.

step 2. $\mathcal{F}$ continuous. Let $\{\mu_n\}$ be a sequence in $\mathcal{B}$ and $t \in J$. From (3.1),we have

$$\begin{aligned} \Big|\mathcal{F}\mu_n(t) - \mathcal{F}\mu(t)\Big| + \kappa \quad &\leq \quad \Big(\Big|\mathcal{F}\mu_n(t) - \mathcal{F}\mu(t)\Big| + \kappa\Big)^{\alpha(\mu_n,\mathcal{F}\mu_n)\cdot\alpha(\mu,\mathcal{F}\mu)} \\ &\leq \quad \theta(\Big|\mu_n(t) - \mu(t)\Big|)\cdot\Big|\mu_n(t) - \mu(t)\Big| + \kappa \\ \Big|\mathcal{F}\mu_n(t) - \mathcal{F}\mu(t)\Big| \quad &\leq \quad \theta(\Big|\mu_n(t) - \mu(t)\Big|)\cdot\Big|u_n(t) - u(t)\Big|. \end{aligned}$$

So,

$$\Big|\Big|\mathcal{F}\mu_n(t) - \mathcal{F}\mu(t)\Big|\Big| \leq \theta(\Big|\Big|\mu_n - \mu\Big|\Big|)\cdot\Big|\Big|\mu_n - \mu\Big|\Big|. \tag{3.9}$$

Since θ is increasing function and $\theta : [0,\infty) \to (0,1]$, hence $0 < \theta(||\mu_n - \mu||) \leq 1$. From (3.9), $\|\mathcal{F}\mu_n - \mathcal{F}\mu\| \leq \|\mu_n - \mu\|$ for $\{\mu_n\} \in \mathcal{B}$ and that is $|\mu_n(t) - \mu(t)| \to 0$.
Hence, $\Big|\mathcal{F}\mu_n(t) - \mathcal{F}\mu(t)\Big| \to 0$, as $n \to \infty$. Consequently $\mathcal{F}$ is continuous.

Moreover, from all conditions of Theorem 3.3 satisfied. We have the there exists $\mu_0 \in C(J,\mathbb{R})$ and equation (3.9), we get that

$$\begin{aligned} \alpha(\mu_n,\mu_{n+1}) \geq 1 \quad &\Rightarrow \quad u(\mu_n(t),\mu_{n+1}(t)) \geq 0 \\ &\Rightarrow \quad u(\mu_{n-1}(t),\mu_n(t)) \geq 0 \\ &\ \ \vdots \\ &\Rightarrow \quad u(\mu_0(t),\mu_1(t)) \geq 0 \\ &\Rightarrow \quad u(\mu_0(t),\mathcal{F}\mu_0(t)) \geq 0 \\ &\Rightarrow \quad \alpha(\mu_0,\mathcal{F}\mu_0) \geq 1. \end{aligned}$$

As a result, the requirements of Theorem 3.1 are met. Thus, we draw the conclusion that $\mu^* \in C(J,\mathbb{R})$ exists and $\mathcal{F}\mu^* = \mu^*$. Hence μ^* is a solution of the problem (1.1). ∎

4. Conclusion

We prove and consider the existence of a solution to a non-linear fractional differential equation with a non-local integral condition by finding new conditions of a fixed point in rectangular metric space for certain α-admissible condition mapping.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly to the writing of this article. All authors read and approved the final manuscript.

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