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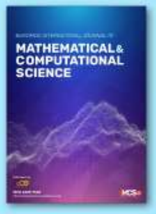

AN ACCELERATED HALPERN-TYPE ALGORITHM FOR SOLVING VARIATIONAL INCLUSION PROBLEMS WITH APPLICATIONS

Abubakar Adamu[1,2], **Duangkamon Kitkuan**[3], **Thidaporn Seangwattana**[4*]

[1] *Mathematics Institute, African University of Science and Technology, PMB 681, Abuja 900107, Nigeria*

[2] *Operational Research Center in Healthcare, Near East University, Nicosia, TRNC, Turkey*

Email: aadamu@aust.edu.ng

[3] *Department of Mathematics, Faculty of Science and Technology, Rambhai Barni Rajabhat University, Chanthaburi 22000, Thailand*

E-mail: duangkamon.k@rbru.ac.th

[4] *Faculty of Science Energy and Environment, King Mongkut's University of Technology North Bangkok, Rayong Campus (KMUTNB), 21120, Rayong, Thailand*

E-mail: thidaporn.s@sciee.kmutnb.ac.th

**Corresponding author.*

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Abstract An inertial Halpern-type forward-backward iterative algorithm for approximating a zero of sum of two accretive operators is introduced and studied. Strong convergence theorem is established in a uniformly convex and q-uniformly smooth real Banach space. The convergence result obtained is applied to convex minimization and image restoration problems. Furthermore, numerical experiments are carried out on some classical test images and personal images degraded with motion blur and random noise. Finally, numerical illustrations in the Banach space, $L_5([-1,1])$ are presented to support the main theorem.

MSC: 26A33, 34A12, 34A43, 34D20

Keywords: α-inverse strongly accretive, m-accretive, image restoration

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1. Introduction

Let E be a real Banach space. Let $A : E \to E$ be a single valued operator and $B : E \multimap E$ be a multi valued operator (possibly nonlinear). Consider the following problem:

$$\text{find} \qquad u \in E \qquad \text{such that} \qquad 0 \in (A + B)u. \tag{1.1}$$

It is well known that problem (1.1) includes, as special cases, variational inequality problems, split feasibility problems, convex minimization problems, equilibrium problems which have applications in machine learning, signal processing, linear inverse problems and image processing.

Iterative algorithms for approximating solutions of the inclusion (1.1) have been studied extensively by numerous authors (see, e.g., [1–3] [11],[12], [13], [14] [15],[16]). Assuming existence of solution, one of the classical methods for approximating solution(s) of (1.1) in the setting of real Hilbert spaces is the well-known *forward-backward algorithm* (FBA) which is an iterative procedure that starts at a point $x_1 \in H$, and generates inductively the sequence $\{x_n\} \subset H$ by:

$$x_{n+1} = \big(I + \lambda_n B\big)^{-1}\big(I - \lambda_n A\big)x_n, \tag{1.2}$$

where $\lambda_n > 0$ is a regularization parameter. The FBA (1.2) as the name implies is based on an explicit forward step with respect to A followed by an implicit backward step with respect to B. Observe that the FBA (1.2) includes, in particular, the *proximal point algorithm* (when $A \equiv 0$). *Weak* convergence of the sequence generated by (1.2) have been established by various authors under suitable conditions (see, e.g., [11]).

In 2012, Takahashi et al [17] introduced and studied a generalization of the FBA in real Hilbert spaces. They proved strong convergence of the sequence of their algorithm to a solution of the inclusion (1.1). In the same year, Lopez et al [18] introduced and studied a Halpern-type FBA in Banach spaces that are uniformly convex and q-uniformly smooth. They proved weak and strong convergence of the sequence of their algorithm to a solution of problem (1.1).

In 2016, Pholasa et al [19] extended the theorem of Takahashi et al [17] from real Hilbert spaces to real Banach spaces that are uniformly convex and q-uniformly smooth. They studied the following algorithm:

Algorithm 1.1. Step 0. Choose an arbitrary point $u, x_1 \in E$, and set $n = 1$.

Step 1. Compute

$$y_n = \alpha_n u + (1 - \alpha_n)\big(I + \lambda_n B\big)^{-1}(I - \lambda_n A)x_n.$$

Step 2. Compute

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)y_n,$$

where $A : E \to E$ is α-*inverse strongly accretive*, $B : E \multimap E$ is m-accretive and, $\{\alpha_n\}, \{\beta_n\} \subset (0, 1)$ $\{\lambda_n\} \subset (0, \infty)$ are sequences satisfying conditions C1-C3 below.

Step 3. Update $n = n + 1$ and go to Step 1.

(C1) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(C2) $\limsup_{n\to\infty} \beta_n < 1$,

(C3) $\liminf_{n\to\infty} \lambda_n \leq \limsup_{n\to\infty} \lambda_n < \left(\frac{\alpha q}{\kappa_q}\right)^{\frac{1}{q-1}}$.

They proved that the sequence generated by Algorithm 1.1 converges strongly to a solution of problem (1.1).

The importance of the efficiency of any iterative algorithm in application cannot be over emphasized. A lot of research efforts have been devoted to improving the speed of convergence of existing algorithms to the desired solutions. One of the methods of doing this is by introducing an *inertial extrapolation term* (see, e.g., [4–6, 8–10]. The motivation for inertial type algorithms comes from the implicit discretization of the second-order differential equation

$$\frac{d^2x(t)}{dt^2} + \beta(t)\frac{dx(t)}{dt} + \nabla F(x(t)) = g(t), \quad 0 < t_0 \leq t \quad \alpha \in [0,1], \tag{1.3}$$

where $\beta(t) = \frac{\lambda}{t^\alpha}$, $F : \mathbb{R}^n \to \mathbb{R}$ is a differentiable convex function admitting at least one minimizer, g is an integrable function. Equation (1.3) describes many models depending on β. For example, if $\alpha = 0$, (1.3) describes the motion of a heavy ball rolling with friction or damping parameter λ. Equation (1.3) is related to inertial optimization algorithms, with various inertia, depending on the choice of the damping function β and the error terms defined by g.

In 2019, Cholamjiak and Shehu [20] introduced and studied an inertial version of the algorithm of Lopez et al [18]. They studied the following algorithm in a uniformly convex and q-uniformly smooth real Banach space E:

Algorithm 1.2. Step 0. Let $\beta \in [0,1)$ and $x_0, x_1 \in E$ be given starting points. Set $n = 1$.

Step 1. Given iterates x_{n-1} and x_n, $n \geq 1$, choose β_n such that $0 \leq \beta_n \leq \bar{\beta_n}$, where

$$\bar{\beta_n} = \begin{cases} \min\left\{\beta, \frac{\epsilon_n}{\|x_n - x_{n-1}\|}\right\}, & x_n \neq x_{n-1}, \\ \beta, & \text{otherwise.} \end{cases}$$

Step 2. Compute

$$\begin{cases} y_n = x_n + \beta_n(x_n - x_{n-1}), \\ x_{n+1} = \alpha_n x_0 + (1-\alpha_n)\big(K^B_{\lambda_n}(y_n - \lambda_n(Ay_n + a_n)) + b_n\big), \quad n \geq 1, \end{cases}$$

where $A : E \to E$ is *α-inverse strongly accretive*, $B : E \multimap E$ is m-accretive and, $\{a_n\}, \{b_n\} \subset E$, $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ $\{\epsilon_n\}, \{\lambda_n\} \subset (0,\infty)$ are sequences satisfying conditions C4-C7.

Step 3. Update $n = n + 1$ and go to Step 1.

(C4) $\lim_{n\to\infty} \|a_n\|/\alpha_n = 0 \quad \lim_{n\to\infty} \|b_n\|/\alpha_n = 0,$

(C5) $\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty,$

(C6) $\liminf_{n\to\infty} \lambda_n \le \limsup_{n\to\infty} \lambda_n < (\alpha q/\kappa_q)^{1/(q-1)},$

(C7) $\epsilon_n = o(\alpha_n)$.

They proved that the sequence $\{x_n\}$ generated by Algorithm 1.2 converges in strongly to a solution of (1.1).

Remark 1.3. We remark here that the choice of the error sequences $\{a_n\}, \{b_n\} \subset E$ may drastically affect the performance of algorithm 1.2.

Motivated by Remark 1.3, the results of Pholasa et al [19] and Cholamjiak and Shehu [20], in this paper, we introduce an inertial version of algorithm 1.1 of Pholasa et al [19] in the setting of real Banach spaces that are uniformly convex and q-uniformly smooth. Furthermore, we proved a theorem that guarantees that the sequence generated by our proposed algorithm converges strongly to a solution of problem (1.1). In addition, we give some applications of our theorem to convex minimization problems and image denoising and debluring problems. Finally, we present some numerical illustrations to support our main theorem and its applications.

2. Preliminaries

In this section will state some important results used in the proof of our main Theorem 3.5. We will assume that the basic notions used are known by the reader (otherwise, see, e.g., page 5 of [21]). The first lemma we will state is the famous subdifferential inequality whose proof can be found in this monograph [22].

Lemma 2.1. [22] *For $q > 1$, let J_q be the generalized duality mapping, then for all $x, y \in E$ there exists $j_q(x+y) \in J_q(x+y)$ such that*

$$\|x+y\|^q \le \|x\|^q + q\langle y, j_q(x+y)\rangle.$$

The next lemma of H.K. Xu will play a crucial role in our proof of Theorem 3.5 after we proved boundedness of the sequence generated by algorithm 3.3.

Lemma 2.2 ([23])**.** *Let E be a uniformly convex real Banach space and let $q > 1$ and $r > 0$. Then there exists a strictly increasing continuous and convex functions $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ such that for all $x, y \in B(0,r) := \{x \in E : \|x\| \le r\}$,*

$$\|\lambda x + (1-\lambda)y\|^q \le \lambda\|x\|^q + (1-\lambda)\|y\|^q - \lambda(1-\lambda)\phi(\|x-y\|).$$

The next two lemmas are some key results established by Lopez et al [18] concerning the study of the forward-backward algorithm on real Banach spaces involving accretive operators. These results are what informed the setting and assumptions of Theorem 3.5.

Lemma 2.3 ([18])**.** *Let E be a q-uniformly smooth real Banach space and let $A : E \to E$ be an α-isa of order q. Then the following inequality holds for all $x, y \in E$*

$$\|(I-\lambda A)x - (I-\lambda A)y\|^q \le \|x-y\|^q - \lambda(\alpha q - \kappa_q\lambda^{q-1})\|Ax - Ay\|^q,$$

where $\kappa_q > 0$ is the q-uniform smoothness coefficient of E. In particular, if $0 < \lambda < (\alpha q - \kappa_q\lambda^{q-1})$ then $(I - \lambda A)$ is nonexpansive.

Remark 2.4. Let $A : E \multimap E$ be an m-accretive map the resolvent $K_\lambda^A : E \to E$ of A is defined by $K_\lambda^A x := \{u \in E : x \in (u + \lambda Au)\}$. It is well-known that K_λ^A is single valued with $F(K_\lambda^A) = A^{-1}0$ and K_λ^A is firmly nonexpansive. In the sequel we shall adopt the following notation:

$$W_\lambda^{A,B} := K_\lambda^B(I - \lambda A) = (I + \lambda B)^{-1}(I - \lambda A), \quad \lambda > 0.$$

The following statements are true, see, e.g., [18]

(i) For $\lambda > 0$, $F(W_\lambda^{A,B}) = (A + B)^{-1}0$.

(ii) For $0 < \lambda \leq \epsilon$ and $x \in E$, $\|x - W_\lambda^{A,B}x\| \leq 2\|x - W_\epsilon^{A,B}x\|$.

Lemma 2.5. [18] *Let E be a uniformly convex and q-uniformly smooth real Banach space and let $A : E \to E$ be an α-isa mapping of order q and $B : E \multimap E$ be an m-accretive mapping. Then given $r > 0$, there exists a continuous, strictly increasing and convex function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi(0) = 0$ such that for all $x, y \in B(0, r)$*

$$\begin{aligned}\|W_\lambda^{A,B}x - W_\lambda^{A,B}y\|^q \leq \|x - y\|^q - \lambda\big(\alpha q - \lambda^{q-1}\kappa_q\big)\|Ax - Ay\|^q \\ - \varphi\big(\|(I - K_\lambda)(I - \lambda A)x - (I - K_\lambda)(I - \lambda A)y\|\big).\end{aligned}$$

The following result is what we will use to conclude that the sequence generated by our algorithm 1.2 converges strongly to a solution of problem 1.1.

Lemma 2.6. [24] *Let $\{d_n\}$ be a sequence of nonnegative real numbers such that*

$$d_{n+1} \leq (1 - \theta_n)d_n + \theta_n\tau_n \quad \text{and} \quad d_{n+1} \leq d_n - \eta_n + \rho_n,$$

where $\{\theta_n\}$ is a sequence in $(0, 1)$, $\{\eta_n\}$ is a sequence of of nonnegative real numbers, $\{\rho_n\}$ and $\{\tau_n\}$ are real sequences such that

(i) $\sum_{n=1}^{\infty} \theta_n = \infty$, *(ii)* $\lim_{n\to\infty} \rho_n = 0$,

(iii) $\lim_{k\to\infty} \eta_{n_k} = 0$ *implies* $\limsup_{k\to\infty} \tau_{n_k} \leq 0$, *for any subsequence* $\{n_k\} \subset \{n\}$.

Then, $\lim_{n\to\infty} d_n = 0$.

The next lemma will be used to conclude that the generated by our algorithm is bounded. Moreover, this result can come handy for prove of boundedness of most inertial algorithms.

Lemma 2.7. [25] *Suppose $\{a_n\}$ and $\{b_n\}$ are two sequences of nonnegative numbers such that $a_{n+1} \leq a_n + b_n$, for all $n \geq 1$. If $\sum_{n=1}^{\infty} b_n$ converges, then $\lim_{n\to\infty} a_n$ exists.*

The next result of Reich is key in establishing strong convergence of Halpern-type algorithms. Similar version for viscosity-type algorithms was established by Cai and Bu (Optim. Lett. 7(2):267-287).

Lemma 2.8. [26] *Let H be a real Hilbert space and let $T : H \to H$ be a nonexpansive mapping with a nonempty fixed point set. For any $u \in H$ and $t \in (0, 1)$ let $\{z_t\}$ be a net defined by $z_t := tu + (1 - t)Tz_t$. Then, $\{z_t\}$ converges strongly to a fixed point of T.*

Remark 2.9. The analytic representations of duality maps and κ_q are known in $L_p(\Lambda)$ and $L_q(\Lambda)$ spaces, $\Lambda \subset \mathbb{R}$, for $p > 1$ and $q > 1$ such that $\frac{1}{p} + \frac{1}{q} = 1$ (see, e.g., page 7 of [21]).

3. Main result

The following assumptions are central in the proof of our main theorem.

Assumption 3.1. The real Banach space E is uniformly convex and q-uniformly smooth, $A : E \to E$ is an α-isa operator of order q, $B : E \multimap E$ is a set-valued m-accretive operator and the solution set $\Omega := (A+B)^{-1}0 \neq \emptyset$.

Assumption 3.2. Choose sequences $\{\beta_n\}$, $\{\gamma_n\} \subset (0,1)$ and $\{\epsilon_n\}, \{\lambda_n\} \subset (0,\infty)$ such that

(A1) $\lim_{n\to\infty} \beta_n = 0$ and $\sum_{n=1}^{\infty} \beta_n = \infty$,

(A2) $\sum_{n=1}^{\infty} \epsilon_n < \infty$ and $\lim_{n\to\infty} \frac{\epsilon_n}{\beta_n} = 0$,

(A3) $0 < \lambda \leq \kappa_q \lambda_n^{q-1} < \alpha q$.

Based on Assumptions 3.1 and 3.2, we now give our algorithm.

Algorithm 3.3. *Inertial Halpern-type forward-backward splitting algorithm.*

Step 0. (Initialization) choose arbitrary points $x_0, x_1 \in E$, and set $n = 1$.

Step 1. Choose α_n such that $0 \leq \alpha_n \leq \bar{\alpha_n}$, where

$$\bar{\alpha_n} = \begin{cases} \min\left\{\alpha, \frac{\epsilon_n}{\|x_n - x_{n-1}\|}\right\}, & x_n \neq x_{n-1}, \\ \alpha, & \text{otherwise.} \end{cases}$$

Step 2. Compute

$$\begin{cases} y_n = x_n + \alpha_n(x_n - x_{n-1}) \\ v_n = \beta_n u + (1-\beta_n) K_{\lambda_n}^B (y_n - \lambda_n A y_n) \\ x_{n+1} = \gamma_n y_n + (1-\gamma_n) v_n. \end{cases}$$

Step 3. Update $n = n + 1$ and go to Step 1.

Lemma 3.4. *Let $\{x_n\}$ be the sequence generated by Algorithm 3.3, then $\{x_n\}$ is bounded.*

Proof. Let $W_{\lambda_n}^{A,B} = K_{\lambda_n}^B (I - \lambda_n A)$ then $W_{\lambda_n}^{A,B}$ is nonexpansive (see, e.g., [21]). Now, using Remark 2.4 (i) and the nonexpansivity of $W_{\lambda_n}^{A,B}$, we have

$$\begin{aligned} \|v_n - z\| &= \|\beta_n u + (1-\beta_n) K_{\lambda_n}^B (y_n - \lambda_n A y_n) - z\| \\ &\leq \beta_n \|u - z\| + (1-\beta_n) \|W_{\lambda_n}^{A,B} y_n - W_{\lambda_n}^{A,B} z\| \\ &\leq \beta_n \|u - z\| + (1-\beta_n)\|y_n - z\|. \end{aligned} \tag{3.1}$$

Thus, using inequality (3.1), we obtain

$$\begin{aligned}
\|x_{n+1}-z\| &= \|\gamma_n y_n + (1-\gamma_n)v_n - z\| \\
&\le \gamma_n\|y_n - z\| + (1-\gamma_n)\|v_n - z\| \\
&\le \gamma_n\|y_n - z\| + (1-\gamma_n)\big(\beta_n\|u-z\| + (1-\beta_n)\|y_n - z\|\big) \\
&= \gamma_n\|y_n - z\| + (1-\gamma_n)\beta_n\|u-z\| + (1-\gamma_n)(1-\beta_n)\|y_n - z\| \\
&= (1-\gamma_n)\beta_n\|u-z\| + (1-\beta_n(1-\gamma_n))\|y_n - z\| \\
&\le (1-\gamma_n)\beta_n\|u-z\| + (1-\beta_n(1-\gamma_n))\|x_n - z\| \\
&\quad + (1-\beta_n(1-\gamma_n))\alpha_n\|x_n - x_{n-1}\| \\
&\le (1-\beta_n(1-\gamma_n))\big(\|x_n - z\| + \epsilon_n\big) + (1-\gamma_n)\beta_n\|u-z\| \\
&\le \max\{\|x_n - z\| + \epsilon_n, \|u-z\|\}.
\end{aligned}$$

If the maximum is $\|u-z\|$, we are done. Else, by Lemma 2.7, $\{\|x_n - z\|\}$ has a limit. This implies that $\{x_n\}$ is bounded. Hence, $\{v_n\}$ and $\{y_n\}$ are also bounded. ∎

Theorem 3.5. *Let $\{x_n\}$ be the sequence generated by Algorithm 3.3. Then $\{x_n\}$ converges strongly to $z \in \Omega$.*

Proof. Let $z \in \Omega$. Using Remark 2.4 (i), Lemmas 2.1 and 2.5, we have

$$\begin{aligned}
\|v_n - z\|^q &= \|\beta_n u + (1-\beta_n)W_{\lambda_n}^{A,B}y_n - z\|^q \\
&\le (1-\beta_n)^q\|W_{\lambda_n}^{A,B}y_n - W_{\lambda_n}^{A,B}z\|^q + q\beta_n\langle u-z, j_q(v_n - z)\rangle \\
&\le (1-\beta_n)^q\Big(\|y_n - z\|^q - \lambda_n(\alpha q - \lambda_n^{q-1}\kappa_q)\|Ay_n - Az\|^q \\
&\quad - (1-\beta_n)^q\varphi(\|y_n - \lambda_n(Ay_n - Az) - W_{\lambda_n}^{A,B}y_n\|)\Big) \\
&\quad + q\beta_n\langle u-z, j_q(v_n - z)\rangle \\
&= (1-\beta_n)^q\|y_n - z\|^q - \lambda_n(1-\beta_n)^q(\alpha q - \lambda_n^{q-1}\kappa_q)\|Ay_n - Az\|^q \\
&\quad - (1-\beta_n)^q\varphi(\|y_n - \lambda_n(Ay_n - Az) - W_{\lambda_n}^{A,B}y_n\|) \\
&\quad + q\beta_n\langle u-z, j_q(v_n - z)\rangle. \qquad (3.2)
\end{aligned}$$

Next, using Lemma 2.2, inequality (3.2) and Lemma 2.1, we obtain

$$\begin{aligned}
\|x_{n+1}-z\|^q &= \|\gamma_n y_n + (1-\gamma_n)v_n - z\|^q \\
&\leq \gamma_n\|y_n - z\|^q + (1-\gamma_n)\|v_n - z\|^q \\
&\leq \gamma_n\|y_n - z\|^q + (1-\gamma_n)\Big((1-\beta_n)^q\|y_n - z\|^q \\
&\quad - \lambda_n(1-\beta_n)^q(\alpha q - \lambda_n^{q-1}\kappa_q)\|Ay_n - Az\|^q \\
&\quad - (1-\beta_n)^q\varphi(\|y_n - \lambda_n(Ay_n - Az) - W_{\lambda_n}^{A,B}y_n\|) \\
&\quad + q\beta_n\langle u - z, j_q(v_n - z)\rangle\Big) \\
&= \gamma_n\|y_n - z\|^q + (1-\gamma_n)(1-\beta_n)^q\|y_n - z\|^q \\
&\quad - \lambda_n(1-\gamma_n)(1-\beta_n)^q(\alpha q - \lambda_n^{q-1}\kappa_q)\|Ay_n - Az\|^q \\
&\quad - (1-\gamma_n)(1-\beta_n)^q\varphi(\|y_n - \lambda_n(Ay_n - Az) - W_{\lambda_n}^{A,B}y_n\|) \\
&\quad + q\beta_n(1-\gamma_n)\langle u - z, j_q(v_n - z)\rangle \\
&\leq (1-(1-\gamma_n)\beta_n)\|y_n - z\|^q \\
&\quad - \lambda_n(1-\gamma_n)(1-\beta_n)^q(\alpha q - \lambda_n^{q-1}\kappa_q)\|Ay_n - Az\|^q \\
&\quad - (1-\gamma_n)(1-\beta_n)^q\varphi(\|y_n - \lambda_n(Ay_n - Az) - W_{\lambda_n}^{A,B}y_n\|) \\
&\quad + q\beta_n(1-\gamma_n)\langle u - z, j_q(v_n - z)\rangle \\
&\leq (1-(1-\gamma_n)\beta_n)\|x_n - z\|^q \\
&\quad + q(1-(1-\gamma_n)\beta_n)\alpha_n\langle x_n - x_{n-1}, j_q(y_n - z)\rangle \\
&\quad - \lambda_n(1-\gamma_n)(1-\beta_n)^q(\alpha q - \lambda_n^{q-1}\kappa_q)\|Ay_n - Az\|^q \\
&\quad - (1-\gamma_n)(1-\beta_n)^q\varphi(\|y_n - \lambda_n(Ay_n - Az) - W_{\lambda_n}^{A,B}y_n\|) \\
&\quad + q\beta_n(1-\gamma_n)\langle u - z, j_q(v_n - z)\rangle. \qquad (3.3)
\end{aligned}$$

Thus, from inequality (3.3), we deduce that

$$\begin{aligned}
\|x_{n+1}-z\|^q &\leq (1-(1-\gamma_n)\beta_n)\|x_n - z\|^q \\
&\quad + q(1-(1-\gamma_n)\beta_n)\alpha_n\langle x_n - x_{n-1}, j_q(y_n - z)\rangle \\
&\quad + q\beta_n(1-\gamma_n)\langle u - z, j_q(v_n - z)\rangle
\end{aligned}$$

and

$$\begin{aligned}
\|x_{n+1}-z\|^q &\leq \|x_n - z\|^q - \lambda_n(1-\gamma_n)(1-\beta_n)^q(\alpha q - \lambda_n^{q-1}\kappa_q)\|Ay_n - Az\|^q \\
&\quad - (1-\gamma_n)(1-\beta_n)^q\varphi(\|y_n - \lambda_n(Ay_n - Az) - W_{\lambda_n}^{A,B}y_n\|) \\
&\quad + q(1-(1-\gamma_n)\beta_n)\alpha_n\langle x_n - x_{n-1}, j_q(y_n - z)\rangle \\
&\quad + q\beta_n(1-\gamma_n)\langle u - z, j_q(v_n - z)\rangle,
\end{aligned}$$

for each $n \geq 1$.

$$\text{Set} \quad d_n = \|x_n - z\|^q, \qquad \theta_n = \beta_n(1-\gamma_n)$$

$$\tau_n = \frac{q(1-(1-\gamma_n)\beta_n)\alpha_n}{\beta_n(1-\gamma_n)}\langle x_n - x_{n-1}, j_q(y_n - z)\rangle + q\langle u - z, j_q(v_n - z)\rangle$$

$$\eta_n = \lambda_n(1-\gamma_n)(1-\beta_n)^q(\alpha q - \lambda_n^{q-1}\kappa_q)\|Ay_n - Az\|^q + (1-\gamma_n)(1-\beta_n)^q\varphi(\|y_n - \lambda_n(Ay_n - Az) - W_{\lambda_n}^{A,B}y_n\|)$$

$$\rho_n = q(1-(1-\gamma_n)\beta_n)\alpha_n\langle x_n - x_{n-1}, j_q(y_n - z)\rangle + q\beta_n(1-\gamma_n)\langle u - z, j_q(v_n - z)\rangle$$

$$d_{n+1} \le (1-\theta_n)d_n + \theta_n\tau_n \quad \text{and} \quad d_{n+1} \le d_n - \eta_n + \rho_n.$$

Observe that $\sum_{n=1}^{\infty}\beta_n = \infty$ implies $\sum_{n=1}^{\infty}\theta_n = \infty$. By the boundedness of $\{y_n\}$ and $\{v_n\}$, and the fact that $\lim_{n\to\infty}\beta_n = 0 = \lim_{n\to\infty}\alpha_n\|x_n - x_{n-1}\|$, we obtain that $\lim_{n\to\infty}\rho_n = 0$.

Next, by Lemma 2.6, it remains to show $\lim_{k\to\infty}\eta_{n_k} = 0$ implies $\limsup_{k\to\infty}\tau_{n_k} \le 0$, for any subsequence $\{n_k\} \subset \{n\}$. Let $\{\eta_{n_k}\}$ be a subsequence of $\{\eta_n\}$ such that $\lim_{k\to\infty}\eta_{n_k} = 0$. Then, by the property of φ, we have

$$\lim_{k\to\infty}\|Ay_{n_k} - Az\| = \lim_{k\to\infty}\|y_{n_k} - \lambda_{n_k}(Ay_{n_k} - Az) - W_{\lambda_{n_k}}^{A,B}y_{n_k}\| = 0.$$

Thus, by the triangle inequality,

$$\lim_{k\to\infty}\|W_{\lambda_{n_k}}^{A,B}y_{n_k} - y_{n_k}\| = 0.$$

Furthermore,

$$\|W_{\lambda_{n_k}}^{A,B}y_{n_k} - x_{n_k}\| \le \|W_{\lambda_{n_k}}^{A,B}y_{n_k} - y_{n_k}\| + \|y_{n_k} - x_{n_k}\| \le \|W_{\lambda_{n_k}}^{A,B}y_{n_k} - y_{n_k}\| + \alpha_{n_k}\|x_{n_k} - x_{n_k-1}\|$$

implies

$$\lim_{k\to\infty}\|W_{\lambda_{n_k}}^{A,B}y_{n_k} - x_{n_k}\| = 0.$$

Also,

$$\|y_{n_k} - v_{n_k}\| \le \|x_{n_k} - v_{n_k}\| + \alpha_{n_k}\|x_{n_k} - x_{n_k-1}\| \le \beta_{n_k}\|x_{n_k} - u\| + (1-\beta_{n_k})\|x_{n_k} - W_{\lambda_{n_k}}^{A,B}y_{n_k}\| + \alpha_{n_k}\|x_{n_k} - x_{n_k-1}\|$$

implies

$$\lim_{k\to\infty}\|y_{n_k} - v_{n_k}\| = 0.$$

By Assumption 3.2 there exists $\lambda > 0$ such that $\lambda_n \ge \lambda$, for all $n \ge 1$. Hence, using Remark 2.4 (ii), we have

$$\|W_{\lambda}^{A,B}y_{n_k} - y_{n_k}\| \le 2\|W_{\lambda_{n_k}}^{A,B}y_{n_k} - y_{n_k}\|.$$

This implies that

$$\limsup_{k\to\infty}\|W_{\lambda}^{A,B}y_{n_k} - y_{n_k}\| \le 2\limsup_{k\to\infty}\|W_{\lambda_{n_k}}^{A,B}y_{n_k} - y_{n_k}\| = 0.$$

So, $\limsup_{k\to\infty}\|W_{\lambda}^{A,B}y_{n_k} - y_{n_k}\| = 0$. Thus, $\lim_{k\to\infty}\|W_{\lambda}^{A,B}y_{n_k} - y_{n_k}\| = 0.$

Observe that

$$\|W_{\lambda}^{A,B}y_{n_k} - v_{n_k}\| \le \|W_{\lambda}^{A,B}y_{n_k} - y_{n_k}\| + \|y_{n_k} - v_{n_k}\|$$

implies

$$\lim_{k\to\infty}\|W_{\lambda}^{A,B}y_{n_k} - v_{n_k}\| = 0.$$

Also,

$$\begin{aligned}\|W_\lambda^{A,B}v_{n_k}-v_{n_k}\| &\le \|W_\lambda^{A,B}v_{n_k}-W_\lambda^{A,B}y_{n_k}\|+\|W_\lambda^{A,B}y_{n_k}-v_{n_k}\|\\ &\le \|v_{n_k}-y_{n_k}\|+\|W_\lambda^{A,B}y_{n_k}-v_{n_k}\|\end{aligned}$$

implies $\lim_{k\to\infty}\|W_\lambda^{A,B}v_{n_k}-v_{n_k}\|=0$.

Now, let $z_t=tu+(1-t)W_\lambda^{A,B}z_t,\ \ t\in(0,1)$. By Lemma 2.8, z_t converges strongly to a $z\in F(W_\lambda^{A,B})=(A+B)^{-1}0$.

By Lemma 2.1 and the fact that $W_\lambda^{A,B}$ is nonexpansive, we obtain

$$\begin{aligned}\|z_t-v_{n_k}\|^q &= \|tu+(1-t)W_\lambda^{A,B}z_t-v_{n_k}\|^q\\ &\le (1-t)^q\|W_\lambda^{A,B}z_t-v_{n_k}\|^q+qt\langle u-v_{n_k},j_q(z_t-v_{n_k})\rangle\\ &\le (1-t)^q\big(\|W_\lambda^{A,B}z_t-W_\lambda^{A,B}v_{n_k}\|+\|W_\lambda^{A,B}v_{n_k}-v_{n_k}\|^q\big)^q\\ &\quad +qt\langle u-v_{n_k},j_q(z_t-v_{n_k})\rangle\\ &\le (1-t)^q\big(\|z_t-v_{n_k}\|+\|W_\lambda^{A,B}v_{n_k}-v_{n_k}\|^q\big)^q\\ &\quad +qt\langle u-v_{n_k},j_q(z_t-v_{n_k})\rangle\\ &\le (1-t)^q\big(\|z_t-v_{n_k}\|+\|W_\lambda^{A,B}v_{n_k}-v_{n_k}\|^q\big)^q+qt\langle u-z_t,j_q(z_t-v_{n_k})\rangle\\ &\quad +qt\langle z_t-v_{n_k},j_q(z_t-v_{n_k}\rangle.\end{aligned}$$

Thus,

$$\begin{aligned}\langle z_t-u,j_q(z_t-v_{n_k})\rangle \le \frac{(1-t)^q}{qt}&\big(\|z_t-v_{n_k}\|+\|W_\lambda^{A,B}v_{n_k}-v_{n_k}\|^q\big)^q\\ &+\frac{(qt-1)}{qt}\|z_t-v_{n_k}\|^q.\end{aligned}$$

Hence,

$$\begin{aligned}\limsup_{k\to\infty}\langle z_t-u,j_q(z_t-v_{n_k}\rangle &\le \frac{(1-t)^q}{qt}C^q+\frac{(qt-1)}{qt}C^q\\ &=\Big(\frac{(1-t)^q+qt-1}{qt}\Big)C^q,\end{aligned}\tag{3.4}$$

where $C=\limsup_{k\to\infty}\|z_t-v_{n_k}\|$. Observe that $\lim_{t\to 0}\dfrac{(1-t)^q+qt-1}{qt}=0$. By the uniform continuity of j_q on bounded sets and the fact that $z_t\to z$, as $t\to 0$, we have

$$\lim_{t\to 0}\|j_q(z_t-v_{n_k})-j_q(z-v_{n_k})\|=0.$$

Thus, by the bicontinuity of $\langle\cdot,\cdot\rangle$, continuity j_q, and the fact that $z_t\to z$, as $t\to 0$, we have that

$$\lim_{t\to 0}\langle z_t-u,j_q(z_t-v_{n_k})\rangle=\langle z-u,j_q(z-v_{n_k})\rangle.$$

From inequality (3.4), we deduce that

$$\limsup_{k\to\infty}\langle z-u,j_q(z-v_{n_k})\rangle\le 0.$$

Furthermore, since

$$\frac{(1-(1-\gamma_{n_k})\beta_{n_k})\alpha_{n_k}q}{(1-\gamma_{n_k})\beta_{n_k}}\langle x_{n_k}-x_{n_k-1}, j_q(y_{n_k}-z)\rangle \leq \frac{(1-(1-\gamma_{n_k})\beta_{n_k})\alpha_{n_k}q}{(1-\gamma_{n_k})\beta_{n_k}}$$
$$\times \|x_{n_k}-x_{n_k-1}\|\|y_{n_k}-z\|^{q-1} \leq \Big(\frac{(1-(1-\gamma_{n_k})\beta_{n_k})q}{(1-\gamma_{n_k})}\|y_{n_k}-z\|^{q-1}\Big)\frac{\epsilon_{n_k}}{\beta_{n_k}},$$

$$\limsup_{k\to\infty}\frac{(1-(1-\gamma_{n_k})\beta_{n_k})\alpha_{n_k}q}{(1-\gamma_{n_k})\beta_{n_k}}\langle x_{n_k}-x_{n-1}, j_q(y_{n_k}-z)\rangle \leq 0.$$

Hence, obtain that $\limsup_{k\to\infty}\tau_{n_k} \leq 0$. Hence, by Lemma 2.6, $\lim_{n\to\infty} d_n = 0$, i.e.,

$$\lim_{n\to\infty} x_n = z \in (A+B)^{-1}0.$$

This completes the proof. ∎

Next, we give a corollary of our main Theorem 3.5 in L_q, $2 < q < \infty$ spaces.

Corollary 3.6. *Let $\{x_n\}$ be the sequence generated by Algorithm 3.3 under the same assumptions with $E = L_q$, $2 < q < \infty$. Then $\{x_n\}$ converges strongly to $z \in \Omega$.*

Proof. Since L_q, $2 < q < \infty$ spaces are uniformly convex and q-uniformly smooth spaces, the proof follows from Theorem 3.5. ∎

4. Applications and Numerical Illustrations

In this section, we shall apply the strong convergence of the inertial Halpern type FBA obtained in section 3 to convex minimization problem and convexly constrained linear inverse problem.

4.1. Application to convex minimization problem

Let H be a real Hilbert space and let $h : H \to \mathbb{R}$ be a convex smooth function and $g : H \to \mathbb{R} \cup \{\infty\}$ be a proper lower-semicontinuous and convex function. We consider the following convex minimization problem:

$$\text{Find} \quad x^* \in H \quad \text{such that} \quad h(x^*) + g(x^*) = \min_{x\in H}\Big\{h(x) + g(x)\Big\}. \tag{4.1}$$

Problem (4.1) is equivalent, by Fermat's rule, to the problem of finding $x^* \in H$ such that

$$0 \in \nabla h(x^*) + \partial g(x^*), \tag{4.2}$$

where ∇h is the gradient of h and ∂g is the subdifferential of g. Set $A = \nabla h$ and $B = \partial g$ in Algorithm 3.3. It is well-known that if ∇h is $(1/\alpha)$-Lipschitz continuous, then it is α-inverse strongly monotone and ∂g is maximal monotone. Hence from Algorithm 3.3 we have the following algorithm:

Algorithm 4.1. *Inertial Halpern-type forward-backward splitting algorithm.*

Step 0. (Initialization) choose arbitrary points $x_0, x_1 \in H$, and set $n = 1$.

Step 1. Choose α_n such that $0 \leq \alpha_n \leq \bar{\alpha_n}$, where

$$\bar{\alpha_n} = \begin{cases} \min\left\{\alpha, \frac{\epsilon_n}{\|x_n - x_{n-1}\|}\right\}, & x_n \neq x_{n-1}, \\ \alpha, & \text{otherwise.} \end{cases}$$

Step 2. Compute

$$\begin{cases} y_n = x_n + \alpha_n(x_n - x_{n-1}) \\ v_n = \beta_n u + (1-\beta_n)K_{\lambda_n}^{\partial g}(I - \lambda_n \nabla h)y_n \\ x_{n+1} = \gamma_n y_n + (1-\gamma_n)v_n. \end{cases}$$

Step 3. Update $n = n + 1$ and go to Step 1.

Theorem 4.2. *Let $\{x_n\}$ be the sequence generated by Algorithm 4.1. Then $\{x_n\}$ converges strongly to $z \in \Omega$.*

Proof. Since Hilbert spaces are uniformly convex and q-uniformly smooth spaces, the proof follows from Theorem 3.5. ∎

4.2. Application to image restoration problems

In this subsection, we focus on using mathematical algorithms in the implementation of image processing tasks on computers. Precisely, we are interested on the classical problems of image restoration: image denoising and debluring. Assume we have a noisy image of dimension $n \times n$ with missing pixels, our objective is to find the closest image to the original image. General image restoration problem can be formulated by the inversion of the following observation model:

$$b = Lx + y,$$

where b is the observed image, x is the unknown image, y is the noise and L is a linear operator that depends on the concerned image recovery problem. It is well-known that regularization methods are used in image restoration problems. The l_1-regularization is a powerful tool in image denoising. The restoration process is given by:

$$\min_x \frac{1}{2}\|Lx - b\|^2 + \lambda\|x\|_1, \tag{4.3}$$

where $\|\cdot\|$ denotes the Euclidean norm, λ is a positive regularization parameter and $\|\cdot\|_1$ is the l_1-regularization term.

Now, we use algorithms 1.1, 1.2 and 4.1 to approximate the solution of the following convex minimization problem:

$$\text{find} \quad u \in H \quad \text{such that } u = \arg\min_{x \in H}\left\{\frac{1}{2}\|Lx - b\|^2 + \lambda_n\|x\|_1\right\}.$$

In algorithm 1.1, we set $\alpha_n = \frac{1}{1000n}$, $\beta_n = \frac{n}{2n+1}$, $\lambda_n = 0.001$, in algorithm 1.2, we set $\alpha_n = \frac{1}{1000n}$, $\beta = 0.5$, $\beta_n = \bar{\beta_n}$, and in algorithm 4.1, we take $\alpha = 0.5$, $alpha_n = \bar{\alpha_n}$, $\beta_n = \frac{1}{1000n}$, $\epsilon_n = \frac{1}{(n+1)^6}$, $\gamma_n = \frac{1}{(n+1)^8}$, $\lambda_n = 0.001$ as our parameters and in all these algorithms, we set $u = x_0 = Lx + b$, $A = \nabla g$ and $B = \partial h$, where $g(x) = \frac{1}{2}\|Lx - b\|^2$, $h(x) = \lambda_n\|x\|_1$. We consider the blur function in MATLAB "fspecial ('motion', 30, 60)" and add random noise ($0.01 \times randn(size(x))$). The test images are Abubakar, Lena and butterfly (see Figure 1) and the stopping criterion of the algorithms is $\frac{\|x_{n+1}-x_n\|}{\|x_{n+1}\|} < 10^{-4}$. As we can see from Figure 1 and Table 1, our proposed algorithm is competitive and promising.

The signal to noise ratio (SNR) is used to measure the quality of the restored images and it is defined as:

$$\text{SNR} := 10 \log \frac{\|x\|^2}{\|x - x_n\|},$$

where x and x_n are the original and estimated image at iteration n, respectively. All algorithms were implemented with Ubuntu 64bits and MATLAB 2018b running on a Zinox laptop with Intel(R) Core(TM) i7 CPU and 4 GB of RAM.

Table 1. Numerical results of SNR in Figure 1

	The Signal to Noise Ratio (SNR)								
	Algorithm 1.1			Algorithm 1.2			Algorithm 4.1		
n	Abubakar	Lena	Butterfly	Abubakar	Lena	Butterfly	Abubakar	Lena	Butterfly
1	23.63	29.25	28.86	21.11	25.35	23.58	21.22	25.52	23.77
10	25.59	32.52	31.25	27.79	35.44	33.76	27.92	35.61	33.92
20	26.62	33.91	32.52	29.15	36.94	35.53	30.03	37.98	36.60
30	27.21	34.64	33.23	29.14	36.81	35.47	31.31	39.31	38.07
40	27.48	34.95	33.55	28.57	36.09	34.78	32.27	40.28	39.15
50	27.54	34.96	33.61	27.92	35.31	34.01	33.04	41.07	40.01
60	27.43	34.79	33.46	27.38	34.66	33.37	33.67	41.73	40.71
70	27.22	34.49	33.20	26.96	34.15	32.87	34.21	42.31	41.31
80	26.96	34.14	32.88	26.63	33.74	32.48	34.68	42.81	41.82
90	26.69	33.79	32.55	26.37	33.41	32.17	35.09	43.26	42.27
100	26.43	33.44	32.24	26.14	33.11	31.91	35.46	43.66	42.67

Remark 4.3. In these applications we just considered, our proposed algorithm 3.3 restored the test images better than algorithms 1.1 of Pholasa et al [19] and 1.2 of Cholamjiak and Shehu [20]. While Algorithms 1.1, 1.2 and 3.3 were given in the setting of Banach spaces, and the applications we have mentioned so far are all in Hilbert spaces, we remark here that the purpose of this applications is to illustrate the performance of the algorithms in this important applications. In the next subsection, we will give an implementation of these algorithms in the setting of the real Banach space $L_5([-1, 1])$.

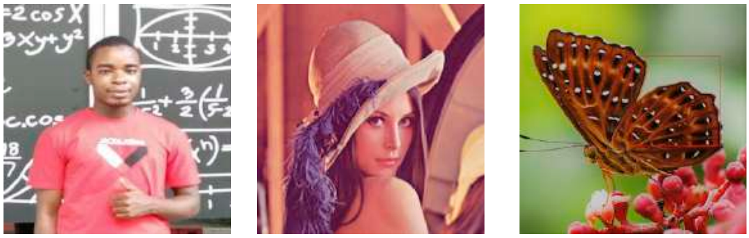
4.3. An Example in $L_5([-1, 1])$

In this subsection, we present a numerical example to compare the convergence of the sequence generated by our algorithm 3.3 and that algorithms 1.1 and 1.2.

Example 4.4.

We consider the Banach space $E = L_5([-1, 1])$, with norm defined by

$$\|x\|_5 := \Big(\int_{-1}^{1} |x(t)|^5 dt\Big)^{\frac{1}{5}} \quad \forall x, y \in E.$$



(A) original images

(B) images degraded by motion blur and random noise

(C) restored images with algorithm 1.1

(D) restored images with algorithm 1.2

(E) restored images with our algorithm 4.1

FIGURE 1. Test images and their restorations via algorithms 1.1, 1.2 and 4.1

Let $A : E \to E,\ B : E \to E$, be defined as

$$Ax(t) := 5x(t) + t + \cos t, \qquad Bx(t) := 2x(t).$$

Then, it is easy to see that A is $\frac{1}{5}$-isa of order 2, B is m-accretive. Furthermore, the solution set $\Omega = (A+B)^{-1}0 = \left\{\frac{-(t+\cos t)}{7}\right\}$. Observe that

$$K_\lambda^B(I - \lambda A)x(t) = \frac{1-5\lambda}{1+2\lambda}x(t) - \frac{\lambda}{1+2\lambda}(t + \cos t), \quad \forall\ \lambda > 0.$$

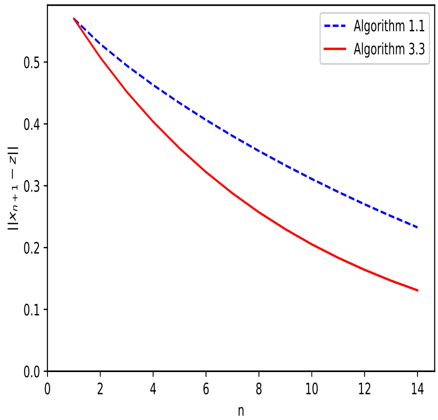
In the Algorithm 1.1, we take $\alpha_n = \frac{1}{1000n}$, $\beta_n = \frac{n}{2n+1}$, $\lambda_n = \frac{1}{64}$ and $u(t) = \frac{t}{2}$, in Algorithm 1.2, we take $\alpha_n = \frac{1}{1000n}$, $\lambda_n = \frac{1}{64}$ and $\bar{\beta_n} = \beta_n$, $\beta = 0.8$, $a_n(t) = b_n(t) = 0$ and, in Algorithm 3.3, we take $\alpha = 0.8$, $\alpha_n = \bar{\alpha_n}$, $\beta_n = \frac{1}{1000n}$, $\epsilon_n = \frac{1}{(n+1)^6}$, $\gamma_n = \frac{1}{(n+1)^8}$, $\lambda_n = \frac{1}{64}$ and $u(t) = \frac{t}{2}$ as our parameters. Clearly, these parameters satisfy the hypothesis of the theorem of Pholasa et al [19], Cholamjiak and Shehu [20] and Theorem 3.5, respectively. Finally, we use a tolerance of 10^{-3} and set maximum number of iterations $n = 15$.

TABLE 2. Table of values choosing $x_0 = 2t^2 + 1$, $x_1 = -t^3$

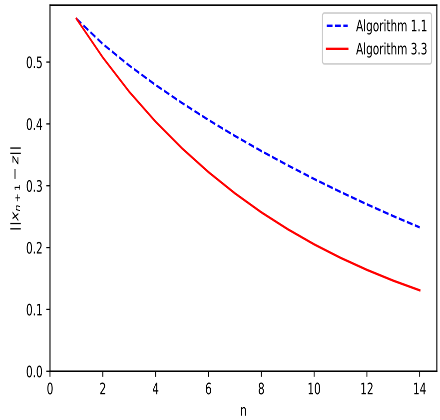
n	Algorithm 1.1 $\|x_n - z\|$	Algorithm 1.2 $\|x_n - z\|$	Algorithm 3.3 $\|x_n - z\|$
1	0.5701	0.5701	0.5701
2	0.5292	0.5084	0.5074
3	0.4944	0.4535	0.4520
5	0.4338	0.3624	0.3605
7	0.3806	0.2897	0.2878
9	0.3331	0.2317	0.2298
11	0.29	0.1853	0.164
13	0.2509	0.1482	0.1309
14	0.2327	0.1325	0.1309

TABLE 3. Table of values choosing $x_0 = t - 4$, $x_1 = \sin t$

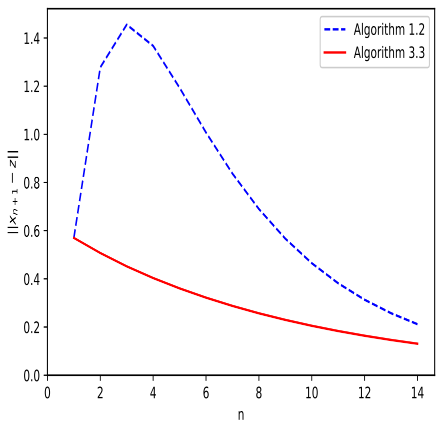
n	Algorithm 1.1 $\|x_n - z\|$	Algorithm 1.2 $\|x_n - z\|$	Algorithm 3.3 $\|x_n - z\|$
1	0.8463	0.8463	0.8463
2	0.7863	0.7574	0.7621
3	0.7361	0.6725	0.6801
5	0.6504	0.5322	0.5432
7	0.5777	0.4221	0.4341
9	0.5147	0.3561	0.3469
11	0.46	0.2684	0.2773
13	0.4123	0.2171	0.2216
14	0.3908	0.1966	0.1981



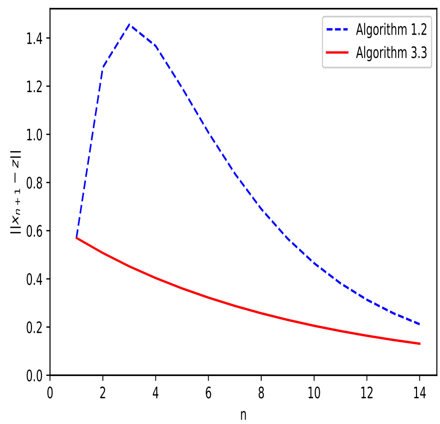
Graph of the first 14 iterates of Algorithms 1.1 and 3.3 choosing $x_0 = 2t^2 + 1$, $x_1 = -t^3$



Graph of the first 14 iterates of Algorithms 1.1 and 3.3 choosing $x_0 = t - 4$, $x_1 = \sin t$



Graph of the first 14 iterates of Algorithms 1.2 and 3.3 choosing $x_0 = 2t^2 + 1$, $x_1 = -t^3$



Graph of the first 14 iterates of Algorithms 1.2 and 3.3 choosing $x_0 = t - 4$, $x_1 = \sin t$

FIGURE 2. Graph of the results from Table 2 and Table 3

5. Conclusion

In this paper, an inertial version of the algorithm of Pholasa et al [19] is introduced and studied. Strong convergence of the sequence of the proposed algorithm is proved in real Banach spaces that are uniformly convex and q-uniformly smooth. Furthermore, the strong convergence result obtained is applied to convex minimization and image restoration problems. Numerical experiments were carried out on some classical test images and personal images degraded with motion blur and random noise. From the results obtained using these images (see Figure 1 and Table 1) the proposed algorithm appears to competitive and promising. Finally, a numerical example is presented in $L_5([-1,1])$ to support the main theorem.

6. Declarations

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6.2. Competing interest

The authors declare that they have no conflict of interest.

6.3. Availability of data and materials

Data sharing is not applicable to this article.

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