Bangmod Int. **J.** Math. **&** **C**omp. **S**ci.
Vol. 3, No. 1-2, 2017; Pages 10 - 15
http://bangmod-jmcs.kmutt.ac.th/
ISSN : 2408-154X

A NEW METHOD FOR SOME FIXED POINT THEOREMS IN FUZZY METRIC SPACES

M. Sangurlu Sezen[1], Hüseyin Işık[2,*] and Duran Turkoglu[3]
[1] *Department of Mathematics, Faculty of Science and Arts, University of Giresun, Gazipaşa, Giresun, TURKEY*
E-mail: msangurlu@gazi.edu.tr
[2] *Department of Mathematics, Faculty of Science and Arts, Muş Alparslan University, Muş 49100, TURKEY*
E-mail: isikhuseyin76@gmail.com
[3] *Department of Mathematics, Faculty of Science, University of Gazi, 06500, Ankara, TURKEY*
E-mail: dturkoglu@gazi.edu.tr

**Corresponding author.*

Abstract We use a new approach for the solution to the existence of a common fixed point for a pair two mappings in fuzzy metric spaces. Our results are different to the usual methods in the literature.

MSC: 47H10, 54H25

Keywords: Fixed point, common fixed point, partial order, fuzzy metric space.

Submission date: 31 March 2017 / Acceptance date: 16 July 2017 /Available online: 1 August 2017

1. Introduction and preliminaries

Fixed point theory is an important issue that is used in many areas. So many authors have worked on this topic. Altun and et al. [2] investigated a new approach for the approximations of solutions to a common fixed point problem in metric spaces. They were interested on the following problem:

$$\text{Find } x \in X \text{ such that} \quad \begin{aligned} x &= Tx, \\ x &= Sx. \end{aligned} \tag{1.1}$$

They provided sufficient conditions for the existence of one and only one solution to the problem (1.1). Also they presented a numerical algorithm in order to approximate such solution. Before the solvability of the problem (1.1) are based on a compatibility condition. Some authors have give relevant examples in different areas [1, 3, 4, 6, 7, 9]. However, there are some major difficulties arise. So, Altun and et al. get a new useful solution to the problem (1.1) without the compatibility condition in [2].

Published by **T**heoretical **and** **C**omputational **S**cience Center **(TaCS)**,
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In this paper, we use this new approach for the solution to problem (1.1) in fuzzy metric spaces. We proved new some common fixed point theorems in fuzzy metric spaces with this new approach.

Definition 1.1 ([10]). A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is called a continuous triangular norm (in short, continuous $t-$norm) if it satisfies the following conditions:

(TN-1) $*$ is commutative and associative;
(TN-2) $*$ is continuous;
(TN-3) $*(a,1) = a$ for every $a \in [0,1]$;
(TN-4) $*(a,b) \leq *(c,d)$ whenever $a \leq c$, $b \leq d$ and $a,b,c,d \in [0,1]$.

An arbitrary $t-$norm $*$ can be extended (by associativity) in a unique way to an nary operator taking for $(x_1, x_2, ..., x_n) \in [0,1]^n$, $n \in \mathbb{N}$, the value $*(x_1, x_2, ..., x_n)$ is defined, in [6], by

$$*_{I=1}^{0} x_i = 1, \;\; *_{I=1}^{n} x_i = *(*_{I=1}^{n-1} x_i, x_n) = *(x_1, x_2, ..., x_n).$$

Definition 1.2 ([5]). A fuzzy metric space is an ordered triple $(X, M, *)$ such that X is a nonempty set, $*$ is a cotinuous t-norm and M is a fuzzy set on $X^2 \times (0,\infty)$, satisfying the following conditions, for all $x,y,z \in X$, $s,t > 0:$

(FM-1) $M(x,y,t) > 0$;
(FM-2) $M(x,y,t) = 1$ iff $x = y$;
(FM-3) $M(x,y,t) = M(y,x,t)$;
(FM-4) $*(M(x,y,t), M(y,z,s)) \leq M(x,z,t+s)$;
(FM-5) $M(x,y,\cdot) : (0,\infty) \to (0,1]$ is continuous.

Definition 1.3 ([8]). Let $(X, M, *)$ be a fuzzy metric space. Then

(i) A sequence $\{x_n\}$ in X is said to converge to x in X, denoted by $x_n \to x$, if and only if $\lim_{n\to\infty} M(x_n, x, t) = 1$ for all $t > 0$, i.e. for each $r \in (0,1)$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n, x, t) > 1 - r$ for all $n \geq n_0$.
(ii) A sequence $\{x_n\}$ is a G-Cauchy sequence if and only if $\lim_{n\to\infty} M(x_n, x_{n+p}, t) = 1$ for any $p > 0$ and $t > 0$.
(iii) The fuzzy metric space $(X, M, *)$ is called G-complete if every G-Cauchy sequence is convergent.

Definition 1.4 ([11]). Let $(X, M, *)$ be a fuzzy metric space and let $\{f_n\}$ be a sequence of self mapping on X. $f_0 : X \to X$ is a given mapping. The sequence $\{f_n\}$ is said to converge uniformly to f_0 if for each $\epsilon \in (0,1)$ and $t > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$M(f_n(x), f_0(x), t) > 1 - \epsilon,$$

for all $n \geq n_0$ and $x \in X$.

In this paper, problem (1.1) is investigated under the following assumptions.

(A1) We suppose that X equipped with a partial order $\preceq$.
(A2) The operator $S : X \to X$ is level closed from the left; that is, the set

$$levS_{\preceq} = \{x \in X : x \preceq Sx\}, \tag{1.2}$$

is nonempty and closed.

(A3) For every $x \in X$, we have

$$\begin{aligned} x \preceq Sx &\Longrightarrow Tx \succeq STx, \\ x \succeq Sx &\Longrightarrow Tx \preceq STx. \end{aligned} \tag{1.3}$$

In order to fix our next assumption, we need to give the following class of mappings.

Definition 1.5 ([12]). Let Ψ be the class of all mappings $\psi : [0, 1] \longrightarrow [0, 1]$ such that

(i) ψ is continuous and nondecreasing,
(ii) $\psi(t) > t$ for all $t \in (0, 1)$.

Lemma 1.6 ([12]). *If $\psi \in \Psi$, then $\psi(1) = 1$.*

Lemma 1.7 ([12]). *If $\psi \in \Psi$, then $\lim_{n\to+\infty} \psi^n(t) = 1$ for all $t \in (0, 1)$.*

(A4) There exists a function $\psi \in \Psi$ such that for every $(x, y) \in X \times X$, we have

$$\left.\begin{array}{l} x \preceq Sx \\ y \succeq Sy \end{array}\right\} \Longrightarrow M(Tx, Ty, t) \geq \psi(M(x, y, t)). \tag{1.4}$$

2. Main Results

Theorem 2.1. *Let $(X, M, *)$ be a complete fuzzy metric space. Suppose that the conditions (A1)-(A4) are satisfied. Then $x^* \in X$ is a unique solution to (1.1) such that for any $x_0 \in levS_{\preceq}$, the Picard sequence $\{T^n x_0\}$ converges to x^*.*

Proof. Let x_0 be an arbitrary element of $levS_{\preceq}$; that is,

$$\begin{aligned} &x_0 \in X \\ &x_0 \preceq Sx_0. \end{aligned}$$

Such an element exists from (A2). From (A3), we have

$$x_1 \succeq Sx_1,$$

where $x_1 = Tx_0$. From (A3), we have

$$x_2 \preceq Sx_2,$$

where $x_2 = Tx_1$. Now, let us consider the Picard sequence $\{x_n\} \subset X$ defined by

$$x_{n+1} = Tx_n, \quad n = 0, 1, 2, ...$$

By induction we get

$$\begin{aligned} &x_{2n} \preceq Sx_{2n}, \\ &x_{2n+1} \succeq Sx_{2n+1}, \quad n = 0, 1, 2, ... \end{aligned} \tag{2.1}$$

Therefore, by (A4), we have

$$M(Tx_{2n}, Tx_{2n+1}, t) \geq \psi(M(x_{2n}, x_{2n+1}, t)), \quad n = 0, 1, 2, ...$$

By (A4), we have

$$M(Tx_{2n+1}, Tx_{2n+2}, t) \geq \psi(M(x_{2n+1}, x_{2n+2}, t)), \quad n = 0, 1, 2, ...$$

As a consequence, we have

$$M(x_{n+1}, x_n, t) \geq \psi(M(x_n, x_{n-1}, t)), \quad n = 0, 1, 2, ... \tag{2.2}$$

From (2.2), since ψ is a nondecreasing function, for every $n = 1, 2, ...,$ we have

$$\begin{aligned} M(x_{n+1}, x_n, t) &\geq \psi(M(x_n, x_{n-1}, t)) \\ &\geq \psi^2(M(x_{n-1}, x_{n-2}, t)) \geq ... \geq \psi^n(M(x_1, x_0, t)). \end{aligned} \tag{2.3}$$

Suppose that

$$M(x_1, x_0, t) = 1.$$

In this case, from (2.1), we have

$$\begin{aligned} &x_0 = x_1 = Tx_0, \\ &x_0 \preceq Sx_0, \\ &x_0 = x_1 \succeq Sx_1 = Sx_0. \end{aligned} \tag{2.4}$$

Since $\preceq$ partial order, this proves that $x_0 \in X$ is a solution to (1.1). Now, we may suppose that $M(x_1, x_0, t) \neq 0$. Let

$$M(x_1, x_0, t) > 0.$$

From (2.3), we have

$$M(x_{n+1}, x_n, t) \geq \psi^n(M(x_1, x_0, t)), \quad n = 0, 1, 2, ... \tag{2.5}$$

Using the (FM-4) and (2.5), for all $m = 1, 2, 3, ...,$ we get

$$\begin{aligned} M(x_n, x_{n+m}, t) &\geq *(M(x_n, x_{n+1}, \frac{t}{m}), M(x_{n+1}, x_{n+2}, \frac{t}{m}), ..., M(x_{n+m-1}, x_{n+m}, \frac{t}{m})) \\ &\geq *(\psi^n(M(x_0, x_1, \frac{t}{m})), \psi^{n+1}(M(x_0, x_1, \frac{t}{m})), ..., \psi^{n+m-1}(M(x_0, x_1, \frac{t}{m}))) \\ &\geq *_{i=0}^{m-1} \psi^{n+i}(M(x_0, x_1, \frac{t}{m})). \end{aligned} \tag{2.6}$$

From Lemma 2, for all $i \in \{0, 1, 2, ..., m-1\}$ we have

$$\lim_{n\to\infty} \psi^{n+i}(M(x_0, x_1, \frac{t}{m})) = 1.$$

That is,

$$\lim_{n\to\infty} M(x_n, x_{n+m}, t) \to 1$$

which implies that $\{x_n\} = \{T^n x_0\}$ is a Cauchy sequence in $(X; M; *)$. Then there is some $x^* \in X$ such that

$$\lim_{n\to\infty} M(x_n, x^*, t) \to 1 \tag{2.7}$$

On the other hand, from (2.1), we have

$$x_{2n} \in levS_{\preceq}, \quad n = 0, 1, 2, ... \tag{2.8}$$

Since $S : X \to X$ is level closed from the left (from (A2)), passing to the limit as $n \to \infty$ and using (2.7), we obtain

$$x^* \in levS_{\preceq},$$

that is

$$x^* \preceq Sx^*. \tag{2.9}$$

From (2.1), (2.9) and (A4), we get

$$M(Tx_{2n+1}, Tx^*, t) \geq \psi(M(x_{2n+1}, x^*, t)), \quad n = 0, 1, 2, ...$$

that is,

$$M(x_{2n+2}, Tx^*, t) \geq \psi(M(x_{2n+1}, x^*, t)), \quad n = 0, 1, 2, ... \tag{2.10}$$

Passing to the limit as $n \to \infty$, using , the property of ψ, we get

$$M(x^*, Tx^*, t) = 1$$

that is,

$$x^* = Tx^*. \tag{2.11}$$

Using (2.9), (2.11) and (A3), we have

$$x^* = Tx^* \succeq STx^* = Sx^*,$$

that is,

$$x^* \succeq Sx^*. \tag{2.12}$$

Since $\preceq$ partial order, inequalities (2.9) and (2.12) yield

$$x^* = Sx^*. \tag{2.13}$$

Further, from (2.11), (2.13) we get that $x^* \in X$ is a solution to problem (1.1).

Now, suppose that $y^* \in X$ is another solution to problem (1.1) with $x^* \neq y^*$. Using (A4) and Definition 6, we get

$$\begin{aligned} M(x^*, y^*, t) &= M(Tx^*, Ty^*, t) \geq \psi(M(x^*, y^*, t)) \\ &> M(x^*, y^*, t), \end{aligned} \tag{2.14}$$

which is a contradiction. Therefore $x^* \in X$ is a unique solution to (1.1). ∎

Observe that Theorem 1 holds true if we replace condition (A2) by the following.

$(A2)'$ The operator $S : X \to X$ is level closed from the right; that is; the set

$$levS_{\succeq} = \{x \in X : x \succeq Sx\} \tag{2.15}$$

is nonempty and closed.

Then, we get the following result.

Theorem 2.2. *Let $(X, M, *)$ be a complete fuzzy metric space. Suppose that Assumptions (A1),$(A2)'$-(A4) are satisfied. Then $x^* \in X$ is a unique solution to 1.1 such that for any $x_0 \in levS_{\succeq}$, the Picard sequence $\{T^n x_0\}$ converges to x^*.*

Taking $S = I_x$ (the identity operator), we get from Theorem 1 (or from Theorem 2) the following fixed point result.

Corollary 2.3. *Let $(X, M, *)$ be a complete fuzzy metric space and $T : X \to X$ be a given mapping. Suppose that there exists $\psi \in \Psi$ such that*

$$M(Tx, Ty, t) \geq \psi(M(x, y, t)), \qquad (x, y) \in X \times X.$$

Then $x^* \in X$ is a unique fixed point of T such that for any $x_0 \in X$, the Picard sequence $\{T^n x_0\}$ converges to x^*.

Conflict of Interests

The author declare that there is no conflict of interests regarding the publication of this paper.

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Bangmod International
Journal of Mathematical Computational Science
ISSN: 2408-154X
Bangmod-JMCS Online @ http://bangmod-jmcs.kmutt.ac.th/

Journal office:
Theoretical and Computational Science Center (TaCS)
Science Laboratory Building, Faculty of Science
King Mongkuts University of Technology Thonburi (KMUTT)
126 Pracha Uthit Road, Bang Mod, Thung Khru, Bangkok, Thailand 10140
Website: http://tacs.kmutt.ac.th/
Email: tacs@kmutt.ac.th