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Gradient Ricci Solitons in δ- Lorentzian Trans-Sasakian manifolds with semi-symmetric metric connection

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Abstract The aim of the present research paper is to study the δ-Lorentzian Trans Sasakian manifolds endowed semi-symmetric metric connections addmitting the gradient Ricci Solitons, η-Ricci Solitons and Ricci Solitions. Initialy, it is shown that the δ-Lorentzian trans Sasakian manifolds with a semi-symmetric-metric connection. We have found the expressions for curvature tensors, Ricci curvature tensors and scalar curvature of the δ-Lorentzian trans Sasakian manifolds with a semi-symmetric-metric and metric connection. Also, we have discussed some results on quasi-projectively flat and ϕ-projectively flat manifolds endowed with a semi-symmetric-metric connection. It shown that the manifold satisfying $\bar{R}.\bar{S} = 0$, $\bar{P}, \bar{S} = 0$. Moreover, we have obtained the conditions for the δ-Lorentzian Trans Sasakian manifolds with a semi-symmetric-metric connection to be conformally flat and ξ-conformally flat.

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1. Introduction

The Study of differentiable manifolds with Lorentizain metric is a natural and interesting topic in differential geometry. In 1996, Ikawa and Erdogan studied Lorentzian Sasakian manifold [20]. Also Lorentzian para contact manifolds were introduced by Matsumoto [23]. Trans Lorentzian para Sasakian manifolds have been used by Gill and Dube [15]. In [40] Yildiz et. al. studied Lorentzian α- Sasakian manifold and Lorentzian β-Kenmotsu manifold studied by Funda et. al. in [39]. After that in 2011, S. S Pujar and V. J. Khairnar [27] have initiated the study of Lorentzian Trans-Sasakian manifolds and

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studied the some basic results with some of its properties. Earlier to this S. S. Pujar [28] has initiated the study of δ-Lorentzian α Sasakian manifolds and δ-Lorentzian β Kenmotsu manifolds[28].

The study of manifolds with indefinite metrics is of interest from the standpoint of physics and relatively. In 1969, Takahashi [35] has introduced the notion of almost contact metric manifolds equipped with pseudo Riemannian metric. These indefinite almost conatct metric manifolds and indefinite Sasakian manifolds are known as (ϵ)-almost contact metric manifolds. The concept of (ε)-Sasakian manifolds was initiated by Bejancu and Duggal [6]. U. C. De and A. Sarkar [11] studied the notion of (ε)-Kenmotsu manifolds. In [38], X. Xufeng and C. Xiaoli studied ε-Sasakian manifolds. Later, S.S. Shukla and D. D. Singh [31] extended the study to (ε)-Trans-Sasakian manifolds with indefinite metric. The semi Riemannian manifolds has the index 1 and the structure vector field ξ is always a time like. Siddiqi et. al. [32] also studied some properties of Indefinite trans-Sasakian manifolds which is closely related to this topic. This motivated the Thripathi and others [34] to introduced (ε)-almost para contact structure where the vector filed ξ is space like or time like according as $(\varepsilon) = 1$ or $(\varepsilon) = -1$.

When M has a Lorentzian metric g, that is, a symmetric non degenerate $(0, 2)$ tensor field of index 1, then M is called a Lorentzian manifold. Since the Lorentzian metric is of index 1, Lorentzian manifold M has not only spacelike vector fields but also timelike and lightlike vector fields. This difference with the Riemannian case give interesting properties on the Lorentzian manifold. A differentiable manifold M has a Lorentzian metric if and only if M has a 1- dimensional distribution. Hence odd dimensional manifold is able to have a Lorentzian metric. Inspired by the above results In 2014, S. M Bhati [8] introduced the notion of δ-Lorentzian Trans Sasakian manifolds.

In 1924, the idea of semi-symmetric linear connection on a differentiable manifold was introduced by A. Friedmann and J. A. Schouten [13]. In 1930, Bartolotti [5] gave a geometrical meaning of such a connection. In 1932, H. A. Hayden [16] defined and studied semi-symmetric metric connection. In 1970, K. Yano [41], started a systematic study of the semi-symmetric metric connection in a Riemannian manifold and this was further studied by various authors such as Sharfuddin Ahmad and S. I. Hussain [30], M. M. Tripathi [33], I. E. Hirică and L. Nicolescu ([17], [18]), G. Pathak and U.C. De [26].

Let ∇ be a linear connection in an n-dimensional differentiable manifold M. The torsion tensor T and the curvature tensor R of ∇ are given respectively by

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y],$$

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

The connection ∇ is said to be symmetric if its torsion tensor T vanishes, otherwise it is non-symmetric. The connection ∇ is said to be metric connection if there is a Riemannian metric g in M such that $\nabla g = 0$, otherwise it is non-metric. It is well known that a linear connection is symmetric and metric if it is the Levi-Civita connection.

A linear connection ∇ is said to be semi-symmetric connection if its torsion tensor T is of the form

$$T(X, Y) = \eta(Y)X - \eta(X)Y,$$

where η is a 1-form.

Semi-symmetric connections play an important role in the study of Riemannian manifolds. There are various physical problems involving the semi-symmetric metric connection. For example, if a man is moving on the surface of the earth always facing one definite point, say Jaruselam or Mekka or the North pole, then this displacement is semi-symmetric and metric [13].

In 1982, Hamilton [19] introduced that the Rici solitons move under the Ricci flow simply by diffeomorphisms of the initial metric that is they are sationary points of the Ricci flow is given by

$$\frac{\partial g}{\partial t} = -2Ric(g). \tag{1.1}$$

Definition 1.1. *A Ricci soliton (g, V, λ) on a Riemannian manifold is defined by*

$$L_V g + 2S + 2\lambda = 0, \tag{1.2}$$

where S is the Ricci tensor, L_V is the Lie derivative along the vector field V on M and λ is a real scalar. Ricci soliton is said to be shrinking, steady or expanding according as $\lambda < 0, \lambda = 0$ and $\lambda > 0$, respectively.

In 1925, Levy [21] obtained the necessary and sufficient conditions for the existence of such tensors. later, R. Sharma [29] initiated the study of Ricci solitons in contact Riemannian geometry . After that, Nagaraja et. al. [24] and others like C. S. Bagewadi et. al. [4] and O. chodosh and others extensively studied Ricci soliton. In 2009, J. T. Cho and m. Kimura [11] introduced the notion of η-Ricci solitons and gave a classification of real hypersurfaces in non-flat complex space forms admitting η-Ricci solitons. Later η-Ricci solitons in (ϵ) almost paracontact metric manifolds have been studied by A. M. Blaga et. al. [3]. A. M. Blaga and various others authors also have been studied η-Ricci solitons in different structures (see [1], [2]) Recently in 2017, K. Venu and G. Nagaraja [37] study the η-Ricci solitns in trans-Sasakian maanifolds with semi-symmetric metric connection. It is natural and interesting to study η-Ricci soliton in δ-Lorentzian Trans-Sasakian manifolds with semi-symmetric metric connection not as real hypersurfaces of complex space forms but a special contact structures. In this paper we derive the condition for a 3 dimensional δ-Lorentzian Trans-Sasakian manifolds with semi-symmetric metric connection as an η-Ricci soliton and derive expression for the scalar curvature.

2. Preliminaries

Let M be an δ-almost contact metric manifold equipped with δ-almost contact metric structure $(\phi, \xi, \eta, g, \delta)$ consisting of a $(1, 1)$ tensor field ϕ, a vector field ξ, a 1-form η and an indefinite metric g such that

$$\phi^2 = X + \eta(X)\xi, \quad \eta \circ \phi = 0, \quad \phi\xi = 0, \tag{2.1}$$

$$\eta(\xi) = -1, \tag{2.2}$$

$$g(\xi, \xi) = -\delta, \tag{2.3}$$

$$\eta(X) = \delta g(X, \xi), \tag{2.4}$$

$$g(\phi X, \phi Y) = g(X, Y) + \delta\eta(X)\eta(Y), \tag{2.5}$$

for all $X, Y \in M$, where δ is such that $\delta^2 = 1$ so that $\delta = \pm 1$. The above structure $(\phi, \xi, \eta, g, \delta)$ on M is called the δ Lorentzian structure on M. If $\delta = 1$ and this is usual Lorentzian structure [27] on M, the vector field ξ is the time like [38], that is M contains a time like vector field.

In [36], Tano classified the connected almost contact metric manifold. For such a manifold the sectional curvature of the plane section containing ξ is constant, say c. He showed that they can be divided into three classes. (1) homogeneous normal contact Riemannian manifolds with $c > 0$. Other two classes can be seen in Tano [36].

In Grey and Harvella [14], the classification of almost Hermitian manifolds, there appears a class W_4 of Hermitian manifolds which are closely related to the conformal Kaehler manifolds. The class $C_6 \oplus C_5$ [25] coincides with the class of trans-Sasakian structure of type (α, β). In fact, the local nature of the two sub classes, namely C_6 and C_5 of trans-Sasakian structures are characterized completely. An almost conatct metric structure on M is called a trans-Sasakian (see [7], [22], [25]) if $(M \times R, J, G)$ belongs to the class W_4, where J is the almost complex structure on $M \times R$ defined by

$$J\left(X, f\frac{d}{dt}\right) = \left(\phi(X) - f\xi, \eta(X)\frac{d}{dt}\right)$$

for all vector fields X on M and smooth functions f on $M \times R$ and G is the product metric on $M \times R$. This may be expressed by the condition

$$(\nabla_X\phi)Y = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X) \tag{2.6}$$

for any vector fields X and Y on M, ∇ denotes the Levi-Civita connection with respect to g, α and β are smooth functions on M. The existence of condition (2.3) is ensure by the above discussion.

With the above literature now we define the δ-Lorentzian trans-Sasakian manifolds [28] as follows.

Definition 2.1. *A δ-Lorentzian manifold with structure $(\phi, \xi, \eta, g, \delta)$ is said to be δ-Lorentzian trans-Sasakian manifold of type (α, β) if it satisfies the condition*

$$(\nabla_X\phi)Y = \alpha(g(X, Y)\xi - \delta\eta(Y)X) + \beta(g(\phi X, Y)\xi - \delta\eta(Y)\phi X) \tag{2.7}$$

for any vector fields X and Y on M.

If $\delta = 1$, then the δ-Lorentzian trans Sasakian manifold is the usual Lorentzian trans Sasakian manifold of type (α, β) [25]. δ-Lorentzian trans Sasakian manifold of type $(0, 0)$, $(0, \beta)$ $(\alpha, 0)$ are the Lorentzian cosymplectic, Lorentzian β-Kenmotsu and Lorentzian α-Sasakian manifolds respectively. In particular if $\alpha = 1$, $\beta = 0$ and $\alpha = 0$, $\beta = 1$, the δ-Lorentzian trans Sasakian manifolds reduces to δ-Lorentzian Sasakian and δ-Lorentzian Kenmotsu manifolds respectively.

Form (2.4), we have

$$\nabla_X\xi = \delta\left\{-\alpha\phi(X) - \beta(X + \eta(X)\xi\right\}, \tag{2.8}$$

and

$$(\nabla_X \eta)Y = \alpha g(\phi X, Y) + \beta[g(X, Y) + \delta\eta(X)\eta(Y)]. \tag{2.9}$$

In a δ-Lorentzian trans Sasakian manifold M, we have the following relations:

$$R(X, Y)\xi = (\alpha^2 + \beta^2)[\eta(Y)X - \eta(X)Y] + 2\alpha\beta[\eta(Y)\phi X - \eta(X)\phi Y] \tag{2.10}$$
$$+ \delta[(Y\alpha)\phi X - (X\alpha)\phi Y + (Y\beta)\phi^2 X - (X\beta)\phi^2 Y],$$
$$R(\xi, Y)X = (\alpha^2 + \beta^2)[\delta g(X, Y)\xi - \eta(X)Y]$$
$$+\delta(X\alpha)\phi Y + \delta g(\phi X, Y)(grad\alpha)$$
$$+\delta(X\beta)(Y + \eta(Y)\xi) - \delta g(\phi Y, \phi X))(grad\beta)$$
$$+2\alpha\beta[\delta g(\phi X, Y)\xi + \eta(X)\phi Y],$$

$$\eta(R(X, Y)Z) = \delta(\alpha^2 + \beta^2)[\eta(X)g(Y, Z) - \eta(Y)g(X, Z) \tag{2.11}$$
$$+2\delta\alpha\beta[-\eta(X)g(\phi Y, Z) + \eta(Y)g(\phi X, Z)]$$
$$-[(Y\alpha)g(\phi X, Z) + (X\alpha)g(Y, \phi Z)]$$
$$-(Y\beta)g(\phi^2 X, Z) + (X\beta)g(\phi^2 Y, Z)],$$

$$S(X, \xi) = [((n-1)(\alpha^2 + \beta^2) - (\xi\beta)]\eta(X) + \delta((\phi X)\alpha) + (n-2)\delta(X\beta), \tag{2.12}$$

$$S(\xi, \xi) = (n-1)(\alpha^2 + \beta^2) - \delta(n-1)(\xi\beta), \tag{2.13}$$

$$Q\xi = (\delta(n-1)(\alpha^2 + \beta^2) - (\xi\beta))\xi + \delta\phi(grad\alpha) - \delta(n-2)(grad\beta), \tag{2.14}$$

where R is curvature tensor, while Q is the Ricci operator given by $S(X, Y) = g(QX, Y)$. Further in an δ-Lorentzian trans Sasakian manifold , we have

$$\delta\phi(grad\alpha) = \delta(n-2)(grad\beta), \tag{2.15}$$

and

$$2\alpha\beta - \delta(\xi\alpha) = 0. \tag{2.16}$$

The ξ-sectional curvature K_ξ of M is the sectional curvature of the plane spanned by ξ and a unit vector field X. From (2.11), we have

$$K_\xi = g(R(\xi, X), \xi, X) = (\alpha^2 + \beta^2) - \delta(\xi\beta). \tag{2.17}$$

It follows from (2.17) that ξ-sectional curvature does not depend on X. From (2.11)

$$g(R(\xi, Y)Z, \xi) = [(\alpha^2 + \beta^2) - \delta(\xi\beta)]g(Y, Z) \tag{2.18}$$
$$+[(\xi\beta) - \delta(\alpha^2 + \beta^2)]\eta(Y)\eta(Z) + [2\alpha\beta + \delta(\delta\alpha)]g(\phi Y, Z),$$

$$C(X, Y)Z = R(X, Y)Z - \frac{1}{(n-2)}[S(Y, Z)X - S(X, Z)Y \tag{2.19}$$
$$+g(Y, Z)QX - g(X, Z)QY] + \frac{r}{(n-1)(n-2)}[g(Y, Z)X - g(X, Z)Y],$$

An affine connection $\bar{\nabla}$ in M i called semi-symmetric connection [13], it its torsion tensor satisfies the following relations

$$\bar{T}(X,Y) = \bar{\nabla}_X Y - \bar{\nabla}_Y X - [X,Y], \tag{2.20}$$

and

$$\bar{T}(X,Y) = \eta(X)Y - \eta(Y)X. \tag{2.21}$$

Moreover, a semi-symmetric connection is called semi-symmetric metric connection if

$$(\bar{g})(X,Y) = 0. \tag{2.22}$$

If ∇ is metric connection and $\bar{\nabla}$ is the semi-symmetric metric connection with non-vanishing torsion tensor T in M, then we have

$$T(X,Y) = \eta * Y)X - \eta(X)Y, \tag{2.23}$$

$$\bar{\nabla}_X Y - \nabla_X Y = \frac{1}{2}[T(X,Y) + T^{'}(X,Y) + T^{'}(X,Y)], \tag{2.24}$$

where

$$g(T(Z,X),Y) = g(T^{'}(X,Y),Z). \tag{2.25}$$

By using (2.4), (2.23) and (2.25), we get

$$g(T^{'}(X,Y),Z) = g(\eta(X)Z - \eta(Z)X,Y),$$
$$g(T^{'}(X,Y),Z) = \eta(X)g(Z,Y) - \delta g(X,Y)g(\xi,Z),$$

$$T^{'}(X,Y) = \eta(X)Y - \delta g(X,Y)\xi, \tag{2.26}$$

$$T^{'}(Y,X) = \eta(Y)X - \delta g(X,Y)\xi. \tag{2.27}$$

From (2.23), (2.24),(2.26) and (2.27), we get

$$\bar{\nabla}_X Y = \nabla_X Y + \eta(Y)X - \delta g(X,Y)\xi.$$

Let M be an-n-dimensional δ-Lorentzian trans-Sasakian manifold and ∇ be the metric connection on M. The relation between the semi-symmetric metric connection $\bar{\nabla}$ and the metric connection ∇ on M is given by

$$\bar{\nabla}_X Y = \nabla_X Y + \eta(Y)X - \delta g(X,Y)\xi. \tag{2.28}$$

3. Curvature tensor on δ-Lorentzian trans-Sasakian mnaifold with semi-symmetric metric connection

Let M be an n-dimensional δ-Lorentzian trans-Sasakian manifold. The curvature tensor $\bar{R}$ of M with respect to the semi-symmetric metric connection $\bar{\nabla}$ is defined by

$$\bar{R}(X,Y)Z = \bar{\nabla}_X \bar{\nabla}_Y Z - \bar{\nabla}_Y \bar{\nabla}_X Z - \bar{\nabla}_{[X,Y]} Z. \tag{3.1}$$

By using (2.4), (2.4), (2.28) and (3,1), we get

$$\begin{aligned} \bar{R}(X,Y)Z = R(X,Y)Z + (\delta)[g(X,Z)Y - g(Y,Z)X] \\ +(\beta+\delta)[g(Y,Z)\eta(X) - g(X,Z)\eta(Y)]\xi \\ -(\beta\delta - 1)[\eta(Y)X - \eta(X)Y]\eta(Z), \end{aligned} \tag{3.2}$$

$$+\alpha[g(\phi X, Z)Y - g(\phi Y, Z)X - g(X, Z)\phi Y + g(Y, Z)\phi X],$$

where

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

is the Riemannian curvature tensor of connection ∇.

Lemma 3.1. *Let M be an n-dimensional δ-Lorentzian trans-Sasakian manifold with a semi-symmetric metric connection, then*

$$(\bar{\nabla}_X \phi)(Y) = \alpha(g(\phi X, Y)\xi - \delta\eta(Y)X + \beta(g(\phi X, Y)\xi - (\delta\beta + \delta)\eta(Y)\phi X, \tag{3.3}$$

$$\bar{\nabla}_X \xi = -(1 + \delta\beta)X - (1 + \delta\beta)\eta(X)\xi - \delta\alpha\phi X, \tag{3.4}$$

$$(\bar{\nabla}_X \eta)Y = \alpha g(\phi X, Y) + (\beta + \delta)g(X, Y) - (1 + \beta\delta)\eta(X)\eta(Y). \tag{3.5}$$

Proof. By the covariant differentiation of ϕY with respect to X, we have

$$\bar{\nabla}_X \phi Y = (\bar{\nabla}_X \phi) + \phi(\bar{\nabla}_X Y).$$

By using (2.1) and (2.28), we have

$$(\bar{\nabla}_X \phi)Y = (\nabla_X \phi)Y - \eta(Y)\phi X.$$

In view of (2.7), the last equation gives

$$(\bar{\nabla}_X \phi)(Y) = \alpha(g(\phi X, Y)\xi - \delta\eta(Y)X + \beta(g(\phi X, Y)\xi - (\delta\beta + \delta)\eta(Y)\phi X.$$

To prove (3.4), we replace $Y = \xi$ in (2.28) and we have

$$\bar{\nabla}_X \xi = \nabla_X \xi + \eta(\xi)X - \delta g(X, \xi)\xi.$$

By using (2.2), (2.4) and (2.8), the above equation gives

$$\bar{\nabla}_X \xi = -(1 + \delta\beta)X - (1 + \delta\beta)\eta(X)\xi - \delta\alpha\phi X.$$

In order to prove (3.5), we differentiate $\eta(Y)$ covariantly with respect to X and using (2.28), we have

$$\bar{\nabla}_X \eta(Y) = (\nabla_X \eta)Y + g(X, Y) - \eta(X)\eta(Y).$$

Using (2.9) in above equation, we get

$$(\bar{\nabla}_X \eta)Y = \alpha g(\phi X, Y) + (\beta + \delta)g(X, Y) - (1 + \beta\delta)\eta(X)\eta(Y).$$

∎

Lemma 3.2. *Let M be an n-dimensional δ-Lorentzian trans-Sasakian manifold with a semi-symmetric metric connection, then*

$$\bar{R}(X, Y)\xi = (\alpha^2 + \beta^2 - \delta\beta)[\eta(X)Y - \eta(Y)X]. \tag{3.6}$$

$$+(2\alpha\beta + \delta\alpha)[\eta(Y)\phi X - \eta(X)\phi Y]$$

$$+\delta[(Y\alpha)\phi X - (-X\alpha)\phi Y - (X\beta)\phi^2 Y + (Y\beta)\phi^2 X].$$

Proof. By replacing $Z = \xi$ in (3.2), we have

$$\bar{R}(X,Y)\xi = R(X,Y)\xi + (\delta)[g(X,\xi)Y - g(Y,\xi)X]$$
$$+(\beta+\delta)[g(Y,\xi)\eta(X) - g(X,\xi)\eta(Y)]\xi$$
$$-(\beta\delta-1)[\eta(Y)X - \eta(X)Y]\eta(\xi).$$
$$+\alpha[g(\phi X,\xi)Y - g(\phi Y,\xi)X - g(X,\xi)\phi Y + g(Y,\xi)\phi X]$$

In view of (2.2), (2.4) and (2.10), the above equation reduces to

$$\bar{R}(X,Y)\xi = (\alpha^2+\beta^2-\delta\beta)[\eta(X)Y - \eta(Y)X].$$
$$+(2\alpha\beta+\delta\alpha)[\eta(Y)\phi X - \eta(X)\phi Y]$$
$$+\delta[(Y\alpha)\phi X - (X\alpha)\phi Y - (X\beta)\Phi^2 Y + (Y\beta)\phi^2 X].$$

∎

Remark 1. Replace $Y = \xi$ and using (3.2), (2.11), (2.2) and (2.4), we obtain

$$\bar{R}(X,\xi)\xi = (\alpha^2+\beta^2-\delta\beta)[-X - \eta(X)Y]. \tag{3.7}$$
$$+(2\alpha\beta+\delta\alpha+\delta(\xi\alpha))[\phi X + \delta(\xi\beta)\phi^2 X].$$

Remark 2. Now, again replace $X = \xi$ in (3.6), using (2.1), (2.2) and (2.4), we obtain

$$\bar{R}(\xi,Y)\xi = (\alpha^2+\beta^2-\delta\beta)[-\eta(Y)\xi - Y]. \tag{3.8}$$
$$-(2\alpha\beta+\delta\alpha+\delta(\xi\alpha))[\phi Y - \delta(\xi\beta)\phi^2 Y].$$

Remark 3. Replace $Y = X$ in (3.8), we get

$$\bar{R}(\xi,X)\xi = -(\alpha^2+\beta^2-\delta\beta)[-X - \eta(X)\xi]. \tag{3.9}$$
$$-(2\alpha\beta+\delta\alpha+\delta(\xi\alpha))[\phi X - \delta(\xi\beta)\phi^2 X].$$

From (3.7) and (3.10), we obtain

$$\bar{R}(X,\xi)\xi = -\bar{R}(\xi,X)\xi. \tag{3.10}$$

Now, contracting X in (3.2), we get

$$\bar{S}(Y,Z) = S(Y,Z) - [(\delta)(n-2)+\beta]g(Y,Z) \tag{3.11}$$
$$-(\beta\delta-1)(n-2)\eta(Z)\eta(Y) - \alpha(n-2)g(\phi Y,Z),$$

where $\bar{S}$ and S are the Ricci tensors of the connections $\bar{\nabla}$ and ∇, respectively on M. This gives

$$\bar{Q}Y = QY - [(\delta)(n-2)+\beta]Y \tag{3.12}$$
$$-(\beta\delta-1)(n-2)\eta(Y)\xi - \alpha(n-2)\phi Y,$$

where $\bar{Q}$ and Q are Ricci operator with respect to the semi-symmetric metric connection and metric connection respectively and define as $g(\bar{Q}Y,Z) = \bar{S}(Y,Z)$ and $g(QY,Z) = S(Y,Z)$ respectively.

Replace $Y = \xi$ in (3.12) and using (2.15), we get

$$\bar{Q}\xi = \delta(n-1)(\alpha^2+\beta^2)\xi - (\xi\beta)\xi - 2\delta(n-2)\xi \tag{3.13}$$
$$+\delta\phi(grad\alpha) - \delta(n-2)(grad\beta) - \beta(n-1)\xi.$$

Putting $Y = Z = e_i$ and taking summation over i, $1 \leq i \leq n-1$ in (3.11), using (2.14) and also the relations $r = S(e_i,e_i) = \sum_{i=1}^{n} \delta_i R(e_i,e_i,e_i,e_i)$, we get

$$\bar{r} = r - (n-1)[(\delta)(n-2)+2\beta], \tag{3.14}$$

where $\bar{r}$ and r are the scalar curvatures of the connections $\bar{\nabla}$ and ∇, respectively on M.
Now, we have the following lemmas.

Lemma 3.3. *Let M be an n-dimensional δ-Lorentzian trans-Sasakian manifold with the semi-symmetric metric connection, then*

$$\bar{S}(\phi Y, Z) = -\delta(\phi^2 Y)\alpha - \delta(n-2)(\phi Y)\beta - \alpha(n-2)g(\phi Y, \phi Z), \tag{3.15}$$

$$\bar{S}(Y, \xi) = [(n-1)(\alpha^2 + \beta^2 - \delta(\xi\beta) - \delta\beta(n-1)]\eta(Y) \tag{3.16}$$

$$+\delta(n-2)(Y\beta) + \delta(\phi Y)\beta,$$

$$\bar{S}(\xi, \xi) = [(n-1)(\alpha^2 + \beta^2 - \delta(\xi\beta) - \delta\beta(n-1)]\eta(Y). \tag{3.17}$$

Proof. By replacing $Y = \phi Y$ in equation (3.11) and using (2.13) and (2.5), we have (3.15). Taking $Y = \xi$ in (3.11) and using (2.13) we get (3.16). (3.17) follows from considering $Y = \xi$ in (3.16) we get (3.17). ∎

Lemma 3.4. *Let M be an n-dimensional δ-Lorentzian trans-Sasakian manifold with the semi-symmetric metric connection, then*

$$\bar{S}(grad\alpha, \xi) = \delta(n-1)(\alpha^2 + \beta^2(\xi\beta) - \beta(n-1)(\xi\alpha) - (\xi\alpha)(\xi\beta) \tag{3.18}$$

$$+\delta(\phi grad\alpha)\alpha + \delta(n-2)g(grad\alpha, grad\beta),$$

$$\bar{S}(grad\beta, \xi) = \delta(n-1)(\alpha^2 + \beta^2(\xi\beta) - \beta(n-1)(\xi\beta) - (\xi\beta)^2 \tag{3.19}$$

$$+\delta(\phi grad\beta)\alpha + \delta(n-2)g(grad\beta)^2.$$

Proof. From equation (3.11) and (3.16) and using $Y = grad\alpha$ we have (3.18) . Similarly taking $\xi = grad\beta$ in (3.11) and using (3.16), we get (3.19).
∎

Using (3.6), (3.13) and (3.16), for constant α and β, we have

$$\bar{R}(X, Y)\xi = (\alpha^2 + \beta^2 - \delta(\xi\beta)[\eta(Y)X - \eta(X)Y], \tag{3.20}$$

$$\bar{S}(X, Y) = [(n-1)(\alpha^2 + \beta^2 - \delta(\xi\beta) - \delta\beta(n-1)]\eta(Y), \tag{3.21}$$

$$\bar{Q}X = \delta(n-1)(\alpha^2 + \beta^2\xi - \delta(\xi\beta)\xi - 2\delta(n-2) - \beta(n-1)\xi. \tag{3.22}$$

4. Quasi-projectively flat δ-Lorentzian trans-Sasakian manifold with semi-symmetric metric connection

Let M be an n-dimensional δ-Lorentzian trans-Sasakian manifold. If there exists a one to one correspondence between each co-ordinate neighborhood of M and a domain in Euclidean space such that any geodesic of δ-Lorentzian trans-Sasakian manifold corresponds to a straight line in the Euclidean space, then M is said to be locally projectively flat. The projective curvature tensor $\bar{P}$ with respect to semi-symmetric metric connection is defined by

$$\bar{P}(X,Y)Z = \bar{R}(X,Y)Z - \frac{1}{(n-1)}[\bar{S}(Y,Z)X - \bar{S}(X,Z)Y]. \tag{4.1}$$

Definition 4.1. *A δ-Lorentzian trans-Sasakian manifold M is said to be quasi-projectively flat with respect to semi-symmetric metric connection, if*

$$g(\bar{P}(\phi X,Y)Z,\phi U) = 0, \tag{4.2}$$

where $\bar{P}$ is the projective curvature tensor with respect to semi-symmetric metric connection.

Now, from (4.1) taking inner product with U, we get

$$g(\bar{P}(X,Y)Z,U) = g(\bar{R}(X,Y)Z,U) - \frac{1}{(n-1)} [\bar{S}(Y,Z)g(X,U) - \bar{S}(X,Z)g(Y,U)]. \tag{4.3}$$

Replace $X = \phi X$ and $U = \phi U$ in (4.3), we get

$$g(\bar{P}(\phi X,Y)Z,\phi U) = g(\bar{R}(\phi X,Y)Z,\phi U) - \frac{1}{(n-1)} [\bar{S}(Y,Z)g(\phi X,\phi U) - \bar{S}(\phi X,Z)g(Y,\phi U)]. \tag{4.4}$$

From (4.2) and (4.4), we have

$$g(\bar{R}(\phi X,Y)Z,\phi U) = \frac{1}{(n-1)}[\bar{S}(Y,Z)g(\phi X,\phi U) - \bar{S}(\phi X,Z)g(Y,\phi U)].$$

Now, using equations (2.1), (2.4), (3.11) and (3.15) in equation (4.5), we have

$$\begin{aligned} g(\bar{R}(\phi X,Y)Z,\phi U) &= \frac{1}{(n-1)}[\bar{S}(Y,Z)g(\phi X,\phi U) - \bar{S}(\phi X,Z)g(Y,\phi U)] \\ &- \frac{(\delta+\beta)}{(n-1)}g(\phi X,Z)g(Y,\phi U) + \frac{(\delta+\beta)}{(n-1)}g(Y,Z)g(\phi X,\phi U) \\ &- \frac{(\delta\beta-1)}{(n-1)}\eta(Y)\eta(Z)g(\phi X,\phi U) + \frac{(\delta\alpha)}{(n-1)}\eta(X)\eta(Z)g(\phi X,\phi U) \\ &- \frac{\alpha}{(n-1)}g(X,Z)g(Y,\phi U) - \frac{\alpha}{(n-1)}g(\phi Y,Z)g(\phi X,\phi U) \\ &+ \alpha g(Y,Z)g(X,\phi U) + \alpha g(\phi X,Z)g(\phi X,\phi U). \end{aligned}$$

Let $\{e_1, e_2 ....e_{n-1}, \xi\}$ be a local orthonormal basis of vector fileds on δ-Lorentzian trans-Sasakian manifold M, then $\{\phi e_1, \phi e_2 ....\phi e_{n-1}, \xi\}$ is also a local orthonormal basis of vector fields on δ-Lorentzian trans-Sasakian manifold M. Now putting $X = U = e_i$ in equation (4.6) and using (2.2), (2.3),(2.19), (3.11) and (3.16), we have

$$\begin{aligned} S(Y,Z) = {} & [(n-2)(\beta+\delta) + \delta(n-1)(\alpha^2+\beta^2) - (n-1)(\xi\beta)]g(Y,Z) \\ & +[\delta(n-2)(\xi\beta) + (n-2)(\beta\delta-1)]\eta(Y)\eta(Z) \\ & -[2\delta(n-1)\alpha\beta + (n-1)(\xi\alpha) - \alpha]g(\phi Y,Z) \\ & -\delta\eta(Y)(\phi Z)\alpha - \delta(n-2)(\xi\beta)\eta(Y). \end{aligned} \tag{4.5}$$

If $\alpha = 0$ and $\beta = constant$ in (4.7), we get

$$S(Y,Z) = [(n-2)(\beta+\delta) + (n-1)\delta\beta^2]g(Y,Z) + (\beta\delta-1)(2-n)\eta(Y)\eta(Z). \tag{4.6}$$

Therefor, we have

$$S(Y,Z) = ag(Y,Z) + b\eta(Y)\eta(Z),$$

where $a = (n-2)(\beta+\delta) + (n-1)\delta\beta^2$ and $b = (\beta\delta-1)(2-n)$.
This results shows that the manifold under the consideration is an η-Einstein manifold. Thus we can state the following theorem:

Theorem 4.2. *An n-dimensional quasi projectively flat δ-Lorentzian trans-Sasakian manifold M with respect to a semi-symmetric metric connection is an η-Einstein manifold if $\alpha = 0$ and $\beta = constant$.*

5. ϕ-Projectively flat δ-Lorentzian trans-Sasakian manifold with semi-symmetric metric connection satisfying

An n-dimensional δ-Lorentzian trans-Sasakian manifold with semi-symmetric metric connection is said to be ϕ-projectively flat if

$$\phi^2(\bar{P}(\phi,X,\phi Y)\phi Z) = 0, \tag{5.1}$$

where $\bar{P}$ is the projective curvature tensor of M n-dimensional δ-Lorentzian trans-Sasakian manifold with respect to a semi-symmetric metric connection. Suppose M be ϕ-projectively flat δ-Lorentzian trans-Sasakian manifold with respect to a semi-symmetric metric connection. It is know that $\phi^2(\bar{P}(\phi,X,\phi Y)\phi Z) = 0$ holds if and only if

$$g(\bar{P}(\phi X,\phi Y)\phi Z,\phi U) = 0, \tag{5.2}$$

for any $X,Y,Z,U \in TM$. Replace $Y = \phi Y$ and $U\phi U$ in (4.4), we have

$$\begin{aligned} g(\bar{P}(\phi X,\phi Y)\phi Z,\phi U) = {} & g(\bar{R}(\phi X,\phi Y)\phi Z,\phi U) - \frac{1}{(n-1)} \\ & [\bar{S}(\phi Y,\phi Z)g(\phi X,\phi U) - \bar{S}(\phi X,\phi Z)g(\phi Y,\phi U)]. \end{aligned} \tag{5.3}$$

From (5.2) and (5.3), we have

$$\begin{aligned} g(\bar{R}(\phi X,\phi Y)\phi Z,\phi U) = {} & \frac{1}{(n-1)}[\bar{S}(\phi Y,\phi Z)g(\phi X,\phi U) \\ & -\bar{S}(\phi X,\phi Z)g(\phi Y,\phi U)]. \end{aligned} \tag{5.4}$$

Now, using (2.1),(2.2),(2.4),(2.5), (3.2) and (3.11) in equation (5.4), we have

$$g(\bar{R}(\phi X,\phi Y)\phi Z,\phi U) = \frac{1}{(n-1)}[\bar{S}(\phi Y,\phi Z)g(\phi X,\phi U) - \bar{S}(\phi X,\phi Z)g(\phi Y,\phi U)]$$

$$-\frac{(\delta+\beta)}{(n-1)}g(\phi Y,\phi Z)g(\phi X,\phi U)+\frac{(\delta+\beta)}{(n-1)}g(\phi X,\phi Z)g(\phi Y,\phi U)$$
$$-\frac{\alpha}{(n-1)}g(Y,\phi Z)g(\phi X,\phi U)-\frac{\alpha}{(n-1)}g(X,\phi YZ)g(\phi X,\phi U)$$
$$+\alpha g(\phi Y,\phi Z)g(X,\phi U)-\alpha g(\phi X,\phi Z)g(Y,\phi U).$$

Let $\{e_1, e_2 \dots e_{n-1}, \xi\}$ be a local orthonormal basis of vector fileds on δ-Lorentzian trans-Sasakian manifold M, then $\{\phi e_1, \phi e_2 .... \phi e_{n-1}, \xi\}$ is also a local orthonormal basis of vector fields on δ-Lorentzian trans-Sasakian manifold M. Now putting $X = U = e_i$ in equation (5.5) and using (2.1)–(2.5), (2.19), (3.11) and (3.16), we have

$$\begin{aligned}S(Y,Z) = {} & [(n-2)(\beta+\delta)+\delta(n-1)(\alpha^2+\beta^2)-(n-1)(\xi\beta)]g(Y,Z)\\ & +[2\delta(n-2)(\xi\beta)+(n-2)(\beta\delta-1)]\eta(Y)\eta(Z)\\ & +[\alpha-2\delta\alpha\beta(n-1)-(n-1)(\xi\alpha)]g(\phi Y,Z)\\ & -[\delta(\phi Z)\alpha+\delta(n-2)(Z\beta)]\eta(Y)-[\delta(\phi Y)\alpha+\delta(n-2)(Y\beta)]\eta(Z),\end{aligned}$$

If $\alpha = 0$ and $\beta = constant$ in (5.6), we get

$$S(Y,Z) = [(n-2)(\beta+\delta)+(n-1)\delta\beta^2]g(Y,Z)+(\beta\delta-1)(2-n)\eta(Y)\eta(Z). \quad (5.5)$$

Therefore,

$$S(Y,Z) = ag(Y,Z)+b\eta(Y)\eta(Z),$$

where $a = (n-2)(\beta+\delta)+(n-1)\delta\beta^2$ and $b = (\beta\delta-1)(2-n)$.
This results shows that the manifold under the consideration is an η-Einstein manifold. Thus we can state the following theorem:

Theorem 5.1. *An n-dimensional ϕ-projectively flat δ-Lorentzian trans-Sasakian manifold M with respect to a semi-symmetric metric connection is an η-Einstein manifold if $\alpha = 0$ and $\beta = constant$.*

6. δ-Lorentzian trans-Sasakian manifold with semi-symmetric metric connection satisfying $\bar{R}.\bar{S} = 0$

Now, suppose that M be an n-dimensional δ-Lorentzian trans-Sasakian manifold with semi-symmetric metric connection satisfying the condition:

$$\bar{R}(X,Y).\bar{S} = 0. \quad (6.1)$$

Then, we have

$$\bar{S}(\bar{R}(X,Y)Z,U)+\bar{S}(Z,\bar{R}(X,Y)U) = 0. \quad (6.2)$$

Now, we replace $X = \xi$ in equation (6.2), using equations (2.11) and (6.2), we have

$$\begin{aligned}&\delta(\alpha^2+\beta^2)g(Y,Z)\bar{S}(\xi,U)-(\alpha^2+\beta^2)\eta(Z)\bar{S}(Y,U)-2\delta\alpha\beta g(\phi Y,Z)\bar{S}(\xi,U) \quad (6.3)\\ &+2\alpha\beta\eta(Z)\bar{S}(\phi Y,U)+\delta(Z\alpha)\bar{S}(\phi Y,U)-\delta g(\phi Y,Z)\bar{S}(grad\alpha,U)\\ &-\delta g(\phi Y,\phi Z)\bar{S}(grad\beta,U)+\delta(Z\beta)\bar{S}(Y,U)-\delta(Z\beta)\eta(Y)\bar{S}(\xi,U)\\ &-\delta g(Y,Z)\bar{S}(\xi,U)+\delta\eta(Z)\bar{S}(Y,U)+\alpha g(\phi Y,Z)\bar{S}(\xi,U)-\delta\alpha\eta(Z)\bar{S}(\phi Y,U)\\ &+\delta(\alpha^2+\beta^2)g(Y,U)\bar{S}(\xi,Z)-(\alpha^2+\beta^2)\eta(U)\bar{S}(Y,Z)-2\delta\alpha\beta g(\phi Y,U)\bar{S}(\xi,Z)\\ &+2\alpha\beta\eta(U)\bar{S}(\phi Y,Z)+\delta(U\alpha)\bar{S}(\phi Y,Z)-\delta g(\phi Y,U)\bar{S}(grad\alpha,Z)\\ &-\delta g(\phi Y,\phi U)\bar{S}(grad\beta,Z)+\delta(U\beta)\bar{S}(Y,Z)-\delta(U\beta)\eta(Y)\bar{S}(\xi,Z)\\ &-\delta g(Y,U)\bar{S}(\xi,Z)+\delta\eta(U)\bar{S}(Y,Z)+\alpha g(\phi Y,U)\bar{S}(\xi,Z)-\delta\alpha\eta(U)\bar{S}(\phi Y,Z) = 0.\end{aligned}$$

Using equations (2.1)–(2.5), (2.13), (2.14), (3.11) and (3.15)–(3.19) in equation (6.3)

$$[(\alpha^2+\beta^2)-\delta(\xi\beta)-\delta\beta]S(Y,Z)$$

$$=[\delta(n-1)(\alpha^2+\beta^2)-2\beta(n-1)(\alpha^2+\beta^2)-2(n-1)(\alpha^2+\beta^2)(\xi\beta)$$

$$+2\delta\beta(n-1)(\xi\beta)-\delta(\xi\beta)^2+(\phi grad\beta)\alpha+(n-2)(grad\beta)^2$$

$$+\delta\beta^2(n-2)+\delta(n-2)(\alpha^2+\beta^2)+\beta(\alpha^2+\beta^2)$$

$$-2\alpha^2\beta(n-2)-\delta\alpha(\xi\alpha)-(n-2)(\xi\beta)-\delta\beta(\xi\beta)$$

$$-\beta(n-2)+\delta\alpha^2(n-2)]g(Y,Z)+[-\delta(\phi grad\beta)\alpha$$

$$-\delta(n-2)(grad\beta)^2+(n-2)(\beta\delta-1)(\alpha^2+\beta^2)$$

$$+2\delta\alpha^2\beta(n-2)+\alpha(n-2)(\xi\alpha)+(\beta+\delta)(n-2)(\xi\beta)$$

$$+\beta(\beta+\delta)(n-2)-\alpha^2(n-2)]\eta(Y)\eta(Z)+[-2\delta\alpha\beta(n-1)(\alpha^2+\beta^2)$$

$$+2(n-2)\alpha\beta^2+2\alpha\beta(n-2)(\xi\beta)-(n-1)(\alpha^2+\beta^2)(\xi\alpha)$$

$$+\delta\beta(n-2)(\xi\alpha)+\delta(\xi\alpha)(\xi\beta)+(\phi grad\alpha)\alpha+(n-2)(g(grad\alpha,grad\beta)$$

$$+\alpha(\alpha^2+\beta^2)-\delta\alpha(\xi\quad beta)-2\alpha\beta(n-2)(\delta)-(n-2)(\delta\alpha)+\alpha(n-2)]g(\phi Y,Z)$$

$$+[\delta(\xi\alpha)+2\alpha\beta-\delta\alpha]S(\phi Y,Z)+[(n-2)(\xi\beta)(Z\beta)$$

$$+[\delta(\alpha^2+\beta^2)(\phi Z)\alpha-\delta(n-2)(\alpha^2+\beta^2)(Z\beta)+(\xi\beta)(\phi Z)\alpha$$

$$\beta(\phi Z)\alpha+\beta(n-2)(Z\beta)]\eta(Y)+[\delta(\alpha^2+\beta^2)(\phi Y)\alpha+\delta(n-2)(\alpha^2+\beta^2)(Y\beta)$$

$$-2\delta\alpha\beta(\phi^2Y)\alpha-2\delta\alpha\beta(n-2)(\phi Y\beta)-\beta(\phi Y)\alpha$$

$$-\beta(n-2)(Y\beta)+\alpha(\phi^2Y)\alpha+\alpha(n-2)(\phi Y\beta)]\eta(Z)$$

$$-(n-2)(Y\beta)(Z\beta)+(n-2)(Z\beta)(\xi\beta).$$

If $\alpha=0$ and $\beta=constant$ in (5.6), we get

$$S(Y,Z)=ag(Y,Z)+b\eta(Y)\eta(Z),$$

where $a=-[\frac{(n-1)\delta\beta^4+(n-2)(grad\beta)^2+(n-2)\delta\beta^2+(n-2)\delta\beta^2-(n-2)\beta+(2n-3)\beta^3}{(\beta+\delta)\beta}]$ and $b=-[\frac{(n-2)(\beta\delta-1)\beta^2+(n-2)(\beta+\delta)\beta-(n-2)\delta(grad\beta^2}{(\beta+\delta)\beta}]$. This show that M is an η-Einstein manifold. Thus,we can state the following theorem:

Theorem 6.1. *An n-dimensional δ-Lorentzian trans-Sasakian manifold M with respect to a semi-symmetric metric connection $\bar{\nabla}$ satisfies $\bar{R}.\bar{S}=0$, then δ-Lorentzian trans-Sasakian manifold M is an η-Einstein manifold if $\alpha=0$ and $\beta=constant$.*

7. δ-Lorentzian trans-Sasakian manifold with semi-symmetric metric connection satisfying $\bar{P}.\bar{S} = 0$

Now, we consider δ-Lorentzian trans-Sasakian manifold with semi-symmetric metric connection satisfying

$$(\bar{P}(X,Y).\bar{S})(Z,U) = 0. \tag{7.1}$$

where $\bar{P}$ is the projective curvature tensor and $\bar{S}$ is the Ricci tensor with semi-symmetric metric connection.Then, we have

$$\bar{S}(\bar{P}(X,Y)Z,U) + \bar{S}(Z,\bar{P}(X,Y)U) = 0. \tag{7.2}$$

Replace $X = \xi$ in the equation (7.2), we get

$$\bar{S}(\bar{P}(\xi,Y)Z,U) + \bar{S}(Z,\bar{P}(\xi,Y)U) = 0. \tag{7.3}$$

Putting $X = \xi$ in (4.1), we get

$$\bar{P}(\xi,Y)Z = \bar{R}(\xi,Y)Z - \frac{1}{(n-1)}[\bar{S}(Y,Z)\xi - \bar{S}(\xi,Z)Y]. \tag{7.4}$$

Using (2.1), (2.2), (2.4), (2.11), (3.2), (3.11), (3.17) and (7.4) in (7.3), we get

$$\begin{aligned}
&\frac{\delta(\alpha^2+\beta^2)(n-1)+(\beta+\delta)(n-2)}{(n-1)}g(Y,Z)\bar{S}(\xi,U) - \frac{1}{(n-1)}S(Y,Z)\bar{S}(\xi,U) \quad (7.5)\\
&-\frac{(n-2)}{(n-1)}(\beta\delta-1)\eta(Y)\eta(Z)\bar{S}(\xi,U) + \frac{\alpha-2\delta\alpha\beta(n-1)}{(n-1)}g(\phi Y,Z)\bar{S}(\xi,U)\\
&-\delta g(\phi Y,Z)\bar{S}(grad\alpha,U) - \delta g(\phi Y,\phi Z)\bar{S}(grad\beta,U) + 2\alpha\beta\eta(Z)\bar{S}(\phi Y,U)\\
&+\delta(Z\alpha)\bar{S}(\phi Y,U) + \delta(Z\beta)\bar{S}(Y,U) - \delta(Z\beta)\eta(Y)\bar{S}(\xi,U) - \delta\alpha\eta(Z)\bar{S}(\phi Y,U)\\
&-\frac{1}{(n-1)}\delta(\xi\beta)\eta(Z)\bar{S}(Y,U)\frac{(n-2)}{(n-1)}\delta(Z\beta)\bar{S}(Y,U) - \frac{1}{(n-1)}\delta(\phi Z)\alpha\bar{S}(Y,U)\\
&\frac{\delta(\alpha^2+\beta^2)(n-1)+(\beta+\delta)(n-2)}{(n-1)}g(Y,U)\bar{S}(\xi,Z) - \frac{1}{(n-1)}S(Y,U)\bar{S}(\xi,Z)\\
&-\frac{(n-2)}{(n-1)}(\beta\delta-1)\eta(Y)\eta(U)\bar{S}(\xi,Z) + \frac{\alpha-2\delta\alpha\beta(n-1)}{(n-1)}g(\phi Y,U)\bar{S}(\xi,Z)\\
&-\delta g(\phi Y,U)\bar{S}(grad\alpha,Z) - \delta g(\phi Y,\phi U)\bar{S}(grad\beta,Z) + 2\alpha\beta\eta(U)\bar{S}(\phi Y,Z)\\
&+\delta(U\alpha)\bar{S}(\phi Y,Z) + \delta(Z\beta)\bar{S}(Y,Z) - \delta(U\beta)\eta(Y)\bar{S}(\xi,Z) - \delta\alpha\eta(U)\bar{S}(\phi Y,Z)\\
&-\frac{1}{(n-1)}\delta(\xi\beta)\eta(Z)\bar{S}(Y,Z)\frac{(n-2)}{(n-1)}\delta(U\beta)\bar{S}(Y,Z) - \frac{1}{(n-1)}\delta(\phi U)\alpha\bar{S}(Y,Z) = 0
\end{aligned}$$

Putting $U = \xi$ and Using (2.1)–(2.5), (3.11)and (3.15)–(3.20) in (7.5), we get

$$\begin{aligned}
&[(\alpha^2+\beta^2) - \delta(\xi\beta) - \delta\beta]S(Y,Z) \quad (7.6)\\
&= [\delta(n-1)(\alpha^2+\beta^2) + (n-2)(\beta\delta)(\alpha^2+\beta^2) - \beta(n-1)(\alpha^2+\beta^2)\\
&-\delta(n-2)(\beta\delta-1) - 2(n-1)(\xi\beta)(\alpha^2+\beta^2) - (n-2)(\beta\delta-1)(\xi\beta)\\
&2\alpha^2\beta(n-2)\delta\alpha(n-2)(\xi\alpha) + \delta\alpha^2(n-2) + \delta\beta(n-1) + \delta(\xi\beta)^2\\
&+(\phi grad\alpha)\alpha + (n-2)(grad\beta)^2]g(Y,Z) + [(n-2)\beta(\beta+\delta) - (n-2)(\alpha^2+\beta^2)\\
&+2(n-2)\delta\alpha^2\beta + \alpha(n-2)(\xi\alpha) + (n-2)(\beta+\delta)(\xi\beta) - \alpha^2(n-2)\\
&-\delta(n-2)(grad\beta)^2 - \delta(\phi grad\beta)\alpha]\eta(Y)\eta(Z) + [\alpha(\alpha^2+\beta^2)\\
&-2\delta\alpha\beta(\alpha^2+\beta^2)(n-1) - 2\alpha\beta^2 n - \delta(\xi\beta) - \delta\beta(\xi\alpha) + 2\alpha\beta(\xi\beta)
\end{aligned}$$

$$-2\delta\alpha\beta(n-2)-(n-1)(\xi\alpha)+\alpha(n-2)-(n-1)(\alpha^2+\beta^2)(\xi\alpha)+(n-1)\delta\beta(\xi\alpha)$$
$$+\delta(\xi\alpha)(\xi\beta)+(\phi grad\alpha)\alpha+)n-2)g(grad\alpha,grad\beta)]g(\phi Y,z)+[\delta\alpha+\delta(\xi\alpha)-\delta\alpha]S(\phi Y,Z)$$
$$+[\delta(n+3)(\alpha^2+\beta^2)(Z\beta)+\beta(n-2)(Z\beta)-delta(\alpha^2+\beta^2)(\phi Z)\alpha$$
$$+(n-1)\beta(\phi Z)\alpha+(\xi\beta)(\phi Z)\alpha)]\eta(Y)+[-2\delta\alpha\beta(\phi^2Y)\alpha-2\delta\alpha\beta(n-2)(\phi Y\beta)$$
$$+\alpha(\phi^2Y)\alpha+\alpha(n-2)(\phi Y\beta)+\delta(\alpha^2+\beta^2)(\phi Y)\alpha+\delta(n-2)(\alpha^2+\beta^2)(Y\beta)$$
$$-\beta(\phi Y)\alpha-\beta(n-2)(Y\beta)]\eta(Z)$$
$$-(Z\alpha)(\phi^2Y)\alpha-(n-2)(Z\beta)(\phi Y\beta)-(Z\beta)(\phi Y)\alpha-\beta(n-2)(Y\beta).$$

If $\alpha=0$ and $\beta=constant$ in (7.6), we get

$$S(Y,Z)=ag(Y,Z)+b\eta(Y)\eta(Z), \tag{7.7}$$

where $a=-[\frac{(n-1)\beta^4+(n-2)\beta^2(\beta\delta)+(n-1)\beta^3-(n-2)\beta(\beta\delta-1)+(n-1)\delta\beta+(n-2)(grad\beta)^2}{\beta(\beta\delta)}]$
and
$b=-[\frac{(n-2)\beta(\beta+\delta)+(n-2)\beta^2-(n-2)\delta(grad\beta)^2}{\beta(\beta+\delta)}]$.

This result showw that the manifold under the consideration is an η-Einstein manifold. Thus we have the following theorem:

Theorem 7.1. *An n-dimensional δ-Lorentzian trans-Sasakian manifold M with respect to a semi-symmetric metric connection $\bar{\nabla}$ satisfies $\bar{P}.\bar{S}=0$, then δ-Lorentzian trans-Sasakian manifold M is an η-Einstein manifold if $\alpha=0$ and $\beta=constant$.*

8. Weyl conformal curvature tensor on δ-Lorentzian trans-Sasakian manifold with semi-symmetric metric connection

The Weyl conformal curvature tensor $\bar{C}$ of type $(1,3)$ of M an n-dimensional δ-Lorentzian trans-Sasakian manifold with semi-symmetric metric connection $\bar{\nabla}$ is given by [42]

$$\bar{C}(X,Y)Z=\bar{R}(X,Y)Z-\frac{1}{(n-2)}[\bar{S}(Y,Z)X-\bar{S}(X,Z)Y \tag{8.1}$$
$$+g(Y,Z)\bar{Q}X-g(X,Z)\bar{Q}Y]+\frac{\bar{r}}{(n-1)(n-2)}[g(Y,Z)X-g(X,Z)Y],$$

where $\bar{Q}$ is the Ricci operator with respect to the semi-symmetric metric connection $\bar{\nabla}$. Let M ba an n-dimensional δ-Lorentzian trans-Sasakian manifold. The Weyl conformal curvature tensor $\bar{C}$ of M with respect to the semi-symmetric metric connection $\bar{\nabla}$ is defined in equation (8.1).

Now, taking inner product with U in (8.1), we get

$$g(\bar{C}(X,Y)Z,U)=g(\bar{R}(X,Y)Z,U)-\frac{1}{(n-2)}[\bar{S}(Y,Z)g(X,U)-\bar{S}(X,Z)g(Y,U) \tag{8.2}$$
$$+g(Y,Z)g(\bar{Q}X,U)-g(X,Z)g(\bar{Q}Y,U)]+\frac{\bar{r}}{(n-1)(n-2)}$$
$$[g(Y,Z)g(X,U)-g(X,Z)g(Y,U)].$$

Using (2.4), (3.2), (3.11), (3.12) and (3.14) in (8.2), we get

$$\bar{C}(X,Y,Z,U) = g(\bar{R}(X,Y)Z,U) - \frac{1}{(n-2)}[S(Y,Z)g(X,U) - S(X,Z)g(Y,U) \quad (8.3)$$
$$+g(Y,Z)g(QX,U) - g(X,Z)g(QY,U)] + \frac{r}{(n-1)(n-2)}$$
$$[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)],$$

where $g(\bar{C}(X,Y)Z,U) = \bar{C}(X,Y,Z,U)$ and $R(X,Y)Z,U) = C(X,Y,Z,U)$ are Weyl curvature tensor with respect to semi-symmetric metric connection respectively, we have

$$\bar{C}(X,Y,Z,U) = C(X,Y,Z,U), \quad (8.4)$$

where

$$C(X,Y,Z,U) = g(\bar{R}(X,Y)Z,U) - \frac{1}{(n-2)}[S(Y,Z)g(X,U) - S(X,Z)g(Y,U) \quad (8.5)$$
$$+g(Y,Z)g(QX,U) - g(X,Z)g(QY,U)] + \frac{r}{(n-1)(n-2)}$$
$$[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)],$$

Theorem 8.1. *The Weyl conformal curvature tensor of a δ-Lorentzian trans-Sasakian manifold M with respect to a metric connection is equal to the Weyl curvature of with respect to the semi-symmetric metric connection.*

9. δ-Lorentzian trans-Sasakian manifold with Weyl conformal flat conditions with semi-symmetric metric connection

Let us consider that the δ-Lorentzian trans-Sasakian manifold M with respect to the semi-symmetric metric connection is Weyl conformally flat, that is $\bar{C} = 0$. Then from equation (8.1), we get

$$\bar{R}(X,Y)Z = \frac{1}{(n-2)}[\bar{S}(Y,Z)X - \bar{S}(X,Z)Y \quad (9.1)$$
$$+g(Y,Z)\bar{Q}X - g(X,Z)\bar{Q}Y] + \frac{\bar{r}}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y],$$

Now, taking the inner product of equation (9.1) with U. then we get

$$g(\bar{R}(X,Y)Z,U) = \frac{1}{(n-2)}[\bar{S}(Y,Z)g(X,U) - \bar{S}(X,Z)g(Y,U) \quad (9.2)$$
$$+g(Y,Z)g(\bar{Q}X,U) - g(X,Z)g(\bar{Q}Y,U)] - \frac{\bar{r}}{(n-1)(n-2)}$$
$$[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)].$$

Using equations (2.4), (3.2), (3.11), (3.12) and (3.14) in equation (9.2), we get

$$g(R(X,Y)Z,U) = \frac{1}{(n-2)}[S(Y,Z)g(X,U) - S(X,Z)g(Y,U) \quad (9.3)$$
$$+g(Y,Z)g(QX,U) - g(X,Z)g(QY,U)] - \frac{r}{(n-1)(n-2)}$$
$$[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)].$$

Putting $X = U = \xi$ in equation (9.3) and using (2.2), (2.3) and (2.4), we get

$$g(R(\xi, Y)Z, \xi) = \frac{1}{(n-2)}[\delta S(Y, Z) - \delta\eta(Y)S(\xi, Z) \\ +g(Y, Z)S(\xi, \xi) - \delta\eta(Z)S(Y, \xi)] - \frac{r}{(n-1)(n-2)} \\ [\delta g(Y, Z) - \eta(Y)\eta(Z)], \tag{9.4}$$

where $g(QY, Z) = S(Y, Z)$.

Now, using equations (2.13), (2.14) and (2.16), we get

$$S(Y, Z) = [(\delta(\alpha^2 + \beta^2) - (\xi\beta)] + \frac{r}{(n-1)}]g(Y, Z) + [\delta(n-4)(\xi\beta) \\ +n(\alpha^2 + \beta^2) - \frac{\delta}{r}(n-1)]\eta(Y)\eta(Z) - [2\delta\alpha\beta(n-2) + (n-2)(\xi\alpha)] \\ g(\phi Y, Z) - [\delta(\phi Z)\alpha + \delta(Z\beta)(n-2)]\eta(Y) - [\delta(\phi Y)\alpha + \delta(n-2)(Y\beta)]\eta(Z). \tag{9.5}$$

If $\alpha = 0$ andd $\beta = constant$ in (7.6), we get

$$S(Y, Z) = [\delta\beta^2 + \frac{r}{(n-1)}]g(Y, Z) + [n\beta^2 - \frac{\delta r}{(n-1)}]\eta(Y)\eta(Z). \tag{9.6}$$

There fore

$$S(Y, Z) = ag(Y, Z) + b\eta(Y)\eta(Z),$$

where $a = [\delta\beta^2 + \frac{r}{(n-1)}]$ and $b = [n\beta^2 - \frac{\delta r}{(n-1)}]$. This shows that M is an η-Einstein manifold. Thus we can state the following theorem:

Theorem 9.1. *Let M ba an n-dimensional Weyl conformally flat δ-Lorentzian trans-Sasakian manifold with respect to semi-symmetric metric connection $\bar{\nabla}$ is an η-Einstein manifold if $\alpha = 0$ and $\beta = constant$.*

Now, taking equation (8.1)

$$\bar{C}(X, Y)Z = \bar{R}(X, Y)Z - \frac{1}{(n-2)}[\bar{S}(Y, Z)X - \bar{S}(X, Z)Y \\ +g(Y, Z)\bar{Q}X - g(X, Z)\bar{Q}Y] + \frac{\bar{r}}{(n-1)(n-2)}[g(Y, Z)X - g(X, Z)Y]. \tag{9.7}$$

Using (2.20), (3.2), (3.11), (3.12) and (3.14) in equation (9.7), we get

$$\bar{C}(X, Y)Z = C(X, Y)Z + \delta[g(X, Z)Y - g(Y, Z)X] \\ +(\delta + \beta)[\eta(X)g(Y, Z) - \eta(Y)g(X, Z)]\xi \\ -(\beta\delta - 1)\eta(Z)[\eta(Y)X - \eta(X)Y] + \alpha[g(\phi X, Z)Y \\ -g(\phi, Z)X - g(Y, Z)\phi X + g(X, Z)\phi Y] + \frac{1}{(n-2)} \\ (\beta\delta - 1)(n-2)\eta(Y)\eta(Z) - ((\delta)(n-2) + \beta)g(Y, Z)X \\ +\alpha(n-2)g(\phi Y, Z)X + ((\delta)(n-2) + \beta)g(X, Z)Y \\ +(\beta\delta - 1)(n-2)\eta(X)\eta(Z)Y - \alpha(n-2)g(\phi X, Z)Y \\ -((\delta)(n-2) + \beta)g(Y, Z)X + (\beta + \delta)(n-2)g(Y, Z)\eta(X)\xi \\ \alpha(n-2)g(Y, Z)\phi X + ((\delta)(n-2) + \beta)g(X, Z)Y \\ -(\beta + \delta)(n-2)g(X, Z)\eta(Y)\xi - \alpha(n-2)g(X, Z)\phi Y] \tag{9.8}$$

$$-\frac{\beta+\delta+(n-2)}{(n-2)}[g(Y,Z)X-g(X,Z)Y].$$

Let X and Y are orthogonal basis to ξ. Putting $Z=\xi$ and using (2.1), (2.2) and (2.4) in (9.8), we get

$$\bar{C}(X,Y)\xi = C(X,Y)\xi.$$

Theorem 9.2. *An n-dimensinal δ-Lorentzian trans-Sasakian manifold M is Weyl ξ-conformally flat with respect to the semi-symmetric metric connection if and only if the manifold is also Weyl ξ-conformally flat with respect to the metric connection provided that the vector fields are horizontal vector fields.*

10. η-Ricci Solitons and Ricci Solitons in δ-Lorentzian trans-Sasakian manifold with semi-symmetric metric connection

Let M be 3-dimensional δ-Lorentzian trans-Sasakian manifold with semi-symmetric metric connection and V be pointwise collinear with ξ *i.e.* $V=b\xi$, where b is a function. Then consider the equation [11]

$$L_V g + 2\bar{S} + 2\lambda g + 2\mu\eta\otimes\eta = 0, \tag{10.1}$$

where L_V is the Lie derivative operator along the vector field V, $\bar{S}$ is the Ricci curvature tensor field of the metric g and λ and μ are real constants. Then equation (10.1) implies,

$$g(\bar{\nabla}_X b\xi, Y) + g(\bar{\nabla}_Y b\xi, X) + 2\bar{S}(X,Y) + 2\lambda g(X,Y) + 2\mu\eta(X)\eta(Y) = 0, \tag{10.2}$$

or

$$bg(\bar{\nabla}_X\xi, Y) + (Xb)\eta(Y) + bg(\bar{\nabla}_Y\xi, X) + (Yb)\eta(X) \tag{10.3}$$
$$+2\bar{S}(X,Y) + 2\lambda g(X,Y) + 2\mu\eta(X)\eta(Y) = 0.$$

Using (3.4) in (10.3), we get

$$bg[-(1+\delta\beta)X - (1+\delta\beta)\eta(X)\xi - \delta\alpha\phi X, Y] + (Xb)\eta(Y) \tag{10.4}$$
$$+bg[-(1+\delta\beta)Y - (1+\delta\beta)\eta(Y)\xi - \delta\alpha\phi Y, X] + (Yb)\eta(X)$$
$$+2\bar{S}(X,Y) + 2\lambda g(X,Y) + 2\mu\eta(X)\eta(Y) = 0.$$

$$-2b(1+\delta\beta)g(X,Y) - 2b(1+\delta\beta)\eta(Y)\eta(X) + (Xb)\eta(Y) + (Yb)\eta(X) \tag{10.5}$$
$$+2\bar{S}(X,Y) + 2\lambda g(X,Y) + 2\mu\eta(X)\eta(Y) = 0.$$

With the substitution of Y with ξ in (10.5) and using (3.21) for constants α and β, it follows that

$$(Xb) + (\xi b)\eta(X) - 4b(1+\delta\beta)\eta(X) + 2[2(\alpha^2+\beta^2-\delta(\xi\beta)) - 2\delta\beta]\eta(X) \tag{10.6}$$
$$+2\lambda\eta(X) + 2\mu\eta(X) = 0.$$

or

$$(Xb) + (\xi b)\eta(X) + \tag{10.7}$$
$$[-4b(1+\delta\beta) + 2(2(\alpha^2+\beta^2-\delta(\xi\beta)) - 2\delta\beta + 2\lambda + 2\mu]\eta(X) = 0.$$

Again replacing $X=\xi$ in (10.7), we obtain

$$\xi b = -[-2b(1+\delta\beta) + (2(\alpha^2+\beta^2-\delta(\xi\beta)) - \delta\beta + \lambda + \mu] \tag{10.8}$$

Putting (10.8) in (10.7), we obtain

$$db = [2b(1 + \delta\beta) - (2(\alpha^2 + \beta^2 - \delta(\xi\beta)) - \delta\beta - \lambda - \mu]\eta. \tag{10.9}$$

By applying d on (10.9), we get

$$[2b(1 + \delta\beta) - (2(\alpha^2 + \beta^2 - \delta(\xi\beta)) - \delta\beta - \lambda - \mu]d\eta = 0. \tag{10.10}$$

Since $d\eta \neq 0$ from (10.10), we have

$$[2b(1 + \delta\beta) - (2(\alpha^2 + \beta^2 - \delta(\xi\beta)) - \delta\beta - \lambda - \mu] = 0. \tag{10.11}$$

By using (10.9) and (10.11), we obtain that b is a constant. Hence from (10.5) it is verified

$$\bar{S}(X, Y) = [b(1 + \delta\beta) - \lambda]g(X, Y) + [b(1 + \delta\beta) - \mu]\eta(X)\eta(Y). \tag{10.12}$$

which implies that M is an η-Einstien manifold. This lead to the following:

Theorem 10.1. *In a 3-dimensional δ-Lorentzian trans-Sasakian manifold with semi-symmetric metric connection, the metric g is an η-Ricci soliton and V is a positive collinear with ξ, then V is a constant multiple of ξ and g is an η-Einstien manifold of the form (10.12) and η-Ricci soliton is expanding or shrinking according as the following relation is positive and negative*

$$\lambda = -[2b(1 + \delta\beta) - (2(\alpha^2 + \beta^2 - \delta(\xi\beta)) - \delta\beta - \mu]. \tag{10.13}$$

For $\mu = 0$, we deduce equation (10.12)

$$\bar{S}(X, Y) = [b(1 + \delta\beta) - \lambda]g(X, Y) + [b(1 + \delta\beta)]\eta(X)\eta(Y). \tag{10.14}$$

Now, we have the following corollary:

Corollary 10.2. *In a 3-dimensional δ-Lorentzian trans-Sasakian manifold with semi-symmetric metric connection, the metric g is a Ricci soliton and V is a positive collinear with ξ, then V is a constant multiple of ξ and g is an η-Einstien manifold and Ricci soliton is shrinking according as the following relation is negative. For $\mu = 0$, (10.13) reduce to*

$$\lambda = -[2b(1 + \delta\beta) - (2(\alpha^2 + \beta^2 - \delta(\xi\beta)) - \delta\beta]. \tag{10.15}$$

Here is an example of η-Ricci soliton on δ-Lorentzian trans-Sasakian manifold with semi-symmetric metric connection.

Example: We consider the three dimensional manifold $M = [(x, y, z) \in \mathbb{R}^3 \mid z \neq 0]$, where (x, y, z) are the cartesian coordinates in R^3. Choosing the vector fields

$$e_1 = z\frac{\partial}{\partial x},\ e_2 = z\frac{\partial}{\partial y},\ e_3 = -z\frac{\partial}{\partial z},$$

which are linearly independent at each point of M. Let g be the Riemannian metric define by

$$g(e_1, e_3) = g(e_2, e_3) = g(e_2, e_2) = 0,\ g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = \delta,$$

where $\delta = \pm 1$. Let η be the 1-form defined by $\eta(Z) = \epsilon g(Z, e_3)$ for any vector field Z on TM. Let ϕ be the $(1, 1)$ tensor field defined by $\phi(e_1) = -e_2,\ \phi(e_2) = e_1,\ \phi(e_3) = 0$. Then by the linearity property of ϕ and g, we have

$$\phi^2 Z = Z + \eta(Z)e_3,\ \eta(e_3) = 1 \text{ and } g(\phi Z, \phi W) = g(Z, W) - \delta\eta(Z)\eta(W)$$

for any vector fields Z, W on M.

Let ∇ be the Levi-Civita connection with respect to the metric g. Then we have

$$[e_1, e_2] = 0, \ [e_1, e_3] = \delta e_1, \ [e_2, e_3] = \delta e_2.$$

The Riemannian connection ∇ with respect to the metric g is given by

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) + g([X, Y], Z) - g([Y, Z], X) + g([Z, X], Y).$$

From above equation which is known as Koszul's formula, we have

$$\nabla_{e_1} e_3 = \delta e_1, \ \nabla_{e_2} e_3 = \delta e_2, \ \nabla_{e_3} e_3 = 0, \tag{10.16}$$
$$\nabla_{e_1} e_2 = 0, \ \nabla_{e_2} e_2 = -\delta e_3, \ \nabla_{e_3} e_2 = 0,$$
$$\nabla_{e_1} e_1 = -\delta e_3, \ \nabla_{e_2} e_1 = 0, \ \nabla_{e_3} e_1 = 0.$$

Using the above relations, for any vector field X on M, we have

$$\nabla_X \xi = \delta \{\beta(X + \eta(X)\xi\},$$

for $\xi \in e_3$, $\alpha = 0$ and $\beta = 1$. Hence the manifold M under consideration is an δ-Lorentzian trans Sasakian of type $(0, 1)$ manifold of dimension three.

Now, we consider at this example for semi-symmetric metric connection from (2.9) and (10.14), we obtain:

$$\bar{\nabla}_{e_1} e_3 = (1 + \delta) e_1, \ \bar{\nabla}_{e_2} e_3 = (1 + \delta) e_2, \ \bar{\nabla}_{e_3} e_3 = 0, \tag{10.17}$$
$$\bar{\nabla}_{e_1} e_2 = 0, \ \bar{\nabla}_{e_2} e_2 = -(1 + \delta) e_3, \ \bar{\nabla}_{e_3} e_2 = 0,$$
$$\bar{\nabla}_{e_1} e_1 = -(1 + \delta) e_3, \ \bar{\nabla}_{e_2} e_1 = 0, \ \bar{\nabla}_{e_3} e_1 = 0.$$

Then the Riemannian and the Ricci curvature tensor fields with respect to semi-symmetric metric connection are given by:

$$\bar{R}(e_1, e_2) e_2 = -(1 + \delta)^2 e_1, \ \bar{R}(e_1, e_3) e_3 = -\delta(1 + \delta) e_2, \ \bar{R}(e_2, e_1) e_1 = -(1 + \delta)^2 e_2,$$
$$\bar{R}(e_2, e_3) e_3 = -\delta(1 + \delta) e_2, \ \bar{R}(e_3, e_1) e_1 = \delta(1 + \delta) e_3, \ \bar{R}(e_3, e_2) e_2 = -\delta(1 + \delta) e_3,$$
$$\bar{S}(e_1, e_1) = \ \bar{S}(e_2, e_2) = -(1 + \delta)(1 + 2\delta), \ \bar{S}(e_3, e_3,) = 2\delta(1 + \delta).$$

From (10.14), for $\lambda = \frac{(1+\delta)^2}{\delta}$ and $\mu = -(1 + \delta)(1 + 3\delta)$, the data (g, ξ, λ, μ) is an η-Ricci soliton on (M, ϕ, ξ, η, g).

11. Gradient Ricci Solitons in 3-dimensional δ-Lorentzian trans-Sasakian manifold with semi-symmetric metric connection (n=3)

If the vector field V is the gradient of a potential function -ψ then g is called a gradient Ricci soliton and (1.2) assume the form

$$\nabla \nabla \psi = S + \lambda g. \tag{11.1}$$

This reduces to

$$\nabla_Y D\psi = QY + \lambda Y, \tag{11.2}$$

where D denoted the gradient operator of g. From (11.2) it follows

$$\bar{R}(X, Y) D\psi = (\bar{\nabla}_X Q) Y - (\bar{\nabla}_Y Q) X. \tag{11.3}$$

Differentiating (3.12) and using (3.22)

$$(\bar{\nabla}_W Q)X = \frac{dr(W)}{2}(X - \eta(X)\xi)) - (\frac{r}{2} - 3(\alpha^2 + \beta^2))(\alpha(g(\phi W, X) + (\beta + \delta)g(W, X) - (1 + \delta\beta)\eta(X)\eta(W)) + \eta(X)\bar{\nabla}_W \xi. \quad (11.4)$$

In (11.4) replacing $W = \xi$, we obtain

$$(\bar{\nabla}_\xi Q)X = \frac{dr(\xi)}{2}(X - \eta(X)\xi)). \quad (11.5)$$

Then we have

$$g((\bar{\nabla}_\xi Q)X - (\bar{\nabla}_X Q)(\xi, \xi) \quad (11.6)$$
$$= g(\frac{dr(\xi)}{2}(X - \eta(X)\xi, \xi)) = \frac{dr(\xi)}{2}(g(X, \xi) - \eta(X))) = 0.$$

Using (11.6) and (11.5), we obtain

$$g(\bar{R}(\xi, X)D\psi, \xi) = 0. \quad (11.7)$$

From (3.20)

$$g(\bar{R}(\xi, Y)D\psi, \xi) = (\alpha^2 + \beta^2 - \delta(\xi\beta)(g(Y, D\psi) - \eta(Y)\eta(D\psi)).$$

Using (11.7), we get

$$(\alpha^2 + \beta^2 - \delta(\xi\beta)(g(Y, D\psi) - \eta(Y)\eta(D\psi)) = 0$$
$$(\alpha^2 + \beta^2 - \delta(\xi\beta)(g(Y, D\psi) - \eta(Y)g(D\psi, \xi)) = 0,$$

or

$$(g(Y, D\psi) - g(Y, \xi)g(D\psi, \xi)) = 0,$$

which implies

$$D\psi = (\xi\psi)\xi, \quad \text{since} \quad \alpha^2 + \beta^2 \neq \delta(\xi\beta). \quad (11.8)$$

Using (11.8) and (11.2)

$$\bar{S}(X, Y) + \lambda g(X, Y) = g(\bar{\nabla}_Y D\psi, X) = g(\bar{\nabla}_Y (\xi\psi)\xi, X)$$
$$= (\xi\psi)g(\bar{\nabla}_Y \xi, X) + Y(\xi\psi)\eta(X)$$
$$= (\xi\psi)g(-\delta\alpha\phi Y - (1 + \delta\beta)Y - (1 + \delta\beta)\eta(Y)\xi, X) + Y(\xi\psi)\eta(X)$$

$$\bar{S}(X, Y) + \lambda g(X, Y) = -\delta\alpha(\xi\psi)g(\phi Y, X) - (1 + \delta\beta)(\xi\psi)g(Y, X) - (1 + \delta\beta)(\xi\psi)\eta(Y)\eta(X) + Y(\xi\psi)\eta(X). \quad (11.9)$$

Putting $X = \xi$ in (11.9) and using (3.21) we get

$$\bar{S}(Y, \xi) + \lambda\eta(Y) = Y(\xi\psi) = [\lambda + 2\delta(1 + \delta\beta) + 2(\alpha^2 + \beta^2 - \delta(\xi\beta)) - 2\delta\beta]\eta(Y). \quad (11.10)$$

Interchanging X and Y in (11.9), we get

$$\bar{S}(X, Y) + \lambda g(X, Y) = -\delta\alpha(\xi\psi)g(Y, \phi X) - (1 + \delta\beta)(\xi\psi)g(X, Y) - (1 + \delta\beta)(\xi\psi)\eta(Y)\eta(X) + X(\xi\psi)\eta(Y). \quad (11.11)$$

Adding (11.9) and (11.11) we get

$$2\bar{S}(X, Y) + 2\lambda g(X, Y) = -2(1 + \delta\beta)(\xi\psi)g(X, Y) + Y(\xi\psi)\eta(X) \quad (11.12)$$

$$-2(1+\delta\beta)(\xi\psi)\eta(X)\eta(Y) + X(\xi\psi)\eta(Y).$$

Using (11.10) in (11.12) we have

$$\bar{S}(X,Y) + \lambda g(X,Y) = -(1+\delta\beta)(\xi\psi)[g(X,Y) - \eta(X)\eta(Y)] \quad (11.13)$$
$$+[\lambda + (1+\delta\beta) + 2(\alpha^2+\beta^2-\delta(\xi\beta)) - 2(\delta\beta)]\eta(X)\eta(Y).$$

Then using (11.2) we have

$$\bar{\nabla}_Y D\psi = -(1+\delta\beta)(\xi\psi)(Y - \eta(Y)\xi) \quad (11.14)$$
$$+[\lambda+(1+\delta\beta)+2(\alpha^2+\beta^2-\delta(\xi\beta))-2(\delta\beta)]\eta(Y)\xi.$$

Using (11.14) we calculate

$$\bar{R}(X,Y)D\psi = \bar{\nabla}_X\bar{\nabla}_Y D\psi - \bar{\nabla}_Y\bar{\nabla}_X D\psi - \bar{\nabla}_{[X,Y]}D\psi$$

$$= -(1+\delta\beta)X(\xi\psi)Y + (1+\delta\beta)Y(\xi\psi)X \quad (11.15)$$
$$-(1+\delta\beta)Y(\xi\psi)\eta(X)\xi + (1+\delta\beta)X(\xi\psi)\eta(Y)\xi$$
$$+[\lambda + (1+\delta\beta) + 2(\alpha^2+\beta^2-\delta(\xi\beta)) - 2(\delta\beta)]((\nabla_X\eta)(Y)\xi - (\nabla_Y\eta)(X)\xi)$$
$$+[\lambda + (1+\delta\beta) + 2(\alpha^2+\beta^2-\delta(\xi\beta)) - 2(\delta\beta)]((\nabla_X\xi)\eta(Y)\xi - (\nabla_Y\xi)\eta(X)).$$

Taking inner product with ξ in (11.15), we get

$$0 = g(\bar{R}(X,Y)D\psi,\xi) = 2\delta\alpha[\lambda + (1+\delta\beta) + 2(\alpha^2+\beta^2-\delta(\xi\beta)) - 2(\delta\beta)]g(\phi Y, X). \quad (11.16)$$

Thus we have $2\delta\alpha[\lambda + (1+\delta\beta) + 2(\alpha^2+\beta^2-\delta(\xi\beta)) - 2(\delta\beta)] = 0$.

Now we consider the following cases:
Case (i) $\delta\alpha = 0$, or
Case (ii) $[\lambda + (1+\delta\beta) + 2(\alpha^2+\beta^2-\delta(\xi\beta)) - 2(\delta\beta)] = 0$,
Case (iii) $\alpha = 0$ and $[\lambda + (1+\delta\beta) + 2(\alpha^2+\beta^2-\delta(\xi\beta)) - 2(\delta\beta)] = 0$.
Case (i) If $\alpha = 0$, the manifold reduces to a δ-Lorentzian β-Kenmotsu manifold with respect to a semi-symmetric metric connection.
Case (ii) Let $[\lambda + (1+\delta\beta) + 2(\alpha^2+\beta^2-\delta(\xi\beta)) - 2(\delta\beta)] = 0$. If we use this in (11.10) we get $Y(\xi\psi) = -(1+\delta\beta)(\xi\psi)\eta(Y)$. Substitute this value in (11.12) we obtain

$$\bar{S}(X,Y) + \lambda g(X,Y) = -(1+\delta\beta)(\xi\psi)g(X,Y) - 2(1+\delta)\eta(X)\eta(Y). \quad (11.17)$$

Now, contracting (11.17), we get

$$\bar{r} + 3\lambda = -3(1+\delta\beta)(\xi\psi) - 2(1+\delta\beta), \quad (11.18)$$

which implies

$$(\xi\psi) = \frac{\bar{r}}{-3(1+\delta\beta)} + \frac{\lambda}{-(1+\delta\beta)} + \frac{2}{-3}. \quad (11.19)$$

If $\bar{r} = constant$, then $(\xi\psi) = constant = k(say)$. Therefore from (11.8) we have $D\psi = (\xi\psi)\xi = k\xi$. This we can write this equation as

$$g(D\psi, X) = k\eta(X), \quad (11.20)$$

which means that $d\psi(X) = k\eta(X)$. Applying d this, we get $kd\eta = 0$. Since $d\eta \neq 0$, we have $k = 0$. Hence we get $D\psi = 0$. This means that $\psi = constant$ Therefore equation (11.1) reduces to

$$\bar{S}(X,Y) = 2(\alpha^2 + \beta^2 - \delta(\xi\beta)) - 2\delta\beta)g(X,Y),$$

that is M is an *Einstein* manifold.
Case (iii) Using $\alpha = 0$ and $[\lambda + (1+\delta\beta) + 2(\alpha^2+\beta^2-\delta(\xi\beta)) - 2(\delta\beta)] = 0$. in (11.10) we obtain $Y(\xi\psi) = -(1+\delta\beta)(\xi\psi)\eta(Y)$. Now as in *Case (ii)* we conclude that the manifold is an *Einstein* manifold.

Thus we have the following :

Theorem 11.1. *If a 3-dimensional δ-Lorentzian trans Sasakian manifold with a semi symmetric metric connection with constant scalar curvature admits gradient Ricci soliton, then the manifold is either a δ-Lorentzian β-Kenmotsu manifold or an Einstein manifold provided $\alpha, \beta = constant$.*

In [12] it was proved that if a 3-dimensional compact connected trans-Sasakian manifold is of constant curvature, then it is either α-Sasakian or β-Kenmotsu. Since for a 3-dimensional Riemannian manifold constant curvature and Einstein manifold are equivalent, therefore from the Theorem 3 we state the following:

Corollary 11.2. *If a compact 3-dimensional δ-Lorentzian trans-Sasakian manifold with a semi-symmetric metric connection with constant scalar curvature admits Ricci soliton, then the manifold is either δ-Lorentzian α-Sasakian or δ-Lorentzian β-Kenmotsu.*

Also in [12], authors proved that a 3-dimensional connected trans-Sasakian manifold is locally ϕ-symmetric if and only if the scalar curvature is constant provided α and β are constants. Hence from Theorem 3 we obtain the following:

Corollary 11.3. *If a locally ϕ-symmetric 3-dimensional connected δ-Lorentzian trans-Sasakian manifold with respect to a semi symmeyric metric connection ith admits gradient Ricci soliton, then manifold is either δ-Lorentzian β-Kenmotsu or Einstein manifold provided $\alpha, \beta = constant$.*

Conflict of Interests

The author declare that there is no conflict of interests regarding the publication of this paper.

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