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Best Proximity Points for a Generalized C−Proximal Almost Weakly Contractive Maps in Partially Ordered Metric Spaces

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Abstract In this paper, we obtain some best proximity point theorems for a generalized C−proximal almost weakly contractive maps in partially ordered metric spaces. Our results generalize the results of Azizi, Moosaei and Zareir [3] by choosing $A = B = X$, where A and B are nonempty subsets of a partially ordered metric space (X, d). We draw some corollaries and give illustrative examples in support of our results.

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1. Introduction and Preliminaries

The famous Banach's contraction principle states that every contraction selfmapping on a complete metric space has a unique fixed point. This principle has been generalized and extended in several ways. Let A and B be nonempty subsets of a metric space (X, d) and let $T : A \to B$ be a non-selfmapping. The equation $Tx = x$ may not have a solution, because of the fact that a solution of the preceding equation demands the non-emptiness of $A \cap B$. Therefore, it is an interesting aspect to seek an approximate solution x that is optimal in the sense that the distance $d(x, Tx)$ is minimum, where $d(A, B) := \inf\{d(x, y) : (x, y) \in A \times B\}$.

A point $x \in A$ is called best proximity point of $T : A \to B$ if $d(x, Tx) = d(A, B)$. A best proximity point becomes a fixed point if the underlying mapping is a selfmapping. Therefore, it can be concluded that best proximity point theorems generalize fixed point theorems in a natural way. The authors [6, 8, 9, 12] and reference therein obtained best proximity point theorems under certain contraction conditions for non-selfmaps. For more works on best proximity point we refer [1, 2, 5, 13] and references therein.

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Our purpose here is to establish best proximity point theorems in the partially ordered metric spaces.

We recall the following notations and definitions. Let $(X, d, \preceq)$ be a partially ordered metric space and let A and B be nonempty subsets of X.

$$A_0 := \{x \in A : d(x, y) = d(A, B) \text{ for some } \ y \in B\},$$
$$B_0 := \{y \in B : d(x, y) = d(A, B) \text{ for some } x \in A\}.$$

Definition 1.1. [7] A mapping $T : A \to B$ is said to be proximally increasing on A_0 if for all $u_1, u_2, x_1, x_2 \in A_0$,

$$\left.\begin{array}{l} x_1 \preceq x_2 \\ d(u_1, Tx_1) = d(A, B) \\ d(u_2, Tx_2) = d(A, B) \end{array}\right\} \Rightarrow u_1 \preceq u_2.$$

Definition 1.2. [11] An altering distance function is a function $\psi : [0, \infty) \to [0, \infty)$ which satisfies:

(i) ψ is continuous and non-decreasing and
(ii) $\psi(t) = 0$ if and only if $t = 0$.

We denote by Ψ the class of altering distance functions.

Definition 1.3. Let (X, d) be a metric space. A function $\phi : X \to \mathbb{R}$ is lower semi-continuous if for any sequence $t_n \subseteq X$ with $t_n \to t$ as $n \to \infty$, then $\phi(t) \leq \liminf_{n\to\infty} \phi(t_n)$.

Definition 1.4. [3] Let $\phi : [0, \infty) \times [0, \infty) \to [0, \infty)$ be a function. We say that the function ϕ has property (P) if the following are satisfied:

(i) ϕ is lower semi-continuous and non-decreasing with respect to both of its components, and
(ii) $\phi(s, t) = 0$ if and only if $s = t = 0$.

We denote by Φ the class of all functions satisfying property (P).

In 2016, Azizi, Moosaei and Zarei [3] proved the existence and uniqueness of fixed points for almost generalized $\mathsf{C}-$ contractive mappings in partially ordered metric spaces.

Definition 1.5. [3] Let $(X, \preceq, d)$ be an ordered metric space. We say that a mapping $f : X \to X$ is an almost generalized $\mathsf{C}-$ contractive if there exist $\xi \geq 0$ and $(\psi, \phi) \in \Psi \times \Phi$ such that

$$\psi(d(fx, fy)) \leq \psi(M(x, y)) - \phi\big(M'(x, y), M''(x, y)\big) + \xi\psi(N(x, y)) \tag{1.1}$$

for all $x, y \in X$ with $x \preceq y$, where

$$M(x, y) = \max\,\{d(x, y), d(x, fx,), d(y, fy), \frac{d(x, fy) + d(y, fx)}{2}\},$$
$$M'(x, y) = \max\,\{d(x, y), d(x, fx), d(x, fy)\},$$
$$M''(x, y) = \max\,\{d(x, y), d(y, fy), d(y, fx)\} \text{ and}$$
$$N(x, y) = \min\,\{d(x, fx), d(y, fx)\}.$$

Theorem 1.6. [3] *Let $(X, \preceq, d)$ be an ordered metric space. Assume that $f : X \to X$ is a non-decreasing (with respect to $\preceq$), continuous and almost generalized $\mathsf{C}-$contractive map. If there exists $x_1 \in X$ such that $x_1 \preceq fx_1$, then f has a fixed point. In particular, if $F(f)$ is totally ordered subset of X, where $F(f)$ denotes the set of all fixed points of f, then f has a unique fixed point.*

Definition 1.7. [10] Let A and B be two nonempty subsets of a metric space (X,d) and $T : A \to B$ be a mapping. We say that T has the RJ property if for any sequence $\{x_n\} \subseteq A$,

$$\left.\begin{array}{l}\lim_{n\to\infty} d(x_{n+1}, Tx_n) = d(A,B)\\ \lim_{n\to\infty} x_n = x\end{array}\right\} \implies x \in A_0.$$

Here we observe that any continuous mapping $T : A \to B$ has the RJ property provided that A and B are nonempty closed subsets of a metric space (X,d).

Lemma 1.8. [4] *Suppose that (X,d) is a metric space. Let $\{x_n\}$ be a sequence in X such that $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$. If $\{x_n\}$ is not a Cauchy sequence, then there exists an $\epsilon > 0$ and sequences of positive integers $\{m_k\}$ and $\{n_k\}$ with $m_k > n_k > k$ such that $d(x_{m_k}, x_{n_k}) \geq \epsilon$, $d(x_{m_k-1}, x_{n_k}) < \epsilon$ and*

(i) $\lim_{k\to\infty} d(x_{m_k-1}, x_{n_k+1}) = \epsilon$,

(ii) $\lim_{k\to\infty} d(x_{m_k}, x_{n_k}) = \epsilon$,

(iii) $\lim_{k\to\infty} d(x_{m_k-1}, x_{n_k}) = \epsilon$.

Remark 1.9. By using the hypotheses of Lemma 1.8 and triangular inequality we can show that $\lim_{k\to\infty} d(x_{m_k-1}, x_{n_k-1}) = \epsilon$ and $\lim_{k\to\infty} d(x_{n_k-1}, x_{m_k}) = \epsilon$.

In the following we define the notion of an almost generalized $\mathsf{C}-$proximal weakly contractive map.

Definition 1.10. Let $(X,d,\preceq)$ be a partially ordered metric space and A, B be nonempty subsets of X. We say that $f : A \to B$ is an almost generalized $\mathsf{C}-$proximal weakly contractive map if there exist $\xi \geq 0$, $\psi \in \Psi$, $\phi \in \Phi$ such that for all $x, y, u, v \in A$ with $x \preceq y$

$$\left.\begin{array}{l}d(u, fx) = d(A,B)\\ d(v, fy) = d(A,B)\end{array}\right\} \implies \psi(d(u,v)) \leq \psi(M(x,y,u,v) \tag{1.2}$$
$$-\phi(M_1(x,y,u,v), M_2(x,y,u,v)) + \xi\psi(N(x,y,u,v)),$$

where

$$M(x,y,u,v) = \max\,\{d(x,y), d(x,u), d(y,v), \frac{d(x,v)+d(y,u)}{2}\},$$
$$M_1(x,y,u,v) = \max\,\{d(x,y), d(x,u), d(x,v)\},$$
$$M_2(x,y,u,v) = \max\,\{d(x,y), d(y,v), d(y,u)\} \text{ and}$$
$$N(x,y,u,v) = \min\,\{d(x,u), d(y,u)\}.$$

Here we observe that if $A = B = X$ in Definition 1.10, then f is an almost generalized $\mathsf{C}-$contractive map.

Example 1.11. Let $X = [0,\infty) \times [0,\infty)$, with the Euclidean metric d. We define a partial order $\preceq$ on X by

$$\preceq := \{((x_1,x_2),(y_1,y_2)) \in X \times X | x_1 = y_1, x_2 = y_2\} \cup \{((0,\tfrac{15}{16}),(0,\tfrac{1}{2^n})), ((0,\tfrac{19}{24}),(0,\tfrac{1}{2^{n+1}})),$$
$$((0,\frac{1}{2^n}),(0,\frac{1}{2^m})), ((0,\frac{1}{2^n}),(0,0)) \mid n,m = 1,2,...,\ m > n\} \cup \{((0,\frac{19}{24}),(0,0)), ((0,\frac{15}{16}),(0,0))\},$$

where $(x_1,x_2) \preceq (y_1,y_2) \iff x_1 \geq y_1$ and $x_2 \geq y_2, \geq$ is the usual order in $\mathbb{R}$.

Let $A = \{0\} \times [0,1] = A_0,\ B = \{\pi\} \times [0,1] = B_0$. We define $f : A \to B$ by

$$f(0,x) = \begin{cases} (\pi, \frac{x}{2}) & \text{if } x \in [0, \frac{3}{4}) \\ (\pi, 2x - \frac{13}{12}) & \text{if } x \in [\frac{3}{4}, 1]. \end{cases}$$

Clearly $d(A,B) = \pi$. To show that f is an almost generalized $\mathsf{C}-$proximal weakly contractive map, we define functions $\psi : [0,\infty) \to [0,\infty)$ and $\phi : [0,\infty) \times [0,\infty) \to [0,\infty)$ by

$$\psi(t) = \begin{cases} \frac{t}{2} & \text{if } t \in [0,1] \\ \frac{t}{1+t} & \text{if } t \geq 1 \end{cases} \quad \text{and } \phi(s,t) = \begin{cases} \frac{s+t}{16} & \text{for all } \ s,t \in [0,1] \\ \frac{1}{4} & \text{otherwise .} \end{cases}$$

Now, let $(0,x), (0,y), (0,u)$ and $(0,v) \in A$ such that

$$\left.\begin{array}{l} (0,x) \preceq (0,y) \\ d((0,u), f(0,x)) = \pi \\ d((0,v), f(0,y)) = \pi. \end{array}\right\} \tag{1.3}$$

Case (i): $(0,x) = (0, \frac{15}{16}),\ (0,y) = (0, \frac{1}{2^n}) :\ n = 1,2,3,\dots, (0,u) = (0, \frac{19}{24}), (0,v) = (0, \frac{1}{2^{n+1}})$.

In this case, we have

$$\begin{aligned} \psi(d((0,u),(0,v))) &= \psi(d((0, \frac{19}{24}), (0, \frac{1}{2^{n+1}}))) = \frac{19}{48} - \frac{1}{2^{n+2}} \\ &\leq \frac{15}{32} - \frac{1}{2^{n+1}} - \left(\frac{15}{128} - \frac{3}{2^{n+5}}\right) + \frac{7}{96} \\ &= \psi(M((0,x),(0,y),(0,u),(0,v))) \\ &\quad - \phi\big(M_1((0,x),(0,y),(0,u),(0,v)), M_2((0,x),(0,y),(0,u),(0,v))\big) \\ &\quad + \xi\psi(N((0,x),(0,y),(0,u),(0,v))), \text{ where } \xi = 1. \end{aligned}$$

Case (ii): $(0,x) = (0, \frac{19}{24}),\ (0,y) = (0, \frac{1}{2^{n+1}}) :\ n = 1,2,3,\dots, (0,u) = (0, \frac{19}{48})$, $(0,v) = (0, \frac{1}{2^{n+2}})$. Now,

$$\begin{aligned} \psi(d((0,u),(0,v))) &= \psi(d((0, \frac{19}{48}), (0, \frac{1}{2^{n+2}}))) = \frac{19}{96} - \frac{1}{2^{n+3}} \\ &\leq \frac{19}{48} - \frac{1}{2^{n+2}} - \left(\frac{83}{768} - \frac{1}{2^{n+6}}\right) + \frac{19}{96} - \frac{1}{2^{n+2}} \\ &= \psi(M((0,x),(0,y),(0,u),(0,v))) \\ &\quad - \phi\big(M_1((0,x),(0,y),(0,u),(0,v)), M_2((0,x),(0,y),(0,u),(0,v))\big) \\ &\quad + \xi\psi(N((0,x),(0,y),(0,u),(0,v))), \text{ where } \xi = 1. \end{aligned}$$

For the other possible cases, the inequality (1.2) holds trivially with $\xi = 1$.

Hence f is an almost generalized $\mathsf{C}-$proximal weakly contractive map.

Remark 1.12. In fact the inequality (1.2) fails to hold when $\xi = 0$ in Example 1.11. For, by choosing $(0,x) = (0, \frac{15}{16}), (0,y) = (0, \frac{1}{2}), (0,u) = (0, \frac{19}{24}),\ (0,v) = (0, \frac{1}{4})$, we have

$$\begin{aligned} \psi(d((0,u),(0,v))) &= \psi(d((0, \frac{19}{24}), (0, \frac{1}{4}))) = \psi(\frac{11}{24}) \not\leq \psi(\frac{7}{16}) - \phi(\frac{11}{16}, \frac{11}{24}) \\ &= \psi(M((0,x),(0,y),(0,u),(0,v))) \\ &\quad - \phi\big(M_1((0,x),(0,y),(0,u),(0,v)), M_2((0,x),(0,y),(0,u),(0,v))\big), \end{aligned}$$

for any $\psi \in \Psi$ and $\phi \in \Phi$.

In Section 2 of this paper, we prove our main results. In Section 3, we draw some corollaries from our results and give examples in support of our results.

2. Main Results

Theorem 2.1. *Let $(X, d, \preceq)$ be a partially ordered complete metric space. Let A, B be non-empty subsets of X. Let $f : A \to B$ be a non-selfmapping such that the following conditions hold:*

(i) f is an almost generalized $\mathsf{C}-$proximal weakly contractive map,
(ii) f is proximally increasing on A_0 and f has the RJ property,
(iii) $f(A_0) \subseteq B_0$,
(iv) there exist elements $x_0, x_1 \in A_0$ such that $d(x_1, fx_0) = d(A, B)$ and $x_0 \preceq x_1$,
(v) if $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to x$ as $n \to \infty$, then $x_n \preceq x$ for all $n \in \mathbb{N}$.

Then there exists $x' \in A_0$ such that $d(x', fx') = d(A, B)$.

Proof. By condition (iv), there exist $x_0,\ x_1 \in A_0$ such that

$$d(x_1, fx_0) = d(A, B) \text{ and } x_0 \preceq x_1. \tag{2.1}$$

Since $f(A_0) \subseteq B_0$, we have $fx_1 \in B_0$ and hence there exists an element $x_2 \in A$ such that

$$d(x_2, fx_1) = d(A, B). \tag{2.2}$$

By definition of A_0 and B_0, it follows that $x_2 \in A_0$. Since f is proximally increasing on A_0, from (2.1) and (2.2), we have $x_1 \preceq x_2$. On continuing this process, we get a sequence $\{x_n\}$ in A_0 such that

$$\left.\begin{array}{l} d(x_n, fx_{n-1}) = d(A, B) \\ d(x_{n+1}, fx_n) = d(A, B) \end{array}\right\},\ n = 1, 2, 3, \dots , \tag{2.3}$$

satisfying

$$x_1 \preceq x_2 \preceq x_3 \preceq \dots \preceq x_n \preceq x_{n+1} \preceq \dots\ n = 1, 2, 3, \dots . \tag{2.4}$$

If $x_{n_0} = x_{n_0+1}$ for some $n_0 \in \mathbb{N}$, then x_{n_0} is the best proximity point of f and hence the conclusion of the theorem follows.

Now, we assume that any consecutive elements of $\{x_n\}$ are distinct. Since f is an almost generalized $\mathsf{C}-$ proximal weakly contractive map, from (2.3) and (2.4), we have

$$\begin{aligned} \psi(d(x_n, x_{n+1})) &\leq \psi(M(x_{n-1}, x_n, x_n, x_{n+1})) \\ &\quad - \phi(M_1(x_{n-1}, x_n, x_n, x_{n+1}), M_2(x_{n-1}, x_n, x_n, x_{n+1})) \\ &\quad + \xi\psi(N(x_{n-1}, x_n, x_n, x_{n+1})), \end{aligned} \tag{2.5}$$

where

$$M(x_{n-1},x_n,x_n,x_{n+1}) = \max\{d(x_{n-1},x_n), d(x_{n-1},x_n), d(x_n,x_{n+1}), \frac{d(x_{n-1},x_{n+1})+d(x_n,x_n)}{2}\},$$
$$M_1(x_{n-1},x_n,x_n,x_{n+1}) = \max\{d(x_{n-1},x_n), d(x_{n-1},x_n), d(x_{n-1},x_{n+1})\},$$
$$M_2(x_{n-1},x_n,x_n,x_{n+1}) = \max\{d(x_{n-1},x_n), d(x_n,x_{n+1}), d(x_n,x_n)\} \text{ and}$$
$$N(x_{n-1},x_n,x_n,x_{n+1}) = \min\{d(x_{n-1},x_n), d(x_n,x_n))\}.$$

Now, we have

$$\begin{aligned} M(x_{n-1},x_n,x_n,x_{n+1}) &= \max\{d(x_{n-1},x_n), d(x_{n-1},x_n), d(x_n,x_{n+1}), \\ &\frac{d(x_{n-1},x_{n+1})+d(x_n,x_n)}{2}\} = \max\{d(x_{n-1},x_n), d(x_n,x_{n+1}), \frac{d(x_{n-1},x_{n+1})}{2}\} \\ &\le \max\{d(x_{n-1},x_n), d(x_n,x_{n+1}), \frac{d(x_{n-1},x_n)+d(x_n,x_{n+1})}{2}\} \\ &= \max\{d(x_{n-1},x_n), d(x_n,x_{n+1})\}, \end{aligned} \tag{2.6}$$

$$M_2(x_{n-1},x_n,x_n,x_{n+1}) = \max\{d(x_{n-1},x_n), d(x_n,x_{n+1})\}, \tag{2.7}$$

$$N(x_{n-1},x_n,x_n,x_{n+1}) = \min\{d(x_{n-1},x_n), d(x_n,x_n)\} = 0. \tag{2.8}$$

From (2.7) and by the non-decreasing property of ϕ, we obtain

$$\begin{aligned} &\phi\big(d(x_{n-1},x_n), \max\{d(x_{n-1},x_n), d(x_n,x_{n+1})\}\big) \\ &\le \phi\big(\max\{d(x_{n-1},x_{n+1}), d(x_{n-1},x_n)\}, \max\{d(x_{n-1},x_n), d(x_n,x_{n+1})\}\big). \end{aligned} \tag{2.9}$$

On combining (2.5), (2.6), (2.8) and (2.9), it follows that

$$\begin{aligned} \psi(d(x_n,x_{n+1})) &\le \psi\big(\max\{d(x_{n-1},x_n), d(x_n,x_{n+1})\}\big) \\ &-\phi\big(d(x_{n-1},x_n), \max\{d(x_{n-1},x_n), d(x_n,x_{n+1})\}\big). \end{aligned} \tag{2.10}$$

If $d(x_n,x_{n+1}) > d(x_{n-1},x_n)$ in (2.10), we get $\phi(d(x_{n-1},x_n), d(x_n,x_{n+1})) = 0$, which yields that $d(x_{n-1},x_n) = d(x_n,x_{n+1}) = 0$,
a contradiction. Therefore $d(x_n,x_{n+1}) \le d(x_{n-1},x_n)$ for all $n \in \mathbb{N}$.
Hence $\{d(x_n,x_{n+1})\}$ is a decreasing sequence of nonnegative real numbers. Thus there exists a real number $r \ge 0$ such that

$$\lim_{n\to\infty} d(x_n,x_{n+1}) = r. \tag{2.11}$$

Suppose $r > 0$. On taking the limit superior as $n \to \infty$ on both sides of (2.10) and by using the properties of ψ and ϕ, we have

$$\begin{aligned} \limsup_{n\to\infty} \psi(d(x_n,x_{n+1})) &\le \limsup_{n\to\infty} \psi\big(\{d(x_{n-1},x_n)\}\big) \\ &- \liminf_{n\to\infty} \phi\big(d(x_{n-1},x_n), \{d(x_{n-1},x_n)\}\big) \end{aligned}$$

and hence $\psi(r) \le \psi(r) - \phi(r,r)$. This implies that $\phi(r,r) = 0$. i.e., $r = 0$.

We now show that the sequence $\{x_n\}$ is Cauchy. Suppose that the sequence $\{x_n\}$ is not Cauchy. Then by Lemma 1.8, there exists an $\epsilon > 0$ for which we can find sequences of

positive integers $\{m_k\}$ and $\{n_k\}$ with $m_k > n_k > k$ such that $d(x_{m_k}, x_{n_k}) \geq \epsilon$, $d(x_{m_k-1}, x_{n_k}) < \epsilon$ and the identities (i)-(iii) of Lemma 1.8 and Remark 1.9 are satisfied.
Now, from (2.3), we have

$$\left.\begin{array}{l} d(x_{n_k}, fx_{n_k-1}) = d(A,B) \\ d(x_{m_k}, fx_{m_k-1}) = d(A,B). \end{array}\right\}$$

Since f is an almost generalized $\mathsf{C}-$proximal weakly contractive map and $x_{n_k} \preceq x_{m_k}$, it follows that

$$\begin{aligned} \psi(d(x_{m_k}, x_{n_k})) &\leq \psi(M(x_{n_k-1}, x_{m_k-1}, x_{n_k}, x_{m_k})) \\ &\quad - \phi(M_1(x_{n_k-1}, x_{m_k-1}, x_{n_k}, x_{m_k}), M_2(x_{n_k-1}, x_{m_k-1}, x_{n_k}, x_{m_k})) \\ &\quad + \xi\psi(N(x_{n_k-1}, x_{m_k-1}, x_{n_k}, x_{m_k})) \\ &= \psi\big(\max\{d(x_{n_k-1}, x_{m_k-1}), d(x_{n_k-1}, x_{n_k}), d(x_{m_k-1}, x_{m_k}), \\ &\quad \frac{d(x_{n_k-1}, x_{m_k}) + d(x_{m_k-1}, x_{n_k})}{2}\} \\ &\quad - \phi\big(\max\{d(x_{n_k-1}, x_{m_k-1}), d(x_{n_k-1}, x_{n_k}), d(x_{n_k-1}, x_{m_k})\}, \\ &\quad \max\{d(x_{n_k-1}, x_{m_k-1}), d(x_{m_k-1}, x_{m_k}), d(x_{m_k-1}, x_{n_k})\}\big) \\ &\quad + \xi\psi\big(\min\{d(x_{n_k-1}, x_{n_k}), d(x_{m_k-1}, x_{n_k})\}\big). \end{aligned} \tag{2.12}$$

On taking limit superior as $k \to \infty$ on both sides of (2.12), by using Lemma 1.8 and Remark 1.9, we get

$$\begin{aligned} \psi(\epsilon) &\leq \psi(\max\{\epsilon, 0, 0, \frac{\epsilon+\epsilon}{2}\}) - \liminf_{k\to\infty} \phi\big(\max\{d(x_{n_k-1}, x_{m_k-1}), d(x_{n_k-1}, x_{n_k}), \\ &\quad d(x_{n_k-1}, x_{m_k})\}, \max\{d(x_{n_k-1}, x_{m_k-1}), d(x_{m_k-1}, x_{m_k}), d(x_{m_k-1}, x_{n_k})\}\big) \\ &\leq \psi(\epsilon) - \phi(\epsilon, \epsilon). \end{aligned}$$

This implies that $\phi(\epsilon, \epsilon) = 0$. i.e., $\epsilon = 0$, a contradiction. Hence $\{x_n\}$ is Cauchy. Since $\{x_n\}$ is a subset of a complete metric space (X, d), then there exists $x' \in X$ such that $\lim_{n\to\infty} x_n = x'$. From RJ property of f, it follows that $x' \in A_0$. Since $f(A_0) \subseteq B_0$, there exists $z \in A_0$ such that $d(z, fx') = d(A, B)$.

Now we prove that $z = x'$. If possible suppose $z \neq x'$. Since $\{x_n\}$ is a decreasing sequence and $x_n \to x'$ as $n \to \infty$, by condition (v), we have $x_n \preceq x'$ for all $n \in \mathbb{N}$. For any $n \in \mathbb{N}$, we have $d(x_{n+1}, fx_n) = d(A, B)$ and $d(z, fx') = d(A, B)$. By using the fact that f is an almost generalized $\mathsf{C}-$proximal weakly contractive map, for any $n \in \mathbb{N}$, it follows that

$$\begin{aligned} \psi(d(x_{n+1}, z)) &\leq \psi(M(x_n, x', x_{n+1}, z)) - \phi(M_1(x_n, x', x_{n+1}, z), \\ &\quad M_2(x_n, x', x_{n+1}, z)) + \xi\psi(N(x_n, x', x_{n+1}, z)) \\ &= \psi\big(\max\{d(x_n, x'), d(x_n, x_{n+1}), d(x', z), \frac{d(x_n, z) + d(x', x_{n+1})}{2}\} \\ &\quad - \phi\big(\max\{d(x_n, x'), d(x_n, x_{n+1}), d(x_n, z)\}, \max\{d(x_n, x'), d(x', z)\}\big) \\ &\quad + \xi\psi\big(\min\{d(x_n, x_{n+1}), d(x', x_{n+1}), d(x', z)\}\big). \end{aligned} \tag{2.13}$$

On taking the limit superior as $n \to \infty$ on both sides of (2.13), we obtain

$$\psi(d(x', z)) \leq \psi(d(x', z)) - \phi(d(x', z), d(x', z)),$$

which implies that
$\phi(d(x', z), d(x', z)) = 0$ and hence $d(x', z) = 0$. i.e., $x' = z$.
Hence x' is the best proximity point of f. ∎

Theorem 2.2. *Let $(X, d, \preceq)$ be a partially ordered complete metric space. Let A, B be a non-empty subsets of X. Let $f : A \to B$ be a non-selfmapping such that the following conditions hold:*

(i) *there exist $\xi \geq 0$, $\psi \in \Psi$, $\phi \in \Phi$ such that for all $x, y, u, v \in A$ with $x \preceq y$*

$$\left.\begin{array}{l} d(u, fx) = d(A, B) \\ d(v, fy) = d(A, B) \end{array}\right\} \implies \psi(d(u, v)) \leq \psi(M'(x, y, u, v))$$

$$-\phi(M_1(x, y, u, v), M_2(x, y, u, v)) + \xi\psi(N'(x, y, u, v)), \tag{2.14}$$

where

$$M'(x, y, u, v) = \max\,\{d(x, y), \frac{d(x, u) + d(y, v)}{2}, \frac{d(x, v) + d(y, u)}{2}\},$$
$$M_1(x, y, u, v) = \max\,\{d(x, y), d(x, u), d(x, v)\},$$
$$M_2(x, y, u, v) = \max\,\{d(x, y), d(y, v), d(y, u)\} \text{ and}$$
$$N'(x, y, u, v) = \min\,\{d(x, u), d(y, u), d(y, v)\},$$

(ii) *f is proximally increasing on A_0 and f has the RJ property,*

(iii) *$f(A_0) \subseteq B_0$,*

(iv) *there exist elements $x_0, x_1 \in A$ such that $d(x_1, fx_0) = d(A, B)$ and $x_0 \preceq x_1$,*

(v) *if $\{x_n\}$ is a nondecreasing sequence in X such that $x_n \to x$ as $n \to \infty$, then $x_n \preceq x$ for all $n \in \mathbb{N}$.*

Then there exists $x' \in A_0$ such that $d(x', fx') = d(A, B)$.

Proof. Since the inequality (2.14) implies the inequality (1.2) the conclusion of this theorem follows from Theorem 2.1. ∎

Lemma 2.3. *In addition to the hypotheses of Theorem 2.2, if x is a best proximity point of f, and x is comparable to some $u \in A_0$, then there exists a sequence $\{u_n\} \subseteq A_0$ such that $d(u_n, fu_{n-1}) = d(A, B)$, u_n is comparable to x for $n = 1,\ 2,\ 3, \ldots$, and $u_n \to x$ as $n \to \infty$.*

Proof. Let x be the best proximity point of f. i.e.,

$$d(x, fx) = d(A, B). \tag{2.15}$$

Let $u \in A_0$ such that x is comparable to u. Now, we set $u_0 = u$. Suppose that either

$$u_0 \preceq x \text{ or } x \preceq u_0.$$

We assume, without loss of generality, that

$$u_0 \preceq x \text{ with } u_0 \neq x. \tag{2.16}$$

Since $f(A_0) \subseteq B_0$ and $u = u_0 \in A_0$, we have $fu_0 \in B_0$. Hence there exists $u_1 \in A_0$ such that

$$d(u_1, fu_0) = d(A, B). \tag{2.17}$$

Since f is proximally increasing on A_0, from (2.15), (2.16) and (2.17), we have $u_1 \preceq x$. On continuing this process we can construct a sequence $\{u_n\}$ in A_0 such that

$$d(u_n, fu_{n-1}) = d(A, B), \tag{2.18}$$

satisfying

$$u_n \preceq x,\ n = 1, 2, 3, \ldots\ . \tag{2.19}$$

Since $u_n \preceq x$, by combining (2.15), (2.18) and by the inequality (2.14), we have

$$\begin{aligned}\psi(d(u_n,x)) &\leq \psi(M'(u_{n-1},x,u_n,x)) - \phi(M_1((u_{n-1},x,u_n,x),\ M_2((u_{n-1},x,u_n,x)) \\ &+ \xi\psi(N'((u_{n-1},x,u_n,x)) = \psi(\max\{d(u_{n-1},x), \\ &\frac{d(u_{n-1},u_n)+d(x,x)}{2}, \frac{d(u_{n-1},x)+d(x,u_n)}{2}\}) \\ &- \phi(\max\{d(u_{n-1},x), d(u_{n-1},u_n), d(u_{n-1},x)\}, \max\{d(u_{n-1},x), d(x,x), \\ &d(x,u_n)\}) \ + \xi\phi(\min\{d(u_{n-1},u_n), d(x,u_n), d(x,x)\}) \\ &\leq \psi(\max\{d(u_{n-1},x), \frac{d(u_{n-1},x)+d(x,u_n)}{2}, \frac{d(u_{n-1},x)+d(x,u_n)}{2}\}) \\ &-\phi(\max\{d(u_{n-1},x), d(u_{n-1},u_n)\}, \max\{d(u_{n-1},x), d(x,u_n)\}). \end{aligned} \tag{2.20}$$

If $d(u_n,x) > d(u_{n-1},x)$, from (2.20), we get

$$\psi(d(u_n,x)) \leq \psi(d(u_n,x)) - \phi(d(u_{n-1},x), d(u_n,x)),$$

this implies that $\phi(d(u_{n-1},x), d(u_n,x)) = 0$. i.e., $d(u_{n-1},x) = d(u_n,x) = 0$,
a contradiction and hence $d(u_{n-1},x)$ is the maximum. Therefore, from (2.20), we obtain

$$\psi(d(u_n,x)) \leq \psi(d(u_{n-1},x)) - \phi(d(u_{n-1},x), d(u_{n-1},x)) < \psi(d(u_{n-1},x)). \tag{2.21}$$

By nondecreasing property of ψ, from (2.21), it follows that
$d(u_n,x) \leq d(u_{n-1},x)$ and hence $\{d(u_n,x)\}$ is a decreasing sequence of nonnegative real numbers. Then there exists $s \geq 0$ such that

$$\lim_{n\to\infty} d(u_n,x) = s. \tag{2.22}$$

If possible suppose $s > 0$. On letting $n \to \infty$ in (2.21), we get $\psi(s) \leq \psi(s) - \phi(s,s)$ this implies that $\phi(s,s) = 0$. i.e., $s = 0$,
a contradiction. Hence $u_n \to x$ as $n \to \infty$. ∎

Theorem 2.4. *In addition to the hypotheses of Theorem 2.2, assume the following. Condition (H): for every $x, y \in A_0$, there exists $u \in A_0$ such that u is comparable to x and y. Then f has a unique best proximity point in A_0.*

Proof. In view of the proof of Theorem 2.2, the set of best proximity points of f is non-empty. Suppose that $x, y \in A_0$ are two distinct best proximity points of f. That is,

$$d(x,fx) = d(A,B) \text{ and } d(y,fy) = d(A,B). \tag{2.23}$$

Case (i): x is comparable to y. i.e., either $x \preceq y$ or $y \preceq x$.

We assume, without loss of generality, that $x \preceq y$. By using the inequality (2.14), we have

$$\begin{aligned}\psi(d(x,y)) &\leq \psi(M(x,y,x,y)) - \phi(M_1(x,y,x,y), M_2(x,y,x,y)) + \xi\psi(N(x,y,x,y)) \\ &= \psi(\max\{d(x,y), \frac{d(x,x), d(y,y)}{2}, \frac{d(x,y)+d(y,x)}{2}\}) \\ &- \phi(\max\{d(x,y), d(x,x), d(x,y)\}, \max\{d(x,y), d(y,y), d(y,x)\}) \\ &+ \xi\phi(\min\{d(x,x), d(y,x), d(y,y)\}) = \psi(d(x,y)) - \phi(d(x,y), d(x,y)).\end{aligned}$$

The above inequality implies that $\phi(d(x,y), d(x,y)) = 0$. i.e., $d(x,y) = 0$ and hence $x = y$.
Case (ii): x is not comparable to y.

By condition (H), there exists $u \in A_0$ such that u is comparable to both x and y. We assume, without loss of generality, that $u \preceq x$ and $u \preceq y$. By Lemma 2.3, it follows that $u_n \to x$ and $u_n \to y$ as $n \to \infty$.

Hence by the uniqueness of limit, we have $x = y$. ∎

3. Corollaries and Examples

If ψ is the the identity map on $[0, \infty)$ in Theorem 2.1, we have the following.

Corollary 3.1. *Let $(X, d, \preceq)$ be a partially ordered complete metric space. Let A, B be a non-empty subsets of X. Let $f : A \to B$ be non-selfmapping satisfying the following condition:*

there exist $\xi \geq 0, \phi \in \Phi$ such that for all $x, y, u, v \in A$ with $x \preceq y$

$$\left.\begin{array}{l} d(u, fx) = d(A, B) \\ d(v, fy) = d(A, B) \end{array}\right\} \implies d(u, v) \leq M(x, y, u, v) - \phi(M_1(x, y, u, v), M_2(x, y, u, v)) + \xi N(x, y, u, v), \tag{3.1}$$

where $M(x, y, u, v)$, $M_1(x, y, u, v)$, $M_2(x, y, u, v)$ and $N(x, y, u, v)$ are as in Definition 1.10. If conditions (ii)-(v) of Theorem 2.1 hold, then there exists $x' \in A_0$ such that $d(x', fx') = d(A, B)$.

Corollary 3.2. *Let $(X, d, \preceq)$ be a partially ordered complete metric space. Let A, B be a non-empty subsets of X. Let $f : A \to B$ be a non-selfmapping satisfying the following condition:*

there exist $\xi \geq 0, \phi \in \Phi$ such that for all $x, y, u, v \in A$ with $x \preceq y$

$$\left.\begin{array}{l} d(u, fx) = d(A, B) \\ d(v, fy) = d(A, B) \end{array}\right\} \implies d(u, v) \leq M'(x, y, u, v) - \phi(M_1(x, y, u, v), M_2(x, y, u, v)) + \xi N'(x, y, u, v), \tag{3.2}$$

where $M'(x, y, u, v)$, $M_1(x, y, u, v)$, $M_2(x, y, u, v)$ and $N'(x, y, u, v)$ are as in Theorem 2.2. If conditions (ii)-(v) of Theorem 2.2 hold, then there exists $x' \in A_0$ such that $d(x', fx') = d(A, B)$.

Proof. Since the inequality (3.2) implies the inequality (3.1), the conclusion of this corollary follows from Corollary 3.2. ∎

If ψ is the the identity map on $[0, \infty)$ and $\xi = 0$ in Theorem 2.1, we have the following.

Corollary 3.3. *Let $(X, d, \preceq)$ be a partially ordered complete metric space. Let A, B be a non-empty subsets of X. Let $f : A \to B$ be a non-selfmapping satisfying the following condition:*

there exists $\phi \in \Phi$ such that for all $x, y, u, v \in A$ with $x \preceq y$

$$\left.\begin{array}{l} d(u, fx) = d(A, B) \\ d(v, fy) = d(A, B) \end{array}\right\} \implies d(u, v) \leq M(x, y, u, v) - \phi(M_1(x, y, u, v), M_2(x, y, u, v)), \tag{3.3}$$

where $M(x, y, u, v)$, $M_1(x, y, u, v)$ and $M_2(x, y, u, v)$ are as in Definition 1.10. If conditions (ii)-(v) of Theorem 2.1 hold, then there exists $x' \in A_0$ such that $d(x', fx') = d(A, B)$.

Corollary 3.4. *Let $(X, d, \preceq)$ be a partially ordered complete metric space. Let A, B be a nonempty subsets of X. Let $f : A \to B$ be a non-selfmapping satisfying the following condition:*

there exists $\phi \in \Phi$ such that for all $x, y, u, v \in A$ with $x \preceq y$

$$\left.\begin{array}{l} d(u, fx) = d(A, B) \\ d(v, fy) = d(A, B) \end{array}\right\} \implies d(u, v) \leq M'(x, y, u, v)$$
$$-\phi(M_1(x, y, u, v), M_2(x, y, u, v)) \qquad (3.4)$$

where $M'(x, y, u, v)$, $M_1(x, y, u, v)$ and $M_2(x, y, u, v)$ are as in Theorem 2.2. If conditions (ii)-(v) of Theorem 2.2 are satisfied, then there exists $x' \in A_0$ such that $d(x', fx') = d(A, B)$.

Proof. Since the inequality (3.4) implies the inequality (3.3), the conclusion of this corollary follows from Corollary 3.3. ∎

The following example is in support of Theorem 2.1.

Example 3.5. Let $X = [0, \infty) \times [0, \infty)$ with the Euclidean metric d. We define a partial order $\preceq$ on X by

$$\preceq := \big\{\big((x_1, x_2), (y_1, y_2)\big) \in X \times X | x_1 = y_1, x_2 = y_2\big\} \cup \ \big\{\big((0, \frac{7}{8}), (0, \frac{1}{2^n})\big), \big((0, \frac{3}{4}), (0, \frac{1}{2^{n+1}})\big),$$
$$\big((0, \frac{1}{2^n}), (0, \frac{1}{2^m})\big), \big((0, \frac{1}{2^n}), (0, 0)\big) \mid n, m = 1, 2, 3, ..., \ m > n\big\}$$
$$\cup \big\{\big((0, \frac{3}{4}), (0, 0)\big), \big((0, \frac{7}{8}), (0, 0)\big)\big\}, \text{ where}$$
$$(x_1, x_2) \preceq (y_1, y_2) \iff x_1 \geq y_1 \text{ and } x_2 \geq y_2 \text{ in the usual sense.}$$

Let $A = \{0\} \times [0, 1] = A_0$, $B = \{2\} \times [0, 1] = B_0$. We define $f : A \to B$ by

$$f(0, x) = \begin{cases} (2, \frac{x}{2}) & \text{if } x \in [0, \frac{3}{4}) \\ (2, 2x - 1) & \text{if } x \in [\frac{3}{4}, 1]. \end{cases}$$

Clearly $d(A, B) = 2$ and $f(A_0) \subseteq B_0$. Now, we choose $x_0 = (0, \frac{1}{2}), x_1 = (0, \frac{1}{4})$, then $d(x_1, fx_0) = d(A, B)$ and $x_0 \preceq x_1$.

Now, we show that f is proximally increasing on A_0. In this regard, let $(0, x), (0, y), (0, u)$ and $(0, v) \in A_0$ such that

$$\left.\begin{array}{l} (0, x) \preceq (0, y) \\ d((0, u), f(0, x)) = 2 \\ d((0, v), f(0, y)) = 2. \end{array}\right\} \qquad (3.5)$$

Case (i): $(0, x) = (0, \frac{7}{8}) \preceq (0, \frac{1}{2^n}) = (0, y) : \ n = 1, 2, 3, ...$.

Since $d((0, u), f(0, x)) = d((0, u), f(0, \frac{7}{8})) = d((0, u), (2, \frac{3}{4})) = 2$, we have

$$u = \frac{3}{4}. \qquad (3.6)$$

From $d((0, v), f(0, y)) = d((0, v), f(0, \frac{1}{2^n})) = d((0, v), (2, \frac{1}{2^{n+1}})) = 2$, we obtain

$$v = \frac{1}{2^{n+1}}. \qquad (3.7)$$

From (3.6) and (3.7), it follows that $(0, u) = (0, \frac{3}{4}) \preceq (0, \frac{1}{2^{n+1}}) = (0, v)$.

Case (ii): $(0,x)=(0,\frac{3}{4})\preceq(0,\frac{1}{2^{n+1}})=(0,y):\ n=1,2,3,\dots$.

Since $d((0,u),f(0,x))=d((0,u),f(0,\frac{3}{4}))=d((0,u),(2,\frac{1}{2}))=2$, we have

$$u=\frac{1}{2}. \tag{3.8}$$

From $d((0,v),f(0,y))=d((0,v),f(0,\frac{1}{2^n}))=d((0,v),(2,\frac{1}{2^{n+1}}))=2$, we obtain

$$v=\frac{1}{2^{n+2}}. \tag{3.9}$$

From (3.8) and (3.9), it follows that $(0,u)=(0,\frac{1}{2})\preceq(0,\frac{1}{2^{n+2}})=(0,v)$.

Case (iii): $(0,x)=(0,\frac{1}{2^n})\preceq(0,\frac{1}{2^m})=(0,y):\ n,m=1,2,3,\dots$, with $m>n$.

Since $d((0,u),f(0,x))=d((0,u),f(0,\frac{1}{2^n}))=d((0,u),(2,\frac{1}{2^{n+1}}))=2$, we have

$$u=\frac{1}{2^{n+1}}. \tag{3.10}$$

Similarly, we get

$$v=\frac{1}{2^{m+1}}. \tag{3.11}$$

From (3.10) and (3.11), it follows that $(0,u)=(0,\frac{1}{2^{n+1}})\preceq(0,\frac{1}{2^{m+1}})=(0,v)$.

Case (iv): $(0,x)=(0,\frac{1}{2^n})\preceq(0,0)=(0,y):\ n=1,2,3,\dots$.

Since $d((0,u),f(0,x))=d((0,u),f(0,\frac{1}{2^n}))=d((0,u),(2,\frac{1}{2^{n+1}}))=2$, we have

$$u=\frac{1}{2^{n+1}}. \tag{3.12}$$

From $d((0,v),f(0,y))=d((0,v),f(0,0))=d((0,v),(2,0))=2$, we obtain

$$v=0. \tag{3.13}$$

From (3.12) and (3.13), it follows that $(0,u)=(0,\frac{1}{2^{n+1}})\preceq(0,0)=(0,v)$.

Case (v): $(0,x)=(0,\frac{3}{4})\preceq(0,0)=(0,y)$.

Since $d((0,u),f(0,x))=d((0,u),f(0,\frac{3}{4}))=d((0,u),(2,\frac{1}{2}))=2$, we have

$$u=\frac{1}{2}. \tag{3.14}$$

Similarly, from $d((0,v),f(0,y))=2$, we get

$$v=0. \tag{3.15}$$

From (3.14) and (3.15), it follows that $(0,u)=(0,\frac{1}{2})\preceq(0,0)=(0,v)$.

Case (vi): $(0,x)=(0,\frac{7}{8})\preceq(0,0)=(0,y)$.

Since $d((0,u),f(0,x))=d((0,u),f(0,\frac{7}{8}))=d((0,u),(2,\frac{3}{4}))=2$, we have

$$u=\frac{3}{4}. \tag{3.16}$$

Similarly, from $d((0,v),f(0,y))=2$, we get

$$v=0. \tag{3.17}$$

From (3.16) and (3.17), it follows that $(0,u)=(0,\frac{3}{4})\preceq(0,0)=(0,v)$.

Hence f is proximally increasing on A_0.

We now show that f satisfies the RJ property. For this purpose, let $\{(0, x_n)\}$ be any sequence in A such that

$$\lim_{n\to\infty}(0, x_n) = (0, x) \text{ and } \lim_{n\to\infty} d((0, x_n), f(0, x_n)) = d(A, B).$$

Case (i): $(0, x_n) \in [0, \frac{3}{4})$ for $n = 1, 2, \ldots$.

$$2 = d(A, B) = \lim_{n\to\infty} d((0, x_{n+1}), f(0, x_n)) = \lim_{n\to\infty} d((0, x_{n+1}), (2, \frac{1}{2}x_n))$$
$$= d((0, x), (2, \frac{1}{2}x)).$$

This implies that $x = 0$. i.e., $(0, 0) \in A_0$.
Case (ii): $(0, x_n) \in [\frac{3}{4}, 1]$ for $n = 1, 2, \ldots$.

$$2 = d(A, B) = \lim_{n\to\infty} d((0, x_{n+1}), f(0, x_n)) = \lim_{n\to\infty} d((0, x_{n+1}), (2, 2x_n - 1))$$
$$= d((0, x), (2, 2x - 1)).$$

This implies that $x = 1$. i.e., $(0, 1) \in A_0$.
Hence in any case $(0, x) \in A_0$ so that f satisfies the RJ property.

Next, we show that f is an almost generalized $\mathsf{C}-$proximal weakly contractive map. We define functions $\psi : [0, \infty) \to [0, \infty)$ and $\phi : [0, \infty) \times [0, \infty) \to [0, \infty)$ by

$$\psi(t) = \begin{cases} \frac{t}{2} & \text{if } t \in [0, 1] \\ \frac{t}{1+t} & \text{if } t \geq 1 \end{cases} \quad \text{and } \phi(s, t) = \begin{cases} \frac{s+t}{8} & \text{for all } s, t \in [0, 1] \\ \frac{1}{2} & \text{otherwise .} \end{cases}$$

Let $(0, x), (0, y), (0, u)$ and $(0, v) \in A$ such that

$$\left.\begin{array}{l} (0, x) \preceq (0, y) \\ d((0, u), f(0, x)) = 2 \\ d((0, v), f(0, y)) = 2. \end{array}\right\}$$

Case (i): $(0, x) = (0, \frac{7}{8}),\ (0, y) = (0, \frac{1}{2^n}),\ (0, u) = (0, \frac{3}{4}), (0, v) = (0, \frac{1}{2^{n+1}}) : n = 1, 2, 3, \ldots$.

In this case,

$$\begin{aligned} \psi(d((0, u), (0, v))) &= \psi(d((0, \frac{3}{4}), (0, \frac{1}{2^{n+1}}))) = \psi(\frac{3}{4} - \frac{1}{2^{n+1}}) = \frac{3}{8} - \frac{1}{2^{n+2}} \\ &\leq \frac{13}{32} - \frac{5}{2^{n+4}} = \psi(\frac{7}{8} - \frac{1}{2^n}) - \phi(\frac{7}{8} - \frac{1}{2^{n+1}}, \frac{7}{8} - \frac{1}{2^n}) + 3 \times \psi(\frac{1}{8}) \\ &= \psi\big(M\big((0, x), (0, y), (0, u), (0, v)\big)\big) - \phi\big(M_1\big((0, x), (0, y), (0, u), (0, v)\big), \\ &\quad M_2\big((0, x), (0, y), (0, u), (0, v)\big)\big) + \xi\psi(N\big((0, x), (0, y), (0, u), (0, v)\big)\big). \end{aligned}$$

Case (ii):$(0, x) = (0, \frac{3}{4}),\ (0, y) = (0, \frac{1}{2^{n+1}}), (0, u) = (0, \frac{1}{2}), (0, v) = (0, \frac{1}{2^{n+2}}) : n = 1, 2, 3, \ldots$.

In this case,

$$\begin{aligned} \psi(d((0, u), (0, v))) &= \psi(d((0, \frac{1}{2}), (0, \frac{1}{2^{n+2}}))) = \psi(\frac{1}{2} - \frac{1}{2^{n+2}}) = \frac{1}{4} - \frac{1}{2^{n+3}} \\ &\leq \frac{15}{16} - \frac{29}{2^{n+5}} = \psi(\frac{3}{4} - \frac{1}{2^{n+1}}) \\ &\quad - \phi(\frac{3}{4} - \frac{1}{2^{n+2}}, \frac{3}{4} - \frac{1}{2^{n+1}}) + 3 \times \psi(\frac{1}{2} - \frac{1}{2^{n+2}}) \\ &= \psi\big(M\big((0, x), (0, y), (0, u), (0, v)\big)\big) \end{aligned}$$

$$- \phi\big(M_1\big((0,x),(0,y),(0,u),(0,v)\big), M_2\big((0,x),(0,y),(0,u),(0,v)\big)\big)$$
$$+ \xi\psi(N\big((0,x),(0,y),(0,u),(0,v)\big)\big).$$

The following are the other possible cases.
Case (iii): $(0,x) = (0,\frac{1}{2^n}),\ (0,y) = (0,\frac{1}{2^m}), (0,u) = (0,\frac{1}{2^{n+1}}),$
$(0,v) = (0,\frac{1}{2^{m+1}}) : n,m = 1,2,3,...$ with $m > n$.
Case (iv): $(0,x) = (0,\frac{1}{2^n}),\ (0,y) = (0,0),\ (0,u) = (0,\frac{1}{2^{n+1}}),\ (0,v) = (0,0) :$
$n = 1,2,3,...$.
Case (v): $(0,x) = (0,\frac{7}{8}),\ (0,y) = (0,0),\ (0,u) = (0,\frac{3}{4}),\ (0,v) = (0,0).$
Case (vi): $(0,x) = (0,\frac{3}{4}),\ (0,y) = (0,0),\ (0,u) = (0,\frac{1}{2}),\ (0,v) = (0,0).$
By considering all the above possible cases, it is trivial to show that the inequality (1.2) holds with $\xi = 3$.

Hence $f,\ \psi$ and ϕ satisfy all the conditions of Theorem 2.1, and $(0,0)$ and $(0,1)$ are two best proximity points of f in A_0.

Remark 3.6. The inequality (1.2) fails to hold when $\xi = 0$ for any $\psi \in \Psi$ and $\phi \in \Phi$. For this purpose, we choose $x = (0,\frac{7}{8}),\ y = (0,\frac{1}{2}),\ u = (0,\frac{3}{4}),\ v = (0,\frac{1}{4})$.

$$\begin{aligned}\psi(d((0,u),(0,v))) &= \psi(d((0,\frac{3}{4}),(0,\frac{1}{4}))) = \psi(\frac{3}{4}-\frac{1}{4}) = \psi(\frac{1}{2})\\ &\not\leq \psi(\frac{3}{8}) - \phi(\frac{5}{8},\frac{3}{8}) = \psi(\frac{7}{8}-\frac{1}{2}) - \phi(\frac{7}{8}-\frac{1}{4},\ \frac{7}{8}-\frac{1}{2})\\ &= \psi\big(M\big((0,x),(0,y),(0,u),(0,v)\big)\big)\\ &\quad - \phi\big(M_1\big((0,x),(0,y),(0,u),(0,v)\big), M_2\big((0,x),(0,y),(0,u),(0,v)\big)\big).\end{aligned}$$

The following example is in support of Theorem 2.4.

Example 3.7. Let $X = \{(2,\frac{1}{2^n}),(1,\frac{1}{2^n}) : n = 1,2,3,...,\ .\}$
$\cup\{(2,0),(2,1),(2,\frac{3}{4}),(1,0),(1,\frac{3}{4}),(1,1)\}$, with the Euclidean metric d. We define a partial order $\preceq$ on X by

$$\begin{aligned}\preceq := &\big\{\big((x_1,x_2),(y_1,y_2)\big) \in X\times X | x_1 = y_1, x_2 = y_2\big\} \cup\ \big\{\big((2,1),(2,\frac{1}{2^n})\big), \big((2,\frac{3}{4}),(2,\frac{1}{2^{n+1}})\big),\\ &\big((2,\frac{1}{2^n}),(2,\frac{1}{2^m})\big), \big((2,\frac{1}{2^n}),(2,0)\big) \mid\ n,m = 1,2,3,...,\text{with } m > n\big\} \cup \big\{\big((2,1),(2,0)\big),\\ &\big((2,\frac{3}{4}),(2,0)\big)\big\}, \text{where } (x_1,x_2) \preceq (y_1,y_2) \iff x_1 \geq y_1,\ x_2 \geq y_2 \text{ and } \geq \text{ is the usual order.}\end{aligned}$$

Let $A = \{(2,\frac{1}{2^n}) :\ n = 1,2,3,...\} \cup \{(2,0),(2,\frac{3}{4}),(2,1)\} = A_0$,
$B = \{(1,\frac{1}{2^n}) :\ n = 1,2,3,...\} \cup \{(1,0),(1,\frac{3}{4}),(1,1)\} = B_0$.
We define $f : A \to B$ by

$$f(2,x) = \begin{cases} (1,\frac{x}{2}) & \text{if } x \in \{\frac{1}{2^n} :\ n = 1,2,3,...\ \} \cup \{0\} \\ (1, x-\frac{1}{4}) & \text{if } x \in \{\frac{3}{4},1\}. \end{cases}$$

Clearly $d(A,B) = 1$ and $f(A_0) \subseteq B_0$. Now, we choose
$x_0 = (2,\frac{1}{2}), x_1 = (2,\frac{1}{4})$, then $d(x_1, fx_0) = d(A,B)$ and $x_0 \preceq x_1$.

Now, we show that f is proximally increasing on A_0. In this case, let $(2,x),(2,y),(2,u)$ and $(2,v) \in A_0$ such that

$$\left.\begin{array}{l}(2,x) \preceq (2,y)\\ d((2,u),f(2,x))=1\\ d((2,v),f(2,y))=1.\end{array}\right\} \tag{3.18}$$

Case (i): $(2,x)=(2,1) \preceq (2,\frac{1}{2^n})=(2,y):\ n=1,2,3,...$

Since $d((2,u),f(2,x))=d((2,u),f(2,1))=d((2,u),(1,\frac{3}{4}))=1$, we have

$$u=\frac{3}{4}. \tag{3.19}$$

From $d((2,v),f(2,y))=d((2,v),f(2,\frac{1}{2^n}))=d((2,v),(1,\frac{1}{2^{n+1}}))=1$, we obtain

$$v=\frac{1}{2^{n+1}}. \tag{3.20}$$

From (3.19) and (3.20), it follows that $(2,u)=(2,\frac{3}{4}) \preceq (2,\frac{1}{2^{n+1}})=(2,v)$.

Case (ii): $(2,x)=(2,\frac{3}{4}) \preceq (2,\frac{1}{2^{n+1}})=(2,y):\ n=1,2,3,...$

Since $d((2,u),f(2,x))=d((2,u),f(2,\frac{3}{4}))=d((2,u),(1,\frac{1}{2}))=1$, we have

$$u=\frac{1}{2}. \tag{3.21}$$

From $d((2,v),f(2,y))=d((2,v),f(2,\frac{1}{2^{n+1}}))=d((2,v),(1,\frac{1}{2^{n+2}}))=1$, we obtain

$$v=\frac{1}{2^{n+2}}. \tag{3.22}$$

From (3.21) and (3.22), it follows that $(2,u)=(2,\frac{1}{2}) \preceq (2,\frac{1}{2^{n+2}})=(2,v)$.

Case (iii): $(2,x)=(2,\frac{1}{2^n}) \preceq (2,\frac{1}{2^n})=(2,y):\ n,m=1,2,3,...$ with $m>n$.

Case (iv): $(2,x)=(2,\frac{1}{2^n}) \preceq (2,0)=(2,y):\ n=1,2,3,...$

Case (v): $(2,x)=(2,1) \preceq (2,0)=(2,y)$.

Case (vi): $(2,x)=(2,\frac{3}{4}) \preceq (2,0)=(2,y)$.

By considering all the above possible cases, it is easy to verify that f is proximally increasing on A_0.

We now show that f satisfies the RJ property. Since A and B are non-empty closed subsets of X and f is continuous, then trivially f satisfies the RJ property.

We now show that the inequality (2.14) holds. We define functions $\psi:[0,\infty)\to[0,\infty)$ and $\phi:[0,\infty)\times[0,\infty)\to[0,\infty)$ by

$$\psi(t)=\begin{cases}\frac{t}{2} & \text{if } t\in[0,1]\\ t-\frac{1}{2} & \text{if } t\geq 1\end{cases} \quad \text{and } \phi(s,t)=\begin{cases}\frac{s+t}{16} & \text{for all } \ s,t\in[0,1]\\ \frac{1}{4} & \text{otherwise .}\end{cases}$$

Let $(2,x),(2,y),(2,u)$ and $(2,v)\in A$ such that

$$\left.\begin{array}{l}(2,x) \preceq (2,y)\\ d((2,u),f(2,x))=2\\ d((2,v),f(2,y))=2.\end{array}\right\} \tag{3.23}$$

Case (i): $(2,x)=(2,1),\ (2,y)=(2,\frac{1}{2^n}),(2,u)=(2,\frac{3}{4}),(2,v)=(2,\frac{1}{2^{n+1}}):n=1,2,3,...$.

In this case,

$$\begin{aligned}\psi(d((2,u),(2,v))) &= \psi(d((2,\frac{3}{4}),(2,\frac{1}{2^{n+1}}))) = \psi(\frac{3}{4}-\frac{1}{2^{n+1}}) = \frac{3}{8}-\frac{1}{2^{n+2}}\\ &\le \frac{3}{8}-\frac{7}{2^{n+5}} = \psi(1-\frac{1}{2^n}) - \phi(1-\frac{1}{2^{n+1}},\ 1-\frac{1}{2^n}) + 1\times\psi(\frac{1}{2^{n+1}})\\ &= \psi\big(M\big((2,x),(2,y),(2,u),(2,v)\big)\big)\\ &\quad -\phi\big(M_1\big((2,x),(2,y),(2,u),(2,v)\big), M_2\big((2,x),(2,y),(2,u),(2,v)\big)\big)\\ &\quad +\xi\psi(N\big((2,x),(2,y),(2,u),(2,v)\big)\big).\end{aligned}$$

By considering all elements of A satisfying (3.23), we can easily show that the inequality (2.14) is satisfied with $\xi = 1$.
Hence f satisfies all the conditions of Theorem 2.4, and $(2, 0)$ is unique best proximity point of f in A_0.

Here we observe that $(2, 1)$ and $(2, \frac{3}{4})$ are not comparable. But there exists $(2, 0)$ which is comparable to both $(2, 1)$ and $(2, \frac{3}{4})$ so that condition H of Theorem 2.4 holds.

Remark 3.8. The inequality (2.14) fails to hold when $\xi = 0$ for any $\psi \in \Psi$ and $\phi \in \Phi$. For, let $x = (2, 1),\ y = (2, \frac{1}{2}),\ u = (2, \frac{3}{4}),\ v = (2, \frac{1}{4})$.

$$\begin{aligned}\psi(d((2,u),(2,v))) &= \psi(d((2,\frac{3}{4}),(2,\frac{1}{4}))) = \psi(\frac{3}{4}-\frac{1}{4}) = \psi(\frac{1}{2})\\ &\nleq \psi(\frac{1}{2}) - \phi(\frac{3}{4},\frac{1}{2}) = \psi(1-\frac{1}{2}) - \phi(1-\frac{1}{4},\ 1-\frac{1}{2})\\ &= \psi\big(M\big((2,x),(2,y),(2,u),(2,v)\big)\big)\\ &\quad -\phi\big(M_1\big((2,x),(2,y),(2,u),(2,v)\big), M_2\big((2,x),(2,y),(2,u),(2,v)\big)\big).\end{aligned}$$

Open Problem: Can we prove the uniqueness of best proximity point of Theorem 2.1 under the assumption 'condition (H)' of Theorem 2.4?

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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