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OSCILLATION OF CERTAIN THIRD ORDER NONLINEAR DIFFERENTIAL EQUATION WITH NEUTRAL TERMS

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Abstract The main goal of this work is to establish some new comparison theorems for oscillation of solutions to the third order nonlinear differential equations with neutral terms of the form

$$\left[r(t)\left[\left(x(t)+\sum_{i=1}^{n}p_i(t)x(\eta_i(t))\right)''\right]^{\gamma}\right]'+q(t)x^{\gamma}(\sigma(t))=0,$$

are presented. We give several Theorems and related examples to illustrate the main results.

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1. Introduction

This work is concerned with oscillation behavior of a class of third order nonlinear neutral differential equation

$$\left[r(t)\left[\left(x(t)+\sum_{i=1}^{n}p_i(t)x(\eta_i(t))\right)''\right]^{\gamma}\right]'+q(t)x^{\gamma}(\sigma(t))=0, \tag{E}$$

where $n>0$ is an integer, $q(t)$, $\sigma(t)$, $p_i(t)$ and $\eta_i(t)$ are continuous differentiable on $[t_0,+\infty)$. Throughout this paper it always assume the following conditions hold:

(C_1) γ is a quotient of odd positive integers, $r(t),q(t)>0$, $0\le p_i(t)\le a_i<\infty$ for $i=1,2,...,n$;

(C_2) $\eta_i\circ\sigma=\sigma\circ\eta_i$, $\eta_i'(t)\ge\lambda_i>0$ for $i=1,2,...,n$; and $\lim\limits_{t\to+\infty}\sigma(t)=\infty$, $\sigma(t)<t$;

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and, moreover

$$\lim_{t\to\infty} R(t) = \infty, \quad R(t) = \int_{t_0}^{t} \frac{1}{r^{1/\gamma}(s)} ds. \tag{1.1}$$

We set

$$Z(t) = x(t) + \sum_{i=1}^{n} p_i(t) x(\eta_i(t)). \tag{1.2}$$

By a solution to (E), we mean a function $x(t)$ in $C^2[T_x, \infty)$ for which $a(t)(Z''(t))^\gamma$ is in $C^1[T_x, \infty)$ and (E) is satisfied on some interval $[T_x, \infty)$, where $T_x \geq t_0$. We consider only solutions $x(t)$ for which $\sup\{|x(t)| : t \geq T\} > 0$ for all $T \geq T_x$. A solution of (E) is called oscillatory if it has arbitrarily large zeros on $[T_x, \infty)$ and otherwise, it is said to be non-oscillatory. The equation itself is called oscillatory if all its solutions are oscillatory.

Of late, much attention is being paid in the research activities related to oscillation and asymptotic behavior of various types of differential equations. As a result innumerable research papers [1-8] as well as several monographs [1, 6] had been published and the references quoted therein.

Recently, C. Zhang et al. [13] studied the oscillatory behavior of the following second order differential equation

$$(r(t)|z'(t)|^{\alpha-1}z'(t))' + q(t)|x(\sigma(t))|^{\alpha-1}x(\sigma(t)) = 0$$

where $z(t)$ is defined in (1.2), under the assumption of (1.1) and T. Li et al. [9] also studied the above equation in the case of $\lim_{t\to\infty} R(t) < \infty$. In third order, B. Baculíková et al. [2], E. Thandapani et al. [12] and J. Dzurina et al. [5] were studied several oscillation results for equation

$$\left(r(t)(z''(t))^\gamma\right)' + q(t)x^\gamma(\sigma(t)) = 0,$$

by the condition of (1.1). B. Baculíková and J. Džurina [3] investigated oscillatory theorems of above neutral equation if $\gamma = 1$ via comparison principle.

In this work, we improve and extend the main results of [3] and use the techniques in [13] to obtain criteria of (E). In the sequel, all inequalities are assumed to hold eventually, that is, for all t large enough. Without loss of generality, we can deal only with the positive solutions of (E).

2. Main Results

For our further reference, let us denote

$$\begin{aligned} Q(t) &= \min\{q(t), q(\eta_1(t)), ..., q(\eta_n(t))\} && (2.1)\\ R_1[t, t_1] &= R(t) - R(t_1) && (2.2)\\ R_2[t, t_1] &= \int_{t_1}^{t} R_1[s, t_1]\, ds && (2.3) \end{aligned}$$

To obtain sufficient conditions for the oscillation of solutions to (E), we need the the following lemmas.

Lemma 2.1. *Let $\gamma \geq 1$. Assume $u_i \geq 0$ for $i = 1, 2, ..., n$. Then*

$$\left(\sum_{i=1}^{n} u_i\right)^\gamma \leq (n+1)^{\gamma-1} \sum_{i=1}^{n} u_i^\gamma \tag{2.4}$$

Proof. Consider a function $g(u) = u^\gamma$. Since $g'' > 0$ for $u > 0$, function $g(u)$ is convex, hence using Jensen's inequality, we obtain (2.4). ∎

We now give oscillation results when (1.1) holds.

Theorem 2.2. *Let $\eta_i(t) \geq t$ for $i = 1, 2, ..., n$ and $\gamma \geq 1$. Assume that*

$$\int_{t_1}^{\infty} \int_{v}^{\infty} \frac{1}{r^{1/\gamma}(u)} \Big(\int_{u}^{\infty} Q(s)\, ds \Big)^{1/\gamma} du\, dv = \infty, \tag{2.5}$$

and the first order delay differential equation

$$w'(t) + \frac{Q(t)}{(n+1)^{\gamma-1}} \frac{\big(R_2[\sigma(t), t_1]\big)^\gamma}{\Big(1 + \sum_{i=1}^{n} a_i^\gamma \lambda_i^{-1}\Big)} w(\sigma(t)) = 0 \tag{2.6}$$

is oscillatory, then every non-oscillatory solution of (E) *satisfies* $\lim_{t\to\infty} x(t) = 0$.

Proof. Assume, for sake of contradiction, $x(t)$ has an eventually positive solution of (E) on $[t_0, \infty)$. Then from (C_1) and (C_2) the corresponding function $Z(t)$ satisfies

$$\begin{aligned} Z^\gamma(\sigma(t)) &= \Big[x(\sigma(t)) + \sum_{i=1}^{n} p_i(\sigma(t)) x(\eta_i(\sigma(t))) \Big]^\gamma \\ &\leq \Big[x(\sigma(t)) + \sum_{i=1}^{n} a_i\, x(\eta_i(\sigma(t))) \Big]^\gamma \\ &\leq \frac{1}{(n+1)^{1-\gamma}} \Big[x^\gamma(\sigma(t)) + \sum_{i=1}^{n} a_i^\gamma\, x^\gamma(\sigma(\eta_i(t))) \Big]. \end{aligned} \tag{2.7}$$

On the other hand, it follows from (E) that

$$\Big(r(t) \big(Z''(t)\big)^\gamma \Big)' + q(t) x^\gamma(\sigma(t)) = 0, \tag{2.8}$$

which yields,

$$\begin{aligned} 0 &= \sum_{i=1}^{n} \frac{a_i^\gamma}{\eta_i'(t)} \Big(r(\eta_i(t)) \big(Z''(\eta_i(t))\big)^\gamma \Big)' + \sum_{i=1}^{n} a_i^\gamma q(\eta_i(t)) x^\gamma(\sigma(\eta_i(t))) \\ &\geq \sum_{i=1}^{n} \frac{a_i^\gamma}{\lambda_i} \Big(r(\eta_i(t)) \big(Z''(\eta_i(t))\big)^\gamma \Big)' + \sum_{i=1}^{n} a_i^\gamma q(\eta_i(t)) x^\gamma(\sigma(\eta_i(t))). \end{aligned} \tag{2.9}$$

Combining (2.8) and (2.9), we are led to,

$$\begin{aligned} \Big(r(t) \big(Z''(t)\big)^\gamma \Big)' &+ \sum_{i=1}^{n} \frac{a_i^\gamma}{\lambda_i} \Big(r(\eta_i(t)) \big(Z''(\eta_i(t))\big)^\gamma \Big)' \\ &+ q(t) x^\gamma(\sigma(t)) + \sum_{i=1}^{n} a_i^\gamma q(\eta_i(t)) x^\gamma(\sigma(\eta_i(t))) \leq 0 \end{aligned}$$

$$\begin{aligned} &\Big(r(t) \big(Z''(t)\big)^\gamma \Big)' + \sum_{i=1}^{n} \frac{a_i^\gamma}{\lambda_i} \Big(r(\eta_i(t)) \big(Z''(\eta_i(t))\big)^\gamma \Big)' \\ &\qquad + \min\{q(t), q(\eta_1(t)), ..., q(\eta_n(t))\} \Big[x^\gamma(\sigma(t)) + \sum_{i=1}^{n} a_i^\gamma\, x^\gamma(\sigma(\eta_i(t))) \Big] \leq 0 \end{aligned}$$

$$\left(r(t)\left(Z''(t)\right)^{\gamma}\right)' + \sum_{i=1}^{n} \frac{a_i^{\gamma}}{\lambda_i}\left(r(\eta_i(t))\left(Z''(\eta_i(t))\right)^{\gamma}\right)' + \frac{Q(t)}{(n+1)^{\gamma-1}} Z^{\gamma}(\sigma(t)) \le 0, \tag{2.10}$$

where $Q(t)$ defined in (2.1). Assumption of (1.1), there exists following cases

$$Z(t) > 0, \quad Z'(t) > 0, \quad Z''(t) > 0, \quad \text{and} \quad (r(t)(Z''(t))^{\gamma})' \le 0, \tag{2.11}$$

or

$$Z(t) > 0, \quad Z'(t) < 0, \quad Z''(t) > 0, \quad \text{and} \quad (r(t)(Z''(t))^{\gamma})' \le 0, \tag{2.12}$$

for $t \ge t_1$, t_1 is large enough. Assume (2.11) holds. Since $Z(t) > 0$ and $y(t) = r(t)(Z''(t))^{\gamma} > 0$ is decreasing, we obtain

$$Z'(t) \ge \int_{t_1}^{t} \frac{1}{r^{1/\gamma}(s)} [r(s)(Z''(s))^{\gamma}]^{1/\gamma}\, ds \ge y^{1/\gamma}(t) \int_{t_1}^{t} \frac{ds}{r^{1/\gamma}(s)} = y^{1/\gamma}(t) R_1[t, t_1].$$

Integrate above from t_1 to t, yields

$$Z(t) \ge \int_{t_1}^{t} y^{1/\gamma}(s) R_1[s, t_1]\, ds \ge y^{1/\gamma}(t) \int_{t_1}^{t} R_1[s, t_1]\, ds.$$

That is,

$$Z^{\gamma}(\sigma(t)) \ge y(\sigma(t)) \left(\int_{t_1}^{\sigma(t)} R_1[s, t_1]\, ds \right)^{\gamma} = y(\sigma(t))\left(R_2[\sigma(t), t_1]\right)^{\gamma}. \tag{2.13}$$

Combining (2.13) together with (2.10), we get that $y(t)$ is a positive solution of

$$\left(y(t) + \sum_{i=1}^{n} \frac{a_i^{\gamma}}{\lambda_i} y(\eta_i(t)) \right)' + \frac{Q(t)}{(n+1)^{\gamma-1}} \left(R_2[\sigma(t), t_1]\right)^{\gamma} y(\sigma(t)) \le 0. \tag{2.14}$$

Let us denote

$$w(t) = y(t) + \sum_{i=1}^{n} \frac{a_i^{\gamma}}{\lambda_i} y(\eta_i(t)). \tag{2.15}$$

Since $y(t)$ decreasing and $\eta_i(t) \ge t$ that

$$w(t) \le \left(1 + \sum_{i=1}^{n} \frac{a_i^{\gamma}}{\lambda_i} \right) y(t).$$

Using latter inequality into (2.14), which yields

$$w'(t) + \frac{Q(t)}{(n+1)^{\gamma-1}} \frac{\left(R_2[\sigma(t), t_1]\right)^{\gamma}}{\left(1 + \sum_{i=1}^{n} a_i^{\gamma} \lambda_i^{-1}\right)} w(\sigma(t)) \le 0,$$

it follows from Theorem 1 in [11], we get $w(t)$ is a positive solution which contradicts to (2.6).

Assume (2.12) hold. Since $Z(t) > 0$ and $Z'(t) < 0$, then there exists $\lim_{t\to\infty} Z(t) = \ell \geq 0$. We claim that $\ell = 0$. If $\ell > 0$, then an integration of (2.10) from t to ∞ leads to

$$\begin{aligned} r(t)(Z''(t))^\gamma + \sum_{i=1}^{n} \frac{a_i^\gamma}{\lambda_i} r(\eta_i(t))(Z''(\eta_i(t)))^\gamma &\geq \int_t^\infty \frac{Q(s)}{(n+1)^{\gamma-1}} Z^\gamma(\sigma(s))\, ds \\ &\geq \frac{\ell^\gamma}{(n+1)^{\gamma-1}} \int_t^\infty Q(s)\, ds. \end{aligned} \tag{2.16}$$

Since $y(t) = r(t)(Z''(t))^\gamma$ decreasing and $\eta_i(t) \geq t$, then

$$r(t)(Z''(t))^\gamma \left(1 + \sum_{i=1}^{n} \frac{a_i^\gamma}{\lambda_i}\right) \geq r(t)(Z''(t))^\gamma + \sum_{i=1}^{n} \frac{a_i^\gamma}{\lambda_i} r(\eta_i(t))(Z''(\eta_i(t)))^\gamma,$$

Which in view of (2.16) provides,

$$\begin{aligned} r(t)(Z''(t))^\gamma &\geq \frac{\ell^\gamma}{(n+1)^{\gamma-1}} \left(1 + \sum_{i=1}^{n} \frac{a_i^\gamma}{\lambda_i}\right)^{-1} \int_t^\infty Q(s)\, ds \\ Z''(t) &\geq \frac{\ell}{(n+1)^{\frac{\gamma-1}{\gamma}}} \left(1 + \sum_{i=1}^{n} \frac{a_i^\gamma}{\lambda_i}\right)^{-1/\gamma} \frac{1}{r^{1/\gamma}(t)} \left[\int_t^\infty Q(s)\, ds\right]^{1/\gamma}. \end{aligned}$$

Integrating again from t to ∞, we have

$$-Z'(t) \geq \frac{\ell}{(n+1)^{\frac{\gamma-1}{\gamma}}} \left(1 + \sum_{i=1}^{n} \frac{a_i^\gamma}{\lambda_i}\right)^{-1/\gamma} \int_t^\infty \frac{1}{r^{1/\gamma}(u)} \left[\int_u^\infty Q(s)\, ds\right]^{1/\gamma} du.$$

Integrating from t_1 to t, we obtain

$$Z(t_1) \geq \frac{\ell}{(n+1)^{\frac{\gamma-1}{\gamma}}} \left(1 + \sum_{i=1}^{n} \frac{a_i^\gamma}{\lambda_i}\right)^{-1/\gamma} \int_{t_1}^t \int_v^\infty \frac{1}{r^{1/\gamma}(v)} \left[\int_u^\infty Q(s)\, ds\right]^{1/\gamma} du\, dv.$$

This contradicts (2.5) and hence $\ell = 0$. ∎

Corollary 2.3. *Let $\eta_i(t) \geq t$ for $i = 1, 2, ..., n$ and $\gamma \geq 1$. Assume that (2.5) holds. Further, assume that*

$$\liminf_{t\to\infty} \int_{\sigma(t)}^t Q(s)(R_2[\sigma(s), t_1])^\gamma\, ds > \frac{(n+1)^{\gamma-1}}{e} \left(1 + \sum_{i=1}^{n} \frac{a_i^\gamma}{\lambda_i}\right) \tag{2.17}$$

then every non-oscillatory solution $x(t)$ of (E) satisfies $\lim_{t\to\infty} x(t) = 0$.

Proof. It follows from Theorem 2.1.1 in [10] that the associated delay differential equation (2.6) also has a positive solution, which contradicts the oscillatory nature of (2.6). ∎

Example 2.4. *Consider,*

$$\left(x(t) + \frac{1}{2}x(t - \frac{3\pi}{2}) + \frac{1}{3}x(t + \frac{\pi}{2})\right)''' + \frac{1}{6}x(t - \pi) = 0 \quad t \geq 1, \tag{2.18}$$

where $r(t) = 1$, $\gamma = 1$, $q(t) = 1/6$, $\sigma(t) = t - \pi$, $p_1(t) = 1/2$, $p_2(t) = 1/3$, $\eta_1 = t - 3\pi/2$ and $\eta_2 = t + \pi/2$. It is not hard to verify all conditions of Corollary 2.3 are satisfied. Hence, every non-oscillatory solution of (2.18) satisfies $\lim_{t\to\infty} x(t) = 0$.

Theorem 2.5. *Let $\sigma(t) \leq \eta_i(t) \leq t$ for $i = 1, 2, ..., n$ and (2.5) holds. Assume that the first order delay differential equation*

$$w'(t) + \frac{Q(t)}{(n+1)^{\gamma-1}} \frac{\left(R_2[\sigma(t), t_1]\right)^{\gamma}}{\left(1 + \sum_{i=1}^{n} a_i^{\gamma} \lambda_i^{-1}\right)} w(\eta^{-1}(\sigma(t))) = 0, \tag{2.19}$$

where $\eta(t) = \min\{\eta_i(t), i = 1, 2, ..., n\}$ and $\eta^{-1}(t)$ is inverse function of $\eta(t)$, is oscillatory, then every non-oscillatory solution of (E) satisfies $\lim_{t\to\infty} x(t) = 0$.

Proof. Assume, for sake of contradiction, $x(t)$ has an eventually positive solution of (E) on $[t_0, \infty)$. Then using the same arguments as in the proof of Theorem 2.2, we assume that (2.11) holds, $y(t) = r(t)(Z''(t))^{\gamma} > 0$ satisfies (2.14). Let us denote $w(t) = y(t) + \sum_{i=1}^{n} \frac{a_i^{\gamma}}{\lambda_i} y(\eta_i(t))$. Since $y(t)$ decreasing and $\eta_i(t) \leq t$ that

$$w(t) = y(\eta(t))(1 + \sum_{i=1}^{n} \frac{a_i^{\gamma}}{\lambda_i}). \tag{2.20}$$

Substituting (2.20) into (2.14), it follows from Theorem 1 in [11], we get $w(t)$ is a positive solution which contradicts to (2.19). Next, Assume (2.12) holds, we get $\lim_{t\to\infty} x(t) = 0$. ∎

Corollary 2.6. *Let $\sigma(t) \leq \eta_i(t) \leq t$ for $i = 1, 2, ..., n$ and $\gamma \geq 1$. Assume that (2.5) hold. Further, assume that*

$$\liminf_{t\to\infty} \int_{\tau^{-1}(\sigma(t))}^{t} Q(s)(R_2[\sigma(s), t_1])^{\gamma}\, ds > \frac{(n+1)^{\gamma-1}}{e} \Big(1 + \sum_{i=1}^{n} \frac{a_i^{\gamma}}{\lambda_i}\Big) \tag{2.21}$$

then every non-oscillatory solution $x(t)$ of (E) satisfies $\lim_{t\to\infty} x(t) = 0$.

Proof. It follows from Theorem 2.1.1 in [10] that the associated delay differential equation (2.6) also has a positive solution, which contradicts the oscillatory nature of (2.19). ∎

Remark. *If $\gamma = 1$ and $n = 1$, then all the above results are reduce to [3].*

Example 2.7. *Consider,*

$$\left[e^t \left[\left(x(t) + \frac{1}{3} x(t/2) + \frac{1}{5} x(2t/3) \right)'' \right]^3 \right] + \frac{e^{2t}}{t^5} x^3(t/2) = 0 \quad t \geq 1, \tag{2.22}$$

where $r(t) = e^t$, $\gamma = 3$, $q(t) = e^{2t}/t^5$, $\sigma(t) = t/2$, $p_1(t) = 1/3$, $p_2(t) = 1/5$, $\eta_1 = t/2$ and $\eta_2 = 2t/3$. It is not hard to verify all conditions of Corollary 2.6 are satisfied. Hence, every non-oscillatory solution of (2.22) satisfies $\lim_{t\to\infty} x(t) = 0$.

Conflict of Interests

The author declare that there is no conflict of interests regarding the publication of this paper.

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