Bangmod Int. **J.** **M**ath. **&** **C**omp. **S**ci.
Vol. 4, No. 1-2, 2018; Pages 59 - 64
http://bangmod-jmcs.kmutt.ac.th/
ISSN : 2408-154X

Common Fixed Point Theorems For Contractive Mappings Under Weak Compatible Condition

[1]**Deepti Thakur and** [2] **Rajinder Sharma**
College of Applied Sciences, Sohar Sultanate of Oman.
E-mails:[1] *thakurdeepti@yahoo.com;* [2] *rajind.math@gmail.com*

[1] *Corresponding author.*

Abstract The objective of this paper is to prove some common fixed point theorems for five and six contractive maps under weak compatible condition without requiring continuity of maps using (EA)-property.

MSC: 47H10, 54H25.

Keywords: Common fixed point, Weakly compatible mappings, Contractive mappings, (EA)-property.

Submission date: 1 November 2018 / Acceptance date: 28 December 2018 /Available online 31 December 2018

1. Introduction

Common fixed points of contractive type mappings is one of the main attractive area in the field of fixed point theory and its applications. Rhoades [1] established common fixed point theorems through comparison among different type of contractive mappings. It is obvious that unless and until the space is assumed to be compact or the strict conditions are replaced by stronger conditions as in ([2],[3],[4]) strict contractive conditions alone doesnt guarantees the existence of common fixed point while the setting up of a metric space. The major breakthrough in the said field came into the light with the generalization of famous Banachs fixed point theorem by Jungck [5] who established common fixed point theorems for commuting maps. By moving one step ahead Sessa [6] defined more generalized form of commutativity, commonly known as weak commutativity and proved the fixed point theorems for the same. Furthermore, Jungck [7] alone came out with a notion of compatibility and thereafter jointly with Rohades [8] introduced the concept of weakly compatible maps and proved some common fixed point theorems for the same. They also emphasized on the point that compatible maps are weakly compatible but converse need not be true. Working in the same line, Pant [9] proved some fixed point theorems for non-compatible mappings and also defined R-weak commutativity. Chugh and Kumar [10] proved an interesting result in metric space for weakly compatible maps

Published by **T**heoretical **a**nd **C**omputational **S**cience Center **(TaCS)**,
King Mongkut's University of Technology Thonburi **(KMUTT)**

Bangmod-JMCS
Available online @ http://bangmod-jmcs.kmutt.ac.th/

without assuming any mappings continuous. Pants ([11],[12]),[13] contributions is also praiseworthy as his work motivated others to explore new horizons in the said field. Amari and Moutawakil [14] made a significant impact by studying a new class of maps satisfying (EA)-property so that compatible and non-compatible maps may be studied together. In this note, we make use of this concept to obtain fixed point theorems for five and six maps without requiring continuity of maps.

2. Preliminaries

Throughout this article, let Y be an arbitrary non-empty set and (X, d) be a metric space.

Definition.[5] Let A and S of a metric space (X, d) be compatible, if and only if $lim_{n\to\infty)} d(ASx_n, SAx_n) = 0$, whenever x_n is a sequence in X such that $lim_{n\to\infty)} Ax_n = lim_{n\to\infty} Sx_n = t$, for some $t \in X$.

Definition.[6] A pair of mappings A and S is called a weakly compatible pair if they commute at coincidence point.

Remark 1. Weakly compatible maps need not be compatible. Let $X = [2, 20]$ and d be the usual metric on X. Define mappings $A, S\colon X \to X$ by $Ax = x$ if $x = 2$ or > 5, $Ax = 6$ if $2 < x \leq 5$, $Sx = x$ if $x = 2, Sx = 12, if\ 2 < x \leq 5, Sx = x-3\ if\ x > 5$. The mappings A and S are non-compatible since sequence $\{x_n\}$ defined by $x_n = 5 + \frac{1}{n}, n \geq 1$. Then $lim_{n\to\infty} Sx_n = 2$, $lim_{n\to\infty} SAx_n = 2$ and $lim_{n\to\infty} SAx_n = 6$ But they are weakly compatible since they commute at coincidence point $x = 2$. Clearly, commuting maps are weakly commuting and weakly commuting mappings are compatible but implications are not reversible.

Definition.[13] Self maps A and S of a metric space (X, d) are $R-$weakly commuting at a point $x \in X$ if $d(ASx, SAx) \leq Rd(Ax, Sx)$, for some $R > 0$. They are point-wise $R-$ weakly commuting on X if for given $x \in X$, there exists $R > 0$ such that $d(ASx, SAx) \leq Rd(Ax, Sx)$. Pant [12] has shown that compatible maps are necessarily point-wise R-weakly commuting but the reverse implication is not true and thereby pinpoints the importance of point-wise R-weakly commuting maps in fixed point consideration and related theorems. Singh and Mishra [16] studied coincidences and fixed points of non-hybrid contractions. Further, Singh and Tomar [17] have noted that compatible maps are more general than R-weakly commuting maps. However, our formulations require only the commutativity of maps just at a coincidence point. Obviously, the commutativity requirements in common fixed point considerations can not be weaker than this.

Definition. Let A and S be maps on Y with values in X. Then A and S are said to satisfy the (EA)-property if there exist a sequence $x_n \in Y$ such that $lim_{n\to\infty} Ax_n = lim_{n\to\infty} Sx_n = t$, for some $t \in X$, If $Y = X$ then we get the definition of (EA)-property for two self maps of X studied by Aamri and Moutawakil [14]. In such a situation, t is called a tangent point by Sastry and Krishna Murthy [15].

3. Main Results

The following is our main result:

Theorem 1. Let (X,d) be a metric space. Furthermore, let $A, B, S, T\, and\, P\colon Y \to X$ be five mappings satisfying the following conditions:

(i)$P(Y) \cup AB(Y) \subset ST(Y)$;

(ii) One of the pairs (P, ST) or (P, AB) satisfies the (EA)-property;
(iii) $d(Px, Py) < max\{d(STx, ABy), d(Px, STx), d(Py, ABy), d(Py, STx), d(Px, By)\}$;
for all $x, y \in X$. Then
(iv) P and ST have a coincidence point, and
(v) P and AB have a coincidence point.

Proof. If the pair (P, AB) satisfies the (EA)-property then there exists a sequence $\{x_n\}$ in Y such that for some $t \in X$. Since $P(Y) \subset ST(Y)$, for each x_n, there exists y_n in Y such that $Px_n = STy_n$ and $STy_n \to t$ as well. We show that $Py_n \to t$. If not, there exist a subsequence $\{Py_{n_{(i)}}\}$ of $\{Py_n\}$, a natural number n, and a real number $r > 0$ such that for some positive integer $k \geq n$, we have $d(Py_k, t) \geq r$, $d(Py_k, Px_k) \geq r$ and
$d(Py_k, Px_k)$
$< max\{d(STy_k, ABx_k), d(Py_k, STy_k), d(Px_k, ABx_k), d(Px_k, STy_k), d(Py_k, ABx_k)\}$
$< d(Py_k, Px_k)$, a contradiction and $Py_n \to t$. Since $t \to P(Y)$ and $P(Y) \subset ST(Y)$, there exist an element $u \in Y$, such that $t = STu$. Now, we show that $Pu = STu$, If $STu \neq Pu$ then by (iii),
$d(Pu, Px_n) < max\{d(STu, ABx_n), d(Pu, STu),$
$d(Px_n, ABx_n), d(Px_n, STu), d(Pu, ABx_n)\}$
Let $n \to \infty$
$d(Pu, STu) < d(Pu, STu)$, a contradiction, therefore $Pu = STu$. This proves (iv).
Since $P(Y) \subset AB(Y)$, there exist a point v in Y such that $Pv = ABv$.
If $ABv \neq Pv$, then by (iii)
$d(Pu, Pv) < max\{d(STu, ABv), d(Pu, STu), d(Pv, ABv), d(Pv, STu), d(Pu, ABv)\}$.
This gives $d(Pu, Pv) < (Pu, Pv)$. Consequently $ABv = Pu = Pv$. This proves (v).

Corollary 1. *Let (X,d) be a metric space. Further, let A, B, S, T and $P\colon Y \to X$ be five mappings. If conditions (i-iii) are satisfied then conclusions (iv) and (v) of Theorem 1 hold. Furthermore, if $Y = X$. Then*
(vi) *P and ST have a common fixed point if the pair (P,ST) is weak compatible.*
(vii) *P and AB have a common fixed point if the pair (P,AB) is weak compatible.*
(viii) *P, A, B, S and T have a unique common fixed point if the pairs (P, ST) and (P, AB) are weakly compatible.*

Proof. As point of coincidence proved in Theorem 1. Now, we proceed for common fixed point and uniqueness as follows. Let $Y = X$, if P and ST commute at their coincidence point u, then $PPu = PSTu = STPu = STSTu$, and by (iii)
$d(Pu, PPu) = d(PPu, Pv) < \max\{(d(STPu, ABv), d(PPu, STPu), d(Pv, ABv),$
$d(Pv, STPu), d(PPu, ABv))\} = d(Pu, PPu)$. *This proves (vi). The proof of (vii) is analogous and the proof of (viii) is immediate.*
Now we extend Theorem1 for six mappings.

Theorem 2. *Let (X,d) be a metric space. Further, let A, B, S, T, P and $Q\colon Y \to X$ be six mappings satisfying the following conditions:*
(ix)*$P(Y) \subset AB(Y)$ and $Q(Y) \subset ST(Y)$;*
(x) *One of the pairs (P, ST) or (Q, AB) satisfies the (EA)-property;*
(xi) *$d(Px, Qy) < max\{d(STx, ABy), d(Px, STx), d(Qy, ABy), d(Qy, STx), d(Px, ABy)\}$;*
for all $x, y \in Y$. Then
(xii) *P and ST have a coincidence point, and*
(xiii) *Q and AB have a coincidence point.*

Proof. *If the pair (Q, AB) satisfies the (EA)-property then there exists a sequence $\{x_n\}$ in Y such that $\lim_{n\to\infty} Qx_n = \lim_{n\to\infty} ABx_n = t$, for some $t \in X$. Since $Q(Y) \subset ST(Y)$, for each x_n, there exists y_n in Y such that $Qx_n = STy_n$ and $STy_n \to t$ as well.*

We show that $Py_n \to t$. If not, there exist a subsequence $\{Py_{n_{(i)}}\}$ of $\{Py_n\}$, a natural number n, and a real number $r > 0$ such that for some positive integer $k \geq n$, we have $d(Py_k, t) \geq r$, $d(Py_k, Qx_k) \geq r$ and

$d(Py_k, Qx_k)$
$< max\{d(STy_k, ABx_k), d(Py_k, STy_k), d(Qx_k, ABx_k), d(Qx_k, STy_k), d(Py_k, ABx_k)\}$
$< d(Py_k, Qx_k)$*, a contradiction and $Py_n \to t$. Since $t \to Q(Y)$ and $Q(Y) \subset ST(Y)$, there exist an element $u \in Y$, such that $t = STu$. Now, we show that $Pu = STu$, If $STu \neq Pu$ then by (xi),*

$d(Pu, Qx_n) < \max\{d(STu, ABx_n), d(Pu, STu),$
$d(Qx_n, ABx_n), d(Qx_n, STu), d(Pu, ABx_n)\}$

Let $n \to \infty$

$d(Pu, STu) < d(Pu, STu)$*, a contradiction, therefore $Pu = STu$. This proves (xii).*
Since $P(Y) \subset AB(Y)$, there exist a point v in Y such that $Pu = ABv$.
If $ABv \neq Qv$, then by (xi)
$d(Pu, Qv) < max\{d(STu, ABv), d(Pu, STu), d(Qv, ABv), d(Qv, STu), d(Pu, ABv)\}$.
This gives $d(Pu, Qv) < (Pu, Qv)$. Consequently $ABv = Pu = Qv$. This proves $(xiii)$.

Corollary 2. *Let (X, d) be a metric space. Further, let A, B, S, T, P and $Q\colon Y \to X$ be six mappings. If conditions (ix-xi) are satisfied then conclusions (xii) and (xiii) of Theorem 2 hold. Furthermore, if $Y = X$. Then*

(xiv) *P and ST have a common fixed point if the pair $(P,\ ST)$ is weak compatible;*
(xv) *Q and AB have a common fixed point if the pair $(Q,\ AB)$ is weak compatible;*
(xvi) *P, A, B, S, T and Q have a unique common fixed point if the pairs (P, ST) and (Q, AB) are weakly compatible.*

Proof. As point of coincidence proved in Theorem 2. Now, we proceed for common fixed point and uniqueness as follows. Let $Y = X$, if P and ST commute at their coincidence point u, then $PPu = PSTu = STPu = STSTu$, and by (xi)
$d(Pu, PPu) = d(PPu, Qv) < \max\{(d(STPu, ABv), d(PPu, STPu),$
$d(Qv, ABv), d(Qv, STPu), d(PPu, ABv))\} = d(Pu, PPu)$.
This proves (xiv). The proof of (xv) is analogous and the proof of (xvi) is immediate.
∎

Example 1. *Let the set $X = [0, 1]$ with the metric d defined by $d(x, y) = |x - y|$, for all $x, y \in X$. Clearly (X, d) is a metric space. Let A, B, S, T, P and Q be defined as $Ax = x, Bx = x/2, Sx = x/5, Tx = x/3, Px = x/6$ and $Qx = 0$, for all $x \in X$ respectively. Then $P(X) = [0, 1/6] \subset [0, 1/2] = AB(X)$, and $Q(X) = \{0\} \subset [0, 1/15] = ST(X)$. Further the pair $\{P, AB\}$ is weak compatible. If $lim_{n\to\infty} x_n = 0$, where $\{x_n\}$ is a sequence in X, such that lim $lim_{n\to\infty} Px_n = lim_{n\to\infty)} ABx_n = 0, 0 \in X$. By similar reason, the pair Q, ST is also weak compatible. Thus all the conditions of Theorem 2 are satisfied and 0 is the unique common fixed point of A, B, S, T, P and Q.*
The following results are slightly more interesting when the above theorem is considered for three maps.

Corollary 3. *Let (X,d) be a metric space and $A, B, S\colon Y \to X$ be three mappings such that*

(xvii) $A(Y) \cup B(Y) \subset S(Y)$;
(xviii) *One of the pair (A,S) or (B,S) satisfies the (EA)-property;*
(xix) $d(Ax, By) < \max\{d(Sx, Sy), d(Ax, Sx), (By, Sy), d(By, Sx), d(Ax, Sy)\}$, *for all* $x, y \in Y$. *Then*
(xx) *A and S have a coincidence point, and*
(xxi) *B and S have a coincidence point.*

Corollary 4. *Let (X,d) be a metric space. Further, let* $A, B, S : Y \to X$ *be three mappings. If conditions (xvii-xix) are satisfied then conclusions (xx) and xxi) of Corollary 3 hold. Furthermore, if* $Y = X$. *Then*
(xxii) *A and S have a common fixed point if the pair (A,S) is weak compatible.*
(xiii) *B and S have a common fixed point if the pair (B,S) is weak compatible.*
(xiv) A, B *and S have a unique common fixed point if the pairs* (A, S) *and* (B, S) *are weakly compatible.*

Conclusion
In this article, we prove some common fixed point theorems for contractive mappings under weak compatible conditions without any appeal to continuity of maps using (EA)-property. The proven results with above mentioned conditions shows that continuity of mappings is not required for existence of fixed point.

Conflict of Interests
The author declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments *The authors are thankful to the anonymous referees for their valuable suggestions /comments to put the paper in its present form.*

References

[1] Rhoades, B. E., A comparison of various definitions of contractive mappings, Trans. Amer. Math. Soc. **226** (1977), 257-290.
[2] Jachymski, J., Common fixed point theorems for some families of maps, Ind. J. Pure & Appl. Math., **25**(1994), 925-937.
[3] Jungck, G., Moon, K. B., Park, S. and Rhoades, B. E., On generalization of the Meir-Keeler type contraction maps, Corrections, J. Math. Anal. Appl., **180** (1993), 221-222.
[4] Pant, R. P., , Common fixed point theorems for contractive maps, J. Math. Anal. Appl., **226** (1998), 251-258.
[5] Jungck, G., Compatible mappings and common fixed points, Internat. J. Math. Math. Sci., **9**(1986), 771-779.
[6] Sessa, S., On weak commutativity condition of mappings in a fixed point consideration, **32** (46), 1986, 149-153.
[7] Jungck, G., Compatible mappings and common fixed points (2), Internat. J. Math. and Math. Sci., **11** (1988), 285-288.
[8] Jungck, G. and Rhoades, B. E., Some fixed point theorems for compatible maps, Internat. J. Math. Sci., **16** (3) (1993), 417-428.
[9] Pant, R. P., R- weak commutativity and common fixed points of non compatible maps, Ganita, **49** (1998), 19-27.

[10] Chugh, R. and Kumar, S., Common fixed points for weakly compatible maps, Proc. Indian Acad. Sci.(Math. Sci.), **111** (2)(2001), 241-247.
[11] Pant, R. P., Discontinuity and common fixed points, J. Math. Anal. Appl., (1999), 284-289.
[12] Pant, R. P., R-weak commutativity and fixed points, Soochoo J. Math., **25** (1999), 37-42.
[13] Pant, R. P. and Pant, V., Common fixed points under strict contractive conditions, J. Math. Anal. Appl., **248** (2000), 327-332.
[14] Aamri, M. and Moutawakil, D. El., Some new common fixed point theorems under strict contractive conditions, J. Math. Anal. Appl., **270** (2002), 181-188.
[15] Sastry, K. P. R. and Krishna Murthy, I. S. R., Common fixed points of two partially commuting tangential self-maps on a metric space, J. Math. Anal. Appl., **250** (2000), 731-734.
[16] Singh, S. L. and Mishra, S. N., Coincidences and fixed points of non-self hybrid contractions, J. Math. Anal. Appl., **256**(2001), 486-497.
[17] Singh, S. L. and Tomar, A., Weaker forms of commuting maps and existence of fixed points, J. Korea Soc. Math. Edu. Ser. B : Pure Appl. Math., **10** (3) (2003),145-161.

Bangmod *International*
J*ournal of* **M***athematical* **C***omputational* **S***cience*
ISSN: 2408-154X
Bangmod-JMCS *Online @ http://bangmod-jmcs.kmutt.ac.th/*

Journal office:
Theoretical and Computational Science Center (TaCS)
Science Laboratory Building, Faculty of Science
King Mongkuts University of Technology Thonburi (KMUTT)
126 Pracha Uthit Road, Bang Mod, Thung Khru, Bangkok, Thailand 10140
Website: http://tacs.kmutt.ac.th/
Email: tacs@kmutt.ac.th