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# Generalization of fixed Point results for ($\alpha^*, \eta^*, \beta$ )-contractive mappings in fuzzy metric spaces

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**Abstract** In this paper, we define generalized modified ($\alpha^*, \eta^*, \beta$ )- contractive mappings and prove the existence of fixed points of such maps in a complete fuzzy metric spaces. Moreover, we present examples in support of the obtained results.

**MSC:** 47H09
**Keywords:** Fixed points, fuzzy metric space, ($\alpha^*, \eta^*$) admissible map, generalized modified ($\alpha^*, \eta^*, \beta$)-contractive mappings.

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## 1. Introduction

From 1975 up to now the theory of fuzzy metric space has been studied by many mathematicians. The first mathematicians, who introduced fuzzy metric space, in 1975 are Kramosil and Michalek [9]. In 1994, George and Veeramani [3] modified the concept of fuzzy metric space introduced by Kramosil and Michalek [9]. In 2002, Gregori and Sapene [7] initiated fuzzy contraction mappings and proved an important fixed point theorem for this class of mappings. In 2008, Mihet [11] introduced $\psi$ contractive mappings in non-Archimedean fuzzy metric spaces. For the last 41 years, the concept of fuzzy metric space and fixed point theorems were studied, generalized and proved by different mathematicians (see [5-13]). In 2012, Samet, Vitero and Vetro [21] introduced the concept of admissible mapping for single valued map, and in the same year Asl, Rezapour and Shahzad [1]extended the concept of admissible for single valued mappings to multi valued mappings. In 2013 Salimi, Latif and Hussain [20] proved a fixed point theorem for $\alpha$-admissible mapping with respect to $\eta$ on a metric space. Soon after, Hussain, Salimi and Latif [8] proved fixed point theorem for single and set valued, ( $\alpha, \eta, \psi$) contractive mappings. Very recently Supak, Cho, Kumam [17] introduced a new contractive condition



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and proved fixed point theorems for modified $(\alpha^*, \eta^*)$ contractive mapping in fuzzy metric space.

## 2. Preliminaries

We begin with some basic definitions and results which will be used in main part of our paper.

**Definition 2.1.** [22] A binary operation $* : [0,1]\times[0,1] \to [0,1]$ is said to be a continuous t-norm if it satisfies the following conditions :

(T1)**:** $*$ is associative and commutative,
(T2)**:** $*$ is continuous,
(T3)**:** $a * 1 = a$ for all $a \in [0,1]$,
(T4)**:** $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0,1]$.

**Remark 2.2.** A t-norm $*$ is called positive, if $a * b > 0$ for all $a, b \in (0,1)$.

Examples of continuous t-norms are Lukasievicz t-norm, i.e, $a *_L b = \max\{a+b-1, 0\}$, product t-norm, i.e, $a * b = ab$ and minimum t-norm, i.e., $a *_M b = \min\{a, b\}$, for $a, b \in [0,1]$.

The concept of fuzzy metric space is defined by George and Veeramani [3] as follows.

**Definition 2.3.** [3] Let $X$ be a nonempty set, $*$ be a continuous t-norm. Assume that a fuzzy set $M : X\times X\times(0,\infty) \to [0,1]$ satisfies the following conditions; for each $x, y, z \in X$ and $t, s > 0$,

(M1)**:** $M(x, y, t) > 0$,
(M2)**:** $M(x, y, t) = 1$ if and only if $x = y$,
(M3)**:** $M(x, y, t) = M(y, x, t)$,
(M4)**:** $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
(M5)**:** $M(x, y, \cdot) : (0,\infty) \to [0,1]$ is continuous.

Then we call $M$ a fuzzy metric on $X$, and we call the 3-tuple $(X, M, *)$ a fuzzy metric space.

**Lemma 2.4.** *[4] Let $(X, M, *)$ be a fuzzy metric space. For all $x, y \in X$, $M(x, y, \cdot)$ is a non-decreasing function.*

**Remark 2.5.** We observe that $0 < M(x, y, t) < 1$ provided $x \neq y$, for all $t > 0$ (see [15]).

Let $(X, M, *)$ be a fuzzy metric space. For $t > 0$, the open ball $B(x, r, t)$ with a center $x \in X$ and radius $0 < r < 1$ is defined by $B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}$. A subset $A \subset X$ is called open if for each $x \in A$, there exist $t > 0$ and $0 < r < 1$ such that $B(x, r, t) \subset A$. Let $\tau$ denote the family of all open subsets of $X$. Then $\tau$ is a topology on $X$, called the topology induced by the fuzzy metric $M$. This topology is metrizable (see [6]).

**Example 2.6.** [3]. Let $(X, d)$ be a metric space. We define $a*b = ab$ (or $a*b = \min\{a, b\}$) for all $a, b \in [0,1]$, and $M : X \times X \times (0,\infty) \to [0,1]$ as

$$M(x, y, t) = \frac{t}{t+d(x,y)} \text{for all} \quad x, y \in X \text{ and } t > 0.$$

Then $(X, M, *)$ is a fuzzy metric space. We call this fuzzy metric $M$ as the fuzzy metric induced by the metric $d$, and this $M$ is known as the standard fuzzy metric.

 

Now we give some examples of fuzzy metric spaces due to Gregori, Morillas and Sapena [5].

**Example 2.7.** [5] Let $(X, d)$ be a metric space and $g : R^+ \to [0, \infty)$, $R^+ = [0, \infty)$ be an increasing continuous function. Define $M : X \times X \times (0, \infty) \to [0, 1]$ as $M(x, y, t) = e^{(\frac{-d(x,y)}{g(t)})}$ for all $x, y \in X$ and $t > 0$. Then $(X, M, *)$ is a fuzzy metric space on $X$ where $*$ is the product t-norm.

**Example 2.8.** [5]. Let $(X, d)$ be a bounded metric space with $d(x, y) < k$ for all $x, y \in X$, where $k$ is fixed constant in $(0, \infty)$ and $g : R^+ \to (k, \infty)$, $R^+ = [0, \infty)$ be an increasing continuous function. Define a function $M : X \times X \times (0, \infty) \to [0, 1]$ as $M(x, y, t) = 1 - \frac{d(x,y)}{g(t)}$ for all $x, y \in X$ and $t > 0$. Then $(X, M, *)$ is a fuzzy metric space, where $*$ is a Lukasievicz t-norm.

**Definition 2.9.** [3] Let $(X, M, *)$ be a fuzzy metric space.

(1) A sequence$\{x_n\}$ in $X$ is said to be convergent to a point $x \in X$ if $\lim_{n\to\infty} M(x_n, x, t) = 1$ for all $t > 0$.

(2) A sequence$\{x_n\}$ in $X$ is called a Cauchy sequence if, for each $0 < \epsilon < 1$ and $t > 0$, there exits $n_0 \in N$ such that $M(x_n, x_m, t) > 1 - \epsilon$ for each $n, m \geq n_0$.

(3) A fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.

(4) A fuzzy metric space in which every sequence has a convergent subsequence is said to be compact.

**Remark 2.10.** In a fuzzy metric space the limit of a convergent sequence is unique.

**Definition 2.11.** [17] Let $(X, M, *)$ be a fuzzy metric space. Then the mapping $M$ is said to be continuous on $X \times X \times (0, \infty)$ if

$$\lim_{n\to\infty} M(x_n, y_n, t_n) = M(x, y, t)$$

when $\{(x_n, y_n, t_n)\}$ is a sequence in $X \times X \times (0, \infty)$ which converges to a point $(x, y, t) \in X \times X \times (0, \infty)$, *i.e.*,

$$\lim_{n\to\infty} M(x_n, x, t) = \lim_{n\to\infty} M(y_n, y, t) = 1 \text{ and } \lim_{n\to\infty} M(x, y, t_n) = M(x, y, t).$$

**Lemma 2.12.** *[18] If $(X, M, *)$ is a fuzzy metric space, then $M$ is a continuous function on $X \times X \times (0, \infty)$.*

The concept of $\alpha$-admissible mapping was introduced by Samet,Vetro and Vetro [21] as follows.

**Definition 2.13.** [21] Let $X$ be a nonempty set, $T : X \to X$ and $\alpha : X \times X \to [0, \infty)$ be maps. We say that $T$ is an $\alpha$-admissible mapping if for all $x, y \in X$, we have $\alpha(x, y) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1$.

In 2013, Salimi, Latif and Hussain [20] modified the concept of $\alpha-$ admissible mapping as follows.

**Definition 2.14.** [20]. Let $X$ be a nonempty set, $T : X \to X$ and $\alpha, \eta : X \times X \to [0, \infty)$. We say that $T$ is an $\alpha$-admissible mapping with respect to $\eta$ if for all $x, y \in X$, we have $\alpha(x, y) \geq \eta(x, y) \Rightarrow \alpha(Tx, Ty) \geq \eta(Tx, Ty)$.

If we take $\eta(x, y) = 1$ for all $x, y \in X$, then $T$ is an $\alpha$-admissible mapping. If we take $\alpha(x, y) = 1$, then we say that $T$ is an $\eta$-subadmissible mapping.

In 2016, Supak, Cho and Kumam[17] introduced the $\alpha^*-$ admissible mappings in fuzzy metric spaces.

**Definition 2.15.** [17] Let $(X, M, *)$ be a fuzzy metric space. A mapping $T : X \to X$ and let $\alpha^* : X \times X \times (0,\infty) \to [0,\infty)$ be a function. We say that $T$ is an $\alpha^*$-admissible mapping if, for all $x, y \in X$ and $t > 0$, $\alpha^*(x,y,t) \geq 1 \Rightarrow \alpha^*(Tx,Ty,t) \geq 1$.

In 2016, Supak, Cho and Kumam[17] introduced the $(\alpha^*, \eta^*)$- admissible mappings in fuzzy metric spaces.

**Definition 2.16.** [17] Let$(X, M, *)$ be a fuzzy metric space. A mapping $T : X \to X$ and let $\alpha^*, \eta^* : X \times X \times (0,\infty) \to [0,\infty)$ be two functions. We say that $T$ is an $(\alpha^*, \eta^*)$- admissible mapping if, for all $x, y \in X$ and $t > 0$, $\alpha^*(x,y,t) \geq \eta^*(x,y,t) \Rightarrow \alpha^*(Tx,Ty,t) \geq \eta^*(Tx,Ty,t)$.

Note that, if $\eta^*(x,y,t) = 1$ then it is clear that $T$ is an $\alpha^*$ admissible mapping. if we take $\alpha^*(x,y,t) = 1$, then we say that $T$ is an $\eta^*$-subadmissible mapping.

We denote

$S = \{\beta = [0,1] \to [1,\infty], |$ for any sequence$\{t_n\} \subset [0,1], \beta(t_n) \to 1 \Rightarrow t_n \to 1\}$.
Note that $S \neq \emptyset$, In fact the map $\beta(t) = \frac{2}{1+t} \in S$.

**Remark 2.17.** For any $\beta \in S$ and $a \in [0,1]$, we have $\beta(a) = 1$ implies that $a = 1$.

**Theorem 2.18.** *[17] Let $(X, M, *)$ be a complete fuzzy metric space. A mapping $T : X \to X$ be $(\alpha^*, \eta^*)$-admissible map. Assume that there exists a function $\beta \in S$ such that*

$$\alpha^*(x,Tx,t)\alpha^*(y,Ty,t) \geq \eta^*(x,Tx,t)\eta^*(y,Ty,t) \Rightarrow M(Tx,Ty,t) \geq \beta(M(x,y,t))N(x,y,t)$$

*where*

$$N(x,y,t) = \min\{M(x,y,t), \max\{M(x,Tx,t), M(y,Ty,t)\}\}$$

*for all $x, y \in X$ and $t > 0$. Suppose that the following conditions hold*

**(a):** *there exists $x_0 \in X$ such that $\alpha^*(x_0,Tx_0,t) \geq \eta^*(x_0,Tx_0,t)$ for all $t > 0$.*

**(b):** *For any sequence $\{x_n\} \subset X$ such that $\alpha^*(x_n,x_{n+1},t) \geq \eta^*(x_n,x_{n+1},t)$, for all $n \in \mathbb{N}, t > 0$ and $x_n \to x$ as $n \to \infty$, then $\alpha^*(x,Tx,t) \geq \eta^*(x,Tx,t)$ for all $t > 0$.*

*Then $T$ has a fixed point.*

Now we introduce the following definition.

**Definition 2.19.** Let $(X, M, *)$ be a fuzzy metric space. Let $T : X \to X$ and let $\alpha^*, \eta^* : X \times X \times (0,\infty) \to [0,\infty)$ be two functions. if there exists a function $\beta \in S$ such that,

$$\begin{aligned} \alpha^*(x,Tx,t)\alpha^*(y,Ty,t) \geq \eta^*(x,Tx,t)\eta^*(y,Ty,t) \Rightarrow M(Tx,Ty,t) \geq \\ \beta(M(x,y,t))N(x,y,t)K(x,y,t), \end{aligned} \tag{2.1}$$

where

$$N(x,y,t) = \min\{M(x,y,t), \max\{M(x,Tx,t), M(y,Ty,t)\}\}.$$

and

$$K(x,y,t) = \max\{M(x,y,t), M(x,Tx,t), M(y,Ty,t), M(x,Ty,t), M(y,Tx,t)\}.$$

Then we say that $T$ is a generalized modified $(\alpha^*, \eta^*, \beta)$ contractive mapping.

**Example 2.20.** Let $X = [0, \frac{2}{3}] \cup [1, \infty)$ and $M(x, y, t) = (\frac{t}{t+1})^{d(x,y)}$, where $d(x, y) = |x - y|$, $*$ be product continuous t-norm. Here $(X, M, *)$ is complete fuzzy metric space. Let $T : X \to X$ be a map defined by

$$Tx = \begin{cases} \frac{x}{2}, & \text{if } x \in [0, \frac{2}{3}] \\ 0, & \text{if } x \in [1, \infty) \end{cases}$$

Let $\alpha^*, \eta^* : X \times X \times (0, \infty) \to [0, \infty)$ defined by

$$\alpha^*(x, y, t) = \begin{cases} 2, & \text{if } x, y \in [0, \frac{2}{3}] \cup \{1\} \\ 0, & \text{otherwise} \end{cases}$$

and

$$\eta^*(x, y, t) = \begin{cases} 1, & \text{if } x, y \in [0, \frac{2}{3}] \cup \{1\} \\ 2, & \text{otherwise} \end{cases}$$

is a generalized modified $(\alpha^*, \eta^*, \beta)$ contractive mapping.

In Section 3, we prove the existence of fixed points for generalized modified $(\alpha^*, \eta^*, \beta)$-mappings in a complete fuzzy metric spaces. we provide some examples to show the validity of our results. Our result generalized the results of ([17]).

## 3. Main Results

The following propositions are needed to establish the main result

**Proposition 3.1.** *Suppose $(X, M, *)$ is fuzzy metric space. Let $\{x_n\}$ be a sequence in $X$ such that*
*$M(x_n, x_{n+1}, t) \to 1$ as $n \to \infty, \forall t > 0$ . If $\{x_n\}$ is not a Cauchy sequence then there exist $0 < \epsilon < 1$, $t_0 > 0$ and sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ with $m(k) > n(k) \geq k$ for each $k \in \mathbb{N}$ such that $M(x_{m(k)}, x_n(k), t_0) \leq 1 - \epsilon$ and*

*(i)* $\lim_{k\to\infty} M(x_{m(k)}, x_{n(k)}, t_0) = 1 - \epsilon$,

*(iii)* $\lim_{k\to\infty} M(x_{m(k)}, x_{n(k)}, \frac{t_0}{4}) = 1 - \epsilon$,

*(v)* $\lim_{k\to\infty} M(x_{m(k)-1}, x_{n(k)}, \frac{t_0}{2}) = 1 - \epsilon$,

*(ii)* $\lim_{k\to\infty} M(x_{m(k)}, x_{n(k)}, \frac{t_0}{2}) = 1 - \epsilon$,

*(iv)* $\lim_{k\to\infty} M(x_{m(k)-1}, x_{n(k)+1}, \frac{t_0}{2}) = 1 - \epsilon$,

*(vi)* $\lim_{k\to\infty} M(x_{m(k)+1}, x_{n(k)+1}, \frac{t_0}{2}) = 1 - \epsilon$.

*Proof.* Suppose that the sequence $\{x_n\}$ is not a Cauchy sequence. Then there exist $\epsilon \in (0, 1)$ and $t_0 > 0$ such that for all $k \geq 1$, there are positive integers $m(k), n(k) \in \mathbb{N}$ with $m(k) > n(k) \geq k$ and

$$M(x_{n(k)}, x_{m(k)}, t_0) \leq 1 - \epsilon. \tag{3.1}$$

Since $M(x,y,\cdot)$ is a non- decreasing map, we have

$$\begin{aligned}M(x_{n(k)},x_{m(k)},\frac{t_0}{8}) &\leq M(x_{n(k)},x_{m(k)},\frac{t_0}{4})\\ &\leq M(x_{n(k)},x_{m(k)},\frac{t_0}{2})\\ &\leq M(x_{n(k)},x_{m(k)},t_0)\\ &\leq 1-\epsilon.\end{aligned}$$

Now, for any $n(k),m(k)$ satisfying (3.1), we have

$$M(x_{n(k)},x_{m(k)},\frac{t_0}{8}) \leq 1-\epsilon.$$

We assume that $m(k)$ is the least positive integer exceeding $n(k)$ and satisfying the above inequality, that is,

$$M(x_{n(k)},x_{m(k)},\frac{t_0}{8}) \leq 1-\epsilon \text{ and } M(x_{n(k)},x_{m(k)-1},\frac{t_0}{8}) > 1-\epsilon.$$

Thus,

$$\begin{aligned}M(x_{n(k)},x_{m(k)-1},t_0) \geq M(x_{n(k)},x_{m(k)-1},\frac{t_0}{2}) &\geq M(x_{n(k)},x_{m(k)-1},\frac{t_0}{4})\\ &\geq M(x_{n(k)},x_{m(k)-1},\frac{t_0}{8})\\ &> 1-\epsilon. \end{aligned}\tag{3.2}$$

we now prove (i). we have,

$$\begin{aligned}1-\epsilon \geq M(x_{n(k)},x_{m(k)},t_0) &\geq M(x_{n(k)},x_{m(k)-1},\frac{t_0}{2}) * M(x_{m(k)-1},x_{m(k)},\frac{t_0}{2})\\ &\geq (1-\epsilon) * M(x_{m(k)-1},x_{m(k)},\frac{t_0}{2}).\end{aligned}\tag{3.3}$$

Since $\lim_{k\to\infty}((1-\epsilon)*M(x_{m(k)-1},x_{m(k)},\frac{t_0}{2})) = (1-\epsilon)*\lim_{k\to\infty} M(x_{m(k)-1},x_{m(k)},\frac{t_0}{2})) = (1-\epsilon)*1 = 1-\epsilon$, from (3.3) it follows that $\lim_{k\to\infty} M(x_{n(k)},x_{m(k)},t_0)$ exists and $\lim_{k\to\infty} M(x_{n(k)},x_{m(k)},t_0) = 1-\epsilon$. Hence (i) holds.
Now, we have

$$\begin{aligned}1-\epsilon \geq M(x_{n(k)},x_{m(k)},\frac{t_0}{2}) &\geq M(x_{n(k)},x_{m(k)-1},\frac{t_0}{4}) * M(x_{m(k)-1},x_{m(k)},\frac{t_0}{4})\\ &\geq (1-\epsilon) * M(x_{m(k)-1},x_{m(k)},\frac{t_0}{4})\end{aligned}\tag{3.4}$$

Since $\lim_{k\to\infty}(1-\epsilon) = \lim_{k\to\infty}((1-\epsilon)*M(x_{m(k)-1},x_{m(k)},\frac{t_0}{4})) = 1-\epsilon$, from (3.37) we have $\lim_{k\to\infty} M(x_{n(k)},x_{m(k)},\frac{t_0}{2})$ exists and $\lim_{k\to\infty} M(x_{n(k)},x_{m(k)},\frac{t_0}{2}) = 1-\epsilon$.
Again,we have

$$\begin{aligned}1-\epsilon \geq M(x_{n(k)},x_{m(k)},\frac{t_0}{4}) &\geq M(x_{n(k)},x_{m(k)-1},\frac{t_0}{8}) * M(x_{m(k)-1},x_{m(k)},\frac{t_0}{8})\\ &> (1-\epsilon) * M(x_{m(k)-1},x_{m(k)},\frac{t_0}{8}).\end{aligned}\tag{3.5}$$

Since $\lim_{k\to\infty}((1-\epsilon)*M(x_{m(k)-1}, x_{m(k)}, \frac{t_0}{8})) = 1-\epsilon$, from (3.5), we have $\lim_{k\to\infty} M(x_{n(k)}, x_{m(k)}, \frac{t_0}{4})$ exists and $\lim_{k\to\infty} M(x_{n(k)}, x_{m(k)}, \frac{t_0}{4}) = 1-\epsilon$.

Therefore (i)-(iii) follows

We now prove (iv). By condition (M4) in fuzzy metric space we have that

$$M(x_{m(k)-1}, x_{n(k)+1}, \frac{t_0}{2}) \geq M(x_{n(k)+1}, x_{n(k)}, \frac{t_0}{8}) * M(x_{n(k)}, x_{m(k)}, \frac{t_0}{4}) * M(x_{m(k)}, x_{m(k)-1}, \frac{t_0}{8}). \quad (3.6)$$

By taking limit inferior as $k \to \infty$, we get

$$\begin{aligned}\liminf_{k\to\infty} M(x_{m(k)-1}, x_{n(k)+1}, \frac{t_0}{2}) \geq \liminf_{k\to\infty} M(x_{n(k)+1}, x_{n(k)}, \frac{t_0}{8}) \quad &* \quad \liminf_{k\to\infty} M(x_{n(k)}, x_{m(k)}, \frac{t_0}{4}) \\ &* \quad \liminf_{k\to\infty} M(x_{m(k)}, x_{m(k)-1}, \frac{t_0}{8}).\end{aligned}$$

Now, using $M(x_n, x_{n+1}, t) \to 1$ as $n \to \infty, \forall t > 0$ and (iii) we obtain

$$\liminf_{k\to\infty} M(x_{m(k)-1}, x_{n(k)+1}, \frac{t_0}{2}) \geq 1 * (1-\epsilon) * 1 = 1-\epsilon. \quad (3.7)$$

Moreover, from the condition (M4) of fuzzy metric space, we have

$$M(x_{n(k)}, x_{m(k)}, t_0) \geq M(x_{n(k)}, x_{n(k)+1}, \frac{t_0}{4}) * M(x_{n(k)+1}, x_{m(k)-1}, \frac{t_0}{2}) * M(x_{m(k)-1}, x_{m(k)}, \frac{t_0}{4}). \quad (3.8)$$

By taking limit inferior as $k \to \infty$ on both sides of the above inequality , we get

$$1-\epsilon \geq \liminf_{k\to\infty} M(x_{n(k)+1}, x_{m(k)-1}, \frac{t_0}{2}). \quad (3.9)$$

From (3.38) and (3.9) we have

$$\liminf_{k\to\infty} M(x_{n(k)+1}, x_{m(k)-1}, \frac{t_0}{2}) = 1-\epsilon. \quad (3.10)$$

Now we take limit superior in (3.8) as $k \to \infty$

$$1-\epsilon \geq \limsup_{k\to\infty} M(x_{n(k)+1}, x_{m(k)-1}, \frac{t_0}{2}). \quad (3.11)$$

From (3.10) and (3.11) we obtain

$$\liminf_{k\to\infty} M(x_{n(k)+1}, x_{m(k)-1}, \frac{t_0}{2}) = \limsup_{k\to\infty} M(x_{n(k)+1}, x_{m(k)-1}, \frac{t_0}{2}) = 1-\epsilon. \quad (3.12)$$

Thus,

$$\lim_{k\to\infty} M(x_{n(k)+1}, x_{m(k)-1}, \frac{t_0}{2}) \quad \text{exists and} \lim_{k\to\infty} M(x_{n(k)+1}, x_{m(k)-1}, \frac{t_0}{2}) = 1-\epsilon.$$

Thus (iv) holds.

(v) We show that $\lim_{k\to\infty} M(x_{m(k)-1}, x_{n(k)}, \frac{t_0}{2}) = 1-\epsilon$.

By using condition (M4) of fuzzy metric space, we have

$$M(x_{m(k)-1}, x_{n(k)}, \frac{t_0}{2}) \geq M(x_{m(k)-1}, x_{m(k)}, \frac{t_0}{4}) * M(x_{m(k)}, x_{n(k)}, \frac{t_0}{4}). \quad (3.13)$$

By taking limit inferior as $k \to \infty$, in (3.13), we get

$$\liminf_{k\to\infty} M(x_{m(k)-1}, x_{n(k)}, \frac{t_0}{2}) \geq \liminf_{k\to\infty} M(x_{m(k)-1}, x_{m(k)}, \frac{t_0}{4}) * \liminf_{k\to\infty} M(x_{m(k)}, x_{n(k)}, \frac{t_0}{4}). \tag{3.14}$$

Now, using $M(x_n, x_{n+1}, t) \to 1$ as $n \to \infty, \forall t > 0$ and (iii) we obtain

$$\liminf_{k\to\infty} M(x_{m(k)-1}, x_{n(k)}, \frac{t_0}{2}) \geq 1 * (1-\epsilon) * 1 = 1 - \epsilon. \tag{3.15}$$

Moreover, from the condition (M4) of fuzzy metric space, we have

$$M(x_{n(k)}, x_{m(k)}, t_0) \geq M(x_{n(k)}, x_{m(k)-1}, \frac{t_0}{2}) * M(x_{m(k)-1}, x_{m(k)}, \frac{t_0}{2}). \tag{3.16}$$

By taking limit inferior as $k \to \infty$ on both sides of the above inequality , we get

$$1 - \epsilon \geq \liminf_{k\to\infty} M(x_{n(k)}, x_{m(k)-1}, \frac{t_0}{2}). \tag{3.17}$$

From (3.15) and (3.17) we have

$$\liminf_{k\to\infty} M(x_{n(k)}, x_{m(k)-1}, \frac{t_0}{2}) = 1 - \epsilon. \tag{3.18}$$

Now we take limit superior as $k \to \infty$ in (3.16) we have

$$1 - \epsilon \geq \limsup_{k\to\infty} M(x_{n(k)}, x_{m(k)-1}, \frac{t_0}{2}). \tag{3.19}$$

From (3.18) and (3.19) we obtain

$$\liminf_{k\to\infty} M(x_{n(k)}, x_{m(k)-1}, \frac{t_0}{2}) = \limsup_{k\to\infty} M(x_{n(k)}, x_{m(k)-1}, \frac{t_0}{2}) = 1 - \epsilon. \tag{3.20}$$

Thus,

$$\lim_{k\to\infty} M(x_{n(k)}, x_{m(k)-1}, \frac{t_0}{2}) \quad \text{exists and} \lim_{k\to\infty} M(x_{n(k)}, x_{m(k)-1}, \frac{t_0}{2}) = 1 - \epsilon.$$

This proves (v).

vi) We prove $\lim_{k\to\infty} M(x_{m(k)+1}, x_{n(k)+1}, \frac{t_o}{2}) = 1 - \epsilon$.

By using condition (M4) of fuzzy metric space, we have

$$M(x_{m(k)+1}, x_{n(k)+1}, \frac{t_0}{2}) \geq M(x_{m(k)+1}, x_{m(k)}, \frac{t_0}{8}) * M(x_{m(k)}, x_{n(k)}, \frac{t_0}{4}) * M(x_{n(k)}, x_{n(k)+1}, \frac{t_0}{8}) \tag{3.21}$$

By taking limit inferior as $k \to \infty$ in (3.21), we get

$$\liminf_{k\to\infty} M(x_{m(k)+1}, x_{n(k)+1}, \frac{t_0}{2}) \geq \liminf_{k\to\infty} M(x_{m(k)+1}, x_{m(k)}, \frac{t_0}{8}) * \liminf_{k\to\infty} M(x_{m(k)}, x_{n(k)}, \frac{t_0}{8}) * \liminf_{k\to\infty} M(x_{n(k)}, x_{n(k)+1}, \frac{t_0}{8}). \tag{3.22}$$

Now, using $M(x_n, x_{n+1}, t) \to 1$ as $n \to \infty, \forall t > 0$ and (iii) we obtain

$$\liminf_{k\to\infty} M(x_{m(k)+1}, x_{n(k)+1}, \frac{t_0}{2}) \geq 1 * (1-\epsilon) * 1 = 1 - \epsilon. \tag{3.23}$$

Moreover, from the condition (M4) in the definition of fuzzy metric space

$$M(x_{n(k)}, x_{m(k)}, t_0) \geq M(x_{n(k)}, x_{n(k)+1}, \frac{t_0}{4}) * M(x_{n(k)+1}, x_{m(k)+1}, \frac{t_0}{2}) * M(x_{m(k)+1}, x_{m(k)}, \frac{t_0}{4}). \tag{3.24}$$

By taking limit inferior as $k \to \infty$ on both sides of the above inequality , we get

$$1 - \epsilon \geq \liminf_{k\to\infty} M(x_{n(k)+1}, x_{m(k)+1}, \frac{t_0}{2}). \tag{3.25}$$

From (3.23) and (3.25) we have

$$\liminf_{k\to\infty} M(x_{n(k)+1}, x_{m(k)+1}, \frac{t_0}{2}) = 1 - \epsilon. \tag{3.26}$$

Now we take limit superior in (3.24) as $k \to \infty$

$$1 - \epsilon \geq \limsup_{k\to\infty} M(x_{n(k)+1}, x_{m(k)+1}, \frac{t_0}{2}). \tag{3.27}$$

From (3.26) and (3.27) we obtain

$$\liminf_{k\to\infty} M(x_{n(k)+1}, x_{m(k)+1}, \frac{t_0}{2}) = \limsup_{k\to\infty} M(x_{n(k)+1}, x_{m(k)+1}, \frac{t_0}{2}) = 1 - \epsilon. \tag{3.28}$$

Thus,

$$\lim_{k\to\infty} M(x_{n(k)+1}, x_{m(k)+1}, \frac{t_0}{2}) \quad \text{exists and} \lim_{k\to\infty} M(x_{n(k)+1}, x_{m(k)+1}, \frac{t_0}{2}) = 1 - \epsilon.$$

Hence (vi) holds.
This completes the proof of Proposition 3.1. ∎

**Proposition 3.2.** *Let $(X, M, *)$ be a fuzzy metric space. Let $T : X \to X$ be a generalized modified $(\alpha^*, \eta^*, \beta)$- contractive mapping. Fix $x_0 \in X$ and define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for $n = 0, 1, 2, \cdots$. If $\alpha^*(x_n, x_{n+1}, t) \geq \eta^*(x_n, x_{n+1}, t)$ $\forall n$ and $\lim_{n\to\infty} M(x_n, x_{n+1}, t) = 1$ then $\{x_n\}$ is Cauchy sequence in $X$.*

*Proof.* Suppose, on the contrary, that $\{x_n\}$ is not a Cauchy sequence. By Proposition 3.1, there exist $0 < \epsilon < 1$, $t_0 > 0$ and sequences of positive integers $\{m(k)\}, \{n(k)\}$ with $m(k) > n(k) \geq k$ for any $k \in \mathbb{N}$ such that

$$\lim_{k\to\infty} M(x_{n(k)}, x_{m(k)}, t_0) = \lim_{k\to\infty} M(x_{n(k)}, x_{m(k)}, \frac{t_0}{2}) = \lim_{k\to\infty} M(x_{n(k)+1}, x_{m(k)+1}, t_0) = 1-\epsilon \tag{3.29}$$

We have

$$\alpha^*(x_{n(k)}, Tx_{n(k)}, \frac{t_0}{2})\alpha^*(x_{m(k)}, Tx_{m(k)}, \frac{t_0}{2}) \geq \eta^*(x_{n(k)}, Tx_{n(k)}, \frac{t_0}{2})\eta^*(x_{m(k)}, Tx_{m(k)}, \frac{t_0}{2}).$$

Hence, from (1), we have

$$M(Tx_{n(k)}, Tx_{m(k)}, \frac{t_0}{2}) \geq \beta(M(x_{n(k)}, x_{m(k)}, \frac{t_0}{2}))N(x_{n(k)}, x_{m(k)}, \frac{t_0}{2})K(x_{n(k)}, x_{m(k)}, \frac{t_0}{2}).$$

Therefore

$$\frac{M\left(x_{n(k)+1}, x_{m(k)+1}, \frac{t_0}{2}\right)}{N\left(x_{n(k)}, x_{m(k)}, \frac{t_0}{2}\right)K\left(x_{n(k)}, x_{m(k)}, \frac{t_0}{2}\right)} \geq \beta\left(M\left(x_{n(k)}, x_{m(k)}, \frac{t_0}{2}\right)\right) \geq 1. \tag{3.30}$$

where
$K(x_{n(k)}, x_{m(k)}, \frac{t_0}{2}) = \max\{M(x_{n(k)}, x_{m(k)}, \frac{t_0}{2}), M(x_{n(k)}, x_{m(k)+1}, \frac{t_0}{2}), M(x_{m(k)}, x_{n(k)+1}, \frac{t_0}{2}),$
$M(x_{n(k)}, x_{n(k)+1}, \frac{t_0}{2}), M(x_{m(k)}, x_{m(k)+1}, \frac{t_0}{2})\}$ and
$N(x_{n(k)}, x_{m(k)}, \frac{t_0}{2}) = \min\{M(x_{n(k)}, x_{m(k)}, \frac{t_0}{2}), \max\{M(x_{n(k)}, x_{n(k)+1}, \frac{t_0}{2}), M(x_{m(k)}, x_{m(k)+1}, \frac{t_0}{2})\}\}$.
Hence we have

$$\lim_{k\to\infty} K(x_{n(k)}, x_{m(k)}, \frac{t_0}{2}) = 1 \quad \text{and} \quad \lim_{k\to\infty} N(x_{n(k)}, x_{m(k)}, \frac{t_0}{2}) = 1 - \epsilon. \tag{3.31}$$

Thus, on taking limits as $k \to \infty$ in (3.30) and by using (3.29) and (3.31), it follows that

$$\frac{1-\epsilon}{1-\epsilon} \geq \lim_{k\to\infty} \beta(M(x_{n(k)}, x_{m(k)}, \frac{t_0}{2})) \geq 1$$

which implies that,

$$\lim_{k\to\infty} \beta(M(x_{n(k)}, x_{m(k)}, \frac{t_0}{2})) = 1.$$

Hence, from the property of $\beta$, we have

$$\lim_{k\to\infty} M(x_{n(k)}, x_{m(k)}, \frac{t_0}{2}) = 1.$$

Thus, $1 - \epsilon = \lim_{k\to\infty} M(x_{n(k)}, x_{m(k)}, \frac{t_0}{2}) = 1.$ so that $\epsilon = 0$, which is a contradiction.
Therefore $\{x_n\}$ is a Cauchy Sequence.
∎

**Theorem 3.3.** *Let $(X, M, *)$ be a complete fuzzy metric space. Let $T : X \to X$ be a generalized modified $(\alpha^*, \eta^*, \beta)$- contractive mapping. Suppose that the following conditions hold:*

**(a):** *$T$ is $(\alpha^*, \eta^*)$ admissible mapping;*
**(b):** *there exists $x_0 \in X$ such that $\alpha^*(x_0, Tx_0, t) \geq \eta^*(x_0, Tx_0, t)$ for all $t > 0$;*
**(c):** *for any sequence $\{x_n\} \subset X$ such that $\alpha^*(x_n, x_{n+1}, t) \geq \eta * (x_n, x_{n+1}, t)$ for all $n \in N, t > 0$ and $x_n \to x$ as $n \to \infty$, then $\alpha^*(x, Tx, t) \geq \eta * (x, Tx, t)$ for all $t > 0$.*

*Then $T$ has a fixed point.*

*Proof.* Let $x_0 \in X$ be such that $\alpha^*(x_0, Tx_0, t) \geq \eta^*(x_0, Tx_0, t)$ for all $t > 0$. Define a sequence $\{x_n\}$ in $X$ such that $x_n = T^n x_0 = Tx_{n-1}$ for all $n \in N$. If $x_n = x_{n+1}$ for some $n \in N$, then $x_n = Tx_n$ and hence $x_n$ is a fixed point of $T$ and we are done.
Assume that $x_n \neq x_{n+1}$ for all $n \in N$. Since $T$ is $(\alpha^*, \eta^*)$ admissible mapping and since $\alpha^*(x_0, Tx_0, t) \geq \eta^*(x_0, Tx_0, t)$ it follows that

$$\alpha^*(x_1, x_2, t) = \alpha^*(Tx_0, Tx_1, t) \geq \eta^*(Tx_0, Tx_1, t) = \eta^*(x_1, x_2, t)$$

so that

$$\alpha^*(x_0, Tx_0, t)\alpha^*(x_1, Tx_1, t) \geq \eta^*(x_0, Tx_0, t)\eta^*(x_1, Tx_1, t).$$

On continuing this process, we have $\alpha^*(x_n, Tx_n, t) \geq \eta^*(x_n, Tx_n, t)$, for all $n \geq 1$ and so we have

$\alpha^*(x_{n-1}, Tx_{n-1}, t)\alpha^*(x_n, Tx_n, t) \geq \eta^*(x_{n-1}, Tx_{n-1}, t)\eta^*(x_n, Tx_n, t)$ for all $n \in N$ and $t > 0$. Now, from the inequality in (1), we have

$$\begin{aligned} M(x_n, x_{n+1}, t) &= M(Tx_{n-1}, Tx_n, t) \\ &\geq \beta(M(x_{n-1}, x_n, t))N(x_{n-1}, x_n, t)K(x_{n-1}, x_n, t) \end{aligned} \tag{3.32}$$

where

$$\begin{aligned} N(x_{n-1}, x_n, t) &= \min\{M(x_{n-1}, x_n, t), \max\{M(x_{n-1}, Tx_{n-1}, t), M(x_n, Tx_n, t)\}\} \\ &= \min\{M(x_{n-1}, x_n, t), \max\{M(x_{n-1}, x_n, t), M(x_n, x_{n+1}, t)\}\}. \end{aligned} \tag{3.33}$$

and

$$\begin{aligned} K(x_{n-1}, x_n, t) &= \max\{M(x_{n-1}, x_n, t), M(x_{n-1}, Tx_{n-1}, t), M(x_n, Tx_n, t)M(x_{n-1}, Tx_n, t), \\ &\quad M(x_n, Tx_{n-1}, t)\}. \end{aligned}$$

Since $M(x_n, Tx_{n-1}, t) = 1$ for all $n \in \mathtt{N}$ and $t > 0$ we have that $K(x_{n-1}, x_n, t) = 1$ for all $n$.
Moreover, since $\min\{a, \max\{a, b\}\} = a$, we have

$$\begin{aligned} N(x_{n-1}, x_n, t) &= \min\{M(x_{n-1}, x_n, t), \max\{M(x_{n-1}, Tx_{n-1}, t), M(x_n, Tx_n, t)\}\} \\ &= M(x_{n-1}, x_n, t). \end{aligned}$$

Hence

$$M(x_n, x_{n+1}, t) \geq \beta(M(x_{n-1}, x_n, t))M(x_{n-1}, x_n, t) \quad \text{for all } n \in N \text{ and } t > 0. \tag{3.34}$$

It follows that the sequence$\{M(x_n, x_{n+1}, t)\}$ is an increasing sequence in $(0, 1]$. Thus, there exists $l_t \in (0, 1]$ such that

$$\lim_{n\to\infty} M(x_n, x_{n+1}, t) = l_t \quad \text{for each} \quad t > 0.$$

We now prove that $l_t = 1$ for each $t > 0$. Let $t > 0$ from (3.34), we have $\frac{M(x_n,x_{n+1},t)}{M(x_{n-1},x_n,t)} \geq \beta M(x_{n-1}, x_n, t) \geq 1$, which implies that $\lim_{n\to\infty} \beta(M(x_{n-1}, x_n, t)) = 1$ . Hence by the property of the function $\beta$ we have $\lim_{n\to\infty} M(x_{n-1}, x_n, t) = 1$,, that is $l_t = 1$.
Thus by Proposition 3.2 we have $\{x_n\}$ is a Cauchy sequence. Since $(X, M, *)$ is complete, there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$ for each $t > 0$. By condition (c) we have $\alpha^*(x^*, Tx^*, t) \geq \eta^*(x^*, Tx^*, t)$. Hence we get that $\alpha^*(x_n, Tx_n, t)\alpha^*(x^*, Tx^*, t) \geq \eta^*(x_n, Tx_n, t)\eta^*(x^*, Tx^*, t)$ for all $n \in \mathtt{N} \cup \{0\}$ and $t > 0$. Now, by applying the inequality (1), we have

$$M(Tx^*, Tx_n, t) = M(Tx^*, x_{n+1}, t) \geq \beta(M(x^*, x_n, t))N(x^*, x_n, t)K(x^*, x_n, t) \tag{3.35}$$

where

$$N(x^*, x_n, t) = \min\{M(x^*, x_n, t), \max\{M(x^*, Tx^*, t), M(x_n, Tx_n, t)\}\}$$

and

$$K(x^*, x_n, t) = \max\{M(x^*, x_n, t), M(x^*, x_{n+1}, t), M(x_{n+1}, Tx^*, t)M(x^*, Tx^*, t), M(x_n, Tx_n, t)\}$$

 

Hence it follows that

$$\lim_{n\to\infty} N(x^*, x_n, t) = 1 \text{ and } \lim_{n\to\infty} K(x^*, x_n, t) = 1. \tag{3.36}$$

On taking limits as $n \to \infty$ in (3.35), we get

$$\lim_{n\to\infty} M(Tx^*, x_{n+1}, t) \geq \lim_{n\to\infty} \beta(M(x^*, x_n, t)) \geq 1.$$

This implies that $\lim_{n\to\infty} M(Tx^*, x_{n+1}, t) = 1$, which shows the sequence $\{x_n\}$ converges to $Tx^*$,but the sequence $\{x_n\}$ converges to $x^*$. Since the limit of a convergent sequence in a fuzzy metric space is unique, we have that $Tx^* = x^*$. ∎

In order to prove the uniqueness of fixed points of Theorem 3.3, we use the following 'Condition(H)': $\alpha^*(x, y, t) = \eta^*(x, y, t)$ if and only if $x = y$. The following are examples of $\alpha^*$ and $\eta^*$ satisfying 'Condition(H)'

**Example 3.4.** Define $\alpha^*$ and $\eta^*$ on $[0, \infty) \times [0, \infty) \times (0, \infty) \to [0, \infty)$ by

$$\alpha^*(x, y, t) = \begin{cases} 5x + t & \text{if } x = y \\ 0 & \text{if } x \neq y \end{cases}$$

and

$$\eta^*(x, y, t) = \begin{cases} 5y + t & \text{if } x = y \\ 1 & \text{if } x \neq y. \end{cases}$$

Then $\alpha^*, \eta^*$ satisfying Condition(H).

**Example 3.5.** Let $X = [0, \infty)$. we define $\alpha^*, \eta^* : X \times X \times (0, \infty) \to [0, \infty)$ by

$$\alpha^*(x, y, t) = \begin{cases} (xy + t)^2 & \text{if } x = y \\ tx & \text{if } x \neq y \end{cases}$$

and

$$\eta^*(x, y, t) = \begin{cases} (xy + t)^2 & \text{if } x = y \\ ty & \text{if } x \neq y. \end{cases}$$

Then $\alpha^*, \eta^*$ satisfying Condition(H).

**Theorem 3.6.** *: In addition to the hypotheses of Theorem 3.3, we assume that 'Condition(H)' holds. Then $T$ has a unique fixed point in $X$.*

*Proof.* Suppose $x$ and $y$ are fixed points of $T$. Thus, $Tx = x$ and $Ty = y$ which implies that $\alpha^*(x, Tx, t) = \alpha^*(x, x, t)$ and $\alpha^*(y, Ty, t) = \alpha^*(y, y, t)$. By condition (H), we have $\alpha^*(x, x, t) = \eta^*(x, x, t)$ and $\alpha^*(y, y, t) = \eta^*(y, y, t)$ for all $t > 0$. This implies that

$$\alpha^*(x, Tx, t)\alpha^*(y, Ty, t) \geq \eta^*(x, Tx, t)\eta^*(y, Ty, t).$$

Since $T$ is a generalized modified $(\alpha^*, \eta^*, \beta)$ contractive mapping, we have that

$$M(x, y, t) = M(Tx, Ty, t) \geq \beta(M(x, y, t))N(x, y, t)K(x, y, t)$$

where $N(x, y, t) = \min\{M(x, y, t), \max\{M(x, Tx, t), M(y, Ty, t)\}\} = M(x, y, t)$ and

$$K(x, y, t) = \max\{M(x, y, t), M(x, Tx, t), M(y, Ty, t), M(x, Ty, t), M(y, Tx, t)\} = 1.$$

This implies that

$$M(x,y,t) \quad \geq \quad \beta(M(x,y,t))M(x,y,t) \geq M(x,y,t)$$

Thus $\beta(M(x,y,t)) = 1$ for all $t > 0$. By Remark 2.17 we get that $M(x,y,t) = 1$ for all $t > 0$. This implies that $x = y$ and the conclusion of the Theorem follows. ∎

**Theorem 3.7.** *Let $T$ be a self map satisfy all the hypotheses of Theorem 3.3. Then for any $x \in X$ with $\alpha^*(x,Tx,t) \geq \eta^*(x,Tx,t)$, the sequence of iterates $\{T^n x\}$ converges to $z$(say) in $X$. Further, $z$ the unique fixed of $T$.*

*Proof.* Let $x, y$ be points in $X$ with $\alpha^*(x,Tx,t) \geq \eta^*(x,Tx,t)$
and

$$\alpha^*(y,Ty,t) \geq \eta^*(y,Ty,t).$$

From the proof of Theorem 3.3, the sequences $\{T^n x\}$ and $\{T^n y\}$ converge to the fixed points of $T$.
Suppose $T^n x \to u$ and $T^n y \to v$. we show that $u = v$. From the hypotheses of Theorem 3.3 we have $\alpha^*(u,Tu,t) \geq \eta^*(u,Tu,t)$ and $\alpha^*(v,Tv,t) \geq \eta^*(v,Tv,t)$. This implies

$$\alpha^*(u,Tu,t)\alpha^*(v,Tv,t) \geq \eta^*(u,Tu,t)\eta^*(v,Tv,t) \text{for all } t > 0. \tag{3.37}$$

Thus

$$M(u,v,t) = M(Tu,Tv,t) \geq \beta(M(u,v,t))N(u,v,t)K(u,v,t).$$

Here we observe that $N(u,v,t) = M(u,v,t)$ and $K(u,v,t) = 1$. By (3.37) we get

$$M(u,v,t) \geq \beta(M(u,v,t))(M(u,v,t)) \geq M(u,v,t).$$

Hence $\beta(M(u,v,t)) = 1$. By Remark 2.17, we have $M(u,v,t) = 1, \quad$ for all $t > 0$. Hence $u = v$. ∎

By taking $\eta^*(x,y,t) = 1$ in Theorem 3.3, we have the following result.

**Corollary 3.8.** *Let $(X, M, *)$ be a complete fuzzy metric space. A mapping $T : X \to X$ be $\alpha^*$-admissible map. Assume that there exists a function $\beta \in S$ such that*

$$\alpha^*(x,Tx,t)\alpha^*(y,Ty,t) \geq 1 \Rightarrow M(Tx,Ty,t) \geq \beta(M(x,y,t))N(x,y,t)K(x,y,t)$$

*where*

$$N(x,y,t) = \min\{M(x,y,t), \max\{M(x,Tx,t), M(y,Ty,t)\}\}$$

*and*

$$K(x,y,t) = \max\{M(x,y,t), M(x,Tx,t), M(y,Ty,t), M(x,Ty,t), M(y,Tx,t)\}$$

*for all $x, y \in X$ and $t > 0$. Suppose that the following conditions hold*

**(a):** *there exists $x_0 \in X$ such that $\alpha^*(x_0,Tx_0,t) \geq 1$ for all $t > 0$.*
**(b):** *For any sequence $\{x_n\} \subset X$ such that $\alpha^*(x_n,x_{n+1},t) \geq 1$, for all $n \in \mathtt{N}, t > 0$ and $x_n \to x$ as $n \to \infty$, then $\alpha^*(x,Tx,t) \geq 1$ for all $t > 0$.*

*Then $T$ has a fixed point.*

**Corollary 3.9.** *Let $(X, M, *)$ be a complete fuzzy metric space. A mapping $T : X \to X$ be $\alpha^*$-admissible map. Assume that there exists a function $\beta \in S$ such that*

$$\frac{1}{\alpha^*(x, Tx, t)\alpha^*(y, Ty, t)} M(Tx, Ty, t) \geq \beta(M(x, y, t))N(x, y, t)K(x, y, t) \tag{3.38}$$

*where*

$$N(x, y, t) = \min\{M(x, y, t), \max\{M(x, Tx, t), M(y, Ty, t)\}\}.$$

*and*

$$K(x, y, t) = \max\{M(x, y, t), M(x, Tx, t), M(y, Ty, t), M(x, Ty, t), M(y, Tx, t)\}$$

*for all $x, y \in X$ and $t > 0$. Suppose that the following conditions hold.*

**(a):** *there exists $x_0 \in X$ such that $\alpha^*(x_0, Tx_0, t) \geq 1$ for all $t > 0$.*

**(b):** *For any sequence $\{x_n\} \subset X$ such that $\alpha^*(x_n, x_{n+1}, t) \geq 1$, for all $n \in \mathtt{N}, t > 0$ and $x_n \to x$ as $n \to \infty$, then $\alpha^*(x, Tx, t) \geq 1$ for all $t > 0$.*

*Then $T$ has a fixed point.*

*Proof.* let $\alpha^*(x, Tx, t)\alpha^*(y, Ty, t) \geq 1$. Thus $\frac{1}{\alpha^*(x,Tx,t)\alpha^*(y,Ty,t)} \leq 1$ and this implies that $\frac{1}{\alpha^*(x,Tx,t)\alpha^*(y,Ty,t)} M(Tx, Ty, t) \leq M(Tx, Ty, t)$. From (3.38) it follows that $M(Tx, Ty, t) \geq \beta(M(x, y, t))N(x, y, t)K(x, y, t)$. Hence by Corollary 3.8 $T$ has a fixed point. ∎

By taking $\alpha^*(x, y, t) = 1$ in Theorem 3.3, we have the following result.

**Corollary 3.10.** *Let $(X, M, *)$ be a complete fuzzy metric space. A mapping $T : X \to X$ be $\eta^*$-sub admissible map. Assume that there exists a function $\beta \in S$ such that*

$$\eta^*(x, Tx, t)\eta^*(y, Ty, t) \leq 1 \Rightarrow M(Tx, Ty, t) \geq \beta(M(x, y, t))N(x, y, t)K(x, y, t)$$

*where*

$$N(x, y, t) = \min\{M(x, y, t), \max\{M(x, Tx, t), M(y, Ty, t)\}\}$$

*and*

$$K(x, y, t) = \max\{M(x, y, t), M(x, Tx, t), M(y, Ty, t), M(x, Ty, t), M(y, Tx, t)\}.$$

*for all $x, y \in X$ and $t > 0$. Suppose that the following conditions hold*

**(a):** *there exists $x_0 \in X$ such that $\eta^*(x_0, Tx_0, t) \leq 1$ for all $t > 0$.*

**(b):** *For any sequence $\{x_n\} \subset X$ such that $\eta^*(x_n, x_{n+1}, t) \leq 1$, for all $n \in \mathtt{N}, t > 0$ and $x_n \to x$ as $n \to \infty$, then $\alpha^*(x, Tx, t) \leq 1$ for all $t > 0$.*

*Then $T$ has a fixed point.*

If $T : X \to X$ is $\eta^*$-sub admissible, then we have the following corollary.

**Corollary 3.11.** *Let $(X, M, *)$ be a complete fuzzy metric space. A mapping $T : X \to X$ be $\eta^*$-sub admissible map. Assume that there exists a function $\beta \in S$ such that*

$$M(Tx, Ty, t) \geq \frac{1}{\eta^*(x, Tx, t)\eta^*(y, Ty, t)} \beta(M(x, y, t))N(x, y, t)K(x, y, t)$$

*where*

$$N(x, y, t) = \min\{M(x, y, t), \max\{M(x, Tx, t), M(y, Ty, t)\}\}$$

*and*

$$K(x, y, t) = \max\{M(x, y, t), M(x, Tx, t), M(y, Ty, t), M(x, Ty, t), M(y, Tx, t)\}.$$

*for all $x, y \in X$ and $t > 0$. Suppose that the following conditions hold*

**(a):** *there exists $x_0 \in X$ such that $\eta^*(x_0, Tx_0, t) \leq 1$ for all $t > 0$.*

**(b):** *For any sequence $\{x_n\} \subset X$ such that $\eta^*(x_n, x_{n+1}, t) \leq 1$, for all $n \in \mathtt{N}, t > 0$ and $x_n \to x$ as $n \to \infty$, then $\alpha^*(x, Tx, t) \leq 1$ for all $t > 0$.*

*Then $T$ has a fixed point.*

If we take $\alpha^*(x, y, t) = 1$ in Corollary3.9 or $\eta^*(x, y, t) = 1$ in Corollary 3.11, we have the following result

**Corollary 3.12.** *Let $(X, M, *)$ be a complete fuzzy metric space. $T$ be a mapping $T : X \to X$ . Assume that there exists a function $\beta \in S$ such that*

$$M(Tx, Ty, t) \geq \beta(M(x, y, t))N(x, y, t)K(x, y, t)$$

*where*

$$N(x, y, t) = \min\{M(x, y, t), \max\{M(x, Tx, t), M(y, Ty, t)\}\}$$

*and*

$$K(x, y, t) = \max\{M(x, y, t), M(x, Tx, t), M(y, Ty, t), M(x, Ty, t), M(y, Tx, t)\}.$$

*for all $x, y \in X$ and $t > 0$. Then $T$ has a fixed point.*

**Remark 3.13.** Theorem 2.18 follows as a corollary to Theorem 3.3 by choosing $K(x, y, t) = 1$ for all $x, y \in X$ and $t > 0$.

Now, we give an example in support Theorem 3.3.

**Example 3.14.** Let $X = [0, \frac{2}{3}] \cup [1, \infty)$ and $M(x, y, t) = (\frac{t}{t+1})^{d(x,y)}$, where $d(x, y) = |x - y|$, $x, y \in X$, $*$ is product continuous t-norm. Here $(X, M, *)$ is complete fuzzy metric space.
Let $T : X \to X$ be a map defined by

$$Tx = \begin{cases} \frac{x}{2} & \text{if } x \in [0, \frac{2}{3}] \\ 0 & \text{if } x \in [1, \infty) \end{cases}$$

Let $\alpha^*, \eta^* : X \times X \times (0, \infty) \to [0, \infty)$ defined by

$$\alpha^*(x, y, t) = \begin{cases} 2 & \text{if } x, y \in [0, \frac{2}{3}] \cup \{1\} \\ 0 & \text{otherwise.} \end{cases}$$

and

$$\eta^*(x, y, t) = \begin{cases} 1 & \text{if } x, y \in [0, \frac{2}{3}] \cup \{1\} \\ 2 & \text{otherwise.} \end{cases}$$

We have $\alpha^*(x, y, t) \geq \eta^*(x, y, t)$ if and only if $x, y \in [0, \frac{2}{3}] \cup \{1\}$. On the other hand , for all $x, y \in [0, \frac{2}{3}] \cup \{1\}$, we have $Tx \leq 1$ and $Ty \leq 1$. This implies that

$$\alpha^*(Tx, Ty, t) \geq \eta^*(Tx, Ty, t).$$

Hence $T$ is $\alpha^*, \eta^*$ admissible mapping.
Moreover, $\alpha^*(\frac{1}{3}, T\frac{1}{3}, t) \geq \eta^*(\frac{1}{3}, T\frac{1}{3}, t)$.
Let $\{x_n\}$ be a sequence in $X$ such that $\alpha^*(x_n, x_{n+1}, t) \geq \eta^*(x_n, x_{n+1}, t)$ for all $n \in \mathtt{N} \cup \{0\}$ and $x_n \to x$ as $n \to \infty$, then $\{x_n\} \subset [0, 1]$, and hence $x \in [0, \frac{2}{3}] \cup \{1\}$ . This implies that $\alpha^*(x, Tx, t) \geq \eta^*(x, Tx, t)$ for all $n \in \mathtt{N}$ and $t > 0$.

Suppose $\alpha^*(x,Tx,t)\alpha^*(y,Ty,t) \geq \eta^*(x,Tx,t)\eta^*(y,Ty,t)$. Then $x,y \in [0,\frac{2}{3}] \cup \{1\}$.

*Case i*: If $x,y \in [0,\frac{2}{3}]$ and $\beta(t) = t^{-\frac{1}{2}}$ then $M(Tx,Ty,t) = \left(\frac{t}{t+1}\right)^{d(Tx,Ty)} = \left(\frac{t}{t+1}\right)^{|Tx-Ty|} = \left(\frac{t}{t+1}\right)^{|\frac{x}{2}-\frac{y}{2}|} = \left(\frac{t}{t+1}\right)^{-|\frac{x}{2}-\frac{y}{2}|}\left(\frac{t}{t+1}\right)^{|x-y|} = \beta(\left(\frac{t}{t+1}\right)^{|x-y|})\left(\frac{t}{t+1}\right)^{d(x,y)} = \beta(M(x,y,t))M(x,y,t) \geq \beta(M(x,y,t))N(x,y,t) \geq \beta(M(x,y,t)N(x,y,t)K(x,y,t)$.

*Case ii*: If $x \in [0,\frac{2}{3}]$ and $y = 1$

Then,

$$M(Tx,Ty,t) = \left(\frac{t}{t+1}\right)^{|\frac{x}{2}-0|} = \left(\frac{t}{t+1}\right)^{\frac{x}{2}}, M(x,y,t) = \left(\frac{t}{t+1}\right)^{1-x},$$

$M(x,Tx,t) = \left(\frac{t}{t+1}\right)^{\frac{x}{2}}, M(y,Ty,t) = \left(\frac{t}{t+1}\right), M(x,Ty,t) = \left(\frac{t}{t+1}\right)^{x}, M(y,Tx,t) = \left(\frac{t}{t+1}\right)^{(1-\frac{x}{2})}$.
Here we have

$$\begin{aligned} N(x,y,t) &= \min\{M(x,y,t), \max\{M(x,Tx,t), M(y,Ty,t)\}\} \\ &= \min\left\{\left(\frac{t}{t+1}\right)^{1-x}, \max\left\{\left(\frac{t}{t+1}\right)^{\frac{x}{2}}, \left(\frac{t}{t+1}\right)\right\}\right\} \\ &= \left(\frac{t}{t+1}\right)^{1-x}. \end{aligned} \tag{3.39}$$

On the other hand

$$\begin{aligned} K(x,y,t) &= \max\{M(x,y,t), M(x,Tx,t), M(y,Ty,t), M(x,Ty,t), M(y,Tx,t)\} \\ &= \max\left\{\left(\frac{t}{t+1}\right)^{1-x}, \left(\frac{t}{t+1}\right)^{x}2, \left(\frac{t}{t+1}\right), \left(\frac{t}{t+1}\right)^{x}, \left(\frac{t}{t+1}\right)^{(1-\frac{x}{2})}\right\} \\ &= \left(\frac{t}{t+1}\right)^{\frac{x}{2}}. \end{aligned} \tag{3.40}$$

Now, $M(Tx,Ty,t) = \left(\frac{t}{t+1}\right)^{\frac{x}{2}} \geq \left(\frac{t}{t+1}\right)^{-\frac{1}{2}+x}\left(\frac{t}{t+1}\right)^{1-x}\left(\frac{t}{t+1}\right)^{\frac{x}{2}}$.
This implies that

$$M(Tx,Ty,t) \geq \beta(M(x,y,t))N(x,y,t)K(x,y,t).$$

It is easy to show that $M(Tx,Ty,t) \geq \beta(M(x,y,t))N(x,y,t)K(x,y,t)$ for $x = 1, y = 1$.
Therefore $T$ satisfies all conditions of Theorem 3.3 with $\beta(t) = t^{-\frac{1}{2}}$ and $\beta \in S$. 0 is the fixed point of $T$.
we now show that contractive condition in Theorem 2.18 fails to hold . For, we choose, $x = \frac{2}{3}$ and $y = 1$ and $t > 0$, we obtain

$$M(T\frac{2}{3},T1,t) = \left(\frac{t}{t+1}\right)^{\frac{1}{3}}, M(\frac{2}{3},1,t) = \left(\frac{t}{t+1}\right)^{\frac{1}{3}}, M(\frac{2}{3},T\frac{2}{3},t) = \left(\frac{t}{t+1}\right)^{\frac{1}{3}}, M(1,T1,t) = \left(\frac{t}{t+1}\right)$$

and

$$\begin{aligned} N(x,y,t) &= \min\{M(x,y,t), \max\{M(x,Tx,t), M(y,Ty,t)\}\} \\ &= \min\left\{\left(\frac{t}{t+1}\right)^{\frac{1}{3}}, \max\left\{\left(\frac{t}{t+1}\right)^{\frac{1}{3}}, \left(\frac{t}{t+1}\right)\right\}\right\} \\ &= \left(\frac{t}{t+1}\right)^{\frac{1}{3}}. \end{aligned} \tag{3.41}$$

If there exist $\beta \in S$ such that $\left(\frac{t}{t+1}\right)^{\frac{1}{3}} = M(T\frac{2}{3}, T1, t) \geq \beta(M(\frac{2}{3}, 1, t))N(\frac{2}{3}, 1, t)$, then $\beta(M(\frac{2}{3}, 1, t)) = 1$. By Remark 2.17 we have $M(\frac{2}{3}, 1, t) = 1$, a contradiction. Hence Remark 3.13 suggests that Theorem 3.3 is a generalization of Theorem 2.18.

**Conflict of Interests**

*The authors declare that there is no conflict of interests regarding the publication of this paper.*

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