**Bangmod** Int. **J.** **M**ath. **&** **C**omp. **S**ci.
Vol. 4, No. 1-2, 2018; Pages 1 - 16
http://bangmod-jmcs.kmutt.ac.th/
**ISSN : 2408-154X**

# Common fixed points of a pair/two pairs of selfmaps satisfying certain contraction condition

[1]**Gutti Venkata Ravindranadh Babu and** [2]**Dasari Ratna Babu**
[1,2]*Department of Mathematics, Andhra University, Visakhapatnam-530 003, India.*
[2]*Permanent Address: Department of Freshmen Engineering, PSCMRCET, Vijayawada-520 001, India.*
*E-mails:* [1] *gvr_babu@hotmail.com;* [2] *ratnababud@gmail.com*

[1]*Corresponding author.*

**Abstract** In this paper, we prove the existence of common fixed points of a pair of weakly compatible/R-weakly commuting of type $(A_g)$/R-weakly commuting of type $(A_f)$ selfmaps satisfying certain contraction condition by using the reciprocal continuity in a complete metric space. Also, we extend it to prove the existence of common fixed points of two pairs of selfmaps on a metric space in which either one of the pairs satisfies the property (E.A) and restricting the completeness of $X$ to its subspace. We provide examples in support of our results.

**MSC:** 47H10; 54H25

**Keywords:** Common fixed points, complete metric space, compatible maps, weakly compatible maps, reciprocal continuous maps, property (E.A).

Submission date: 14 February 2018 / Acceptance date: 18 December 2018 /Available online 31 December 2018


## 1. Introduction

The development of fixed point theory is based on the generalization of contraction conditions in one direction or/and generalization of ambient spaces of the operator under consideration on the other. Let $T$ be a selfmap on a metric space $(X, d)$. If $x \in X$, we write $Tx$ for the image of $x$ under $T$. The $T-$iterates $x, Tx, T^2x, \ldots$ define $T-$orbit at $x \in X$ which we denote it by $O_T(x)$. Banach contraction principle plays an important role in solving nonlinear equations, and it is one of the most useful results in fixed point theory. Banach contraction principle has been generalized in various ways either by using contractive conditions or by generalizing the ambient space. In the direction of generalization of contraction conditions, in 1976, Jungck [3] established fixed point theorem for pair of commuting selfmappings.

**Theorem 1.1.** [3] *Let $(X, d)$ be a complete metric space. $f, g$ be commuting selfmaps of*



Published by **T**heoretical **and** **C**omputational **S**cience Center **(TaCS)**,
King Mongkut's University of Technology Thonburi **(KMUTT)**

*$X$. Assume that*
(i) $f(X) \subseteq g(X)$
(ii) *there exists $k \in [0, 1)$ such that*
$$d(fx, fy) \leq kd(gx, gy) \tag{1.1.1}$$
*for all $x, y \in X$. If $g$ is continuous, then $f$ and $g$ have a unique common fixed point.*

Several researchers generalized Theorem 1.1 either by replacing (1.1.1) by a weaker contraction condition and/or dropping the continuity of $g$. The following is one such result due to Das and Naik [2] .

**Theorem 1.2.** [2] *Let $(X, d)$ be a complete metric space. $f, g$ be commuting selfmaps of $X$. Assume that*
(i) $f(X) \subseteq g(X)$
(ii) *there exists $k \in [0, 1)$ such that*
$$d(fx, fy) \leq k \max\{d(gx, gy), d(gx, fx), d(gy, fy), d(gx, fy), d(gy, fx)\} \tag{1.2.1}$$
*for all $x, y \in X$. If $g$ is continuous, then $f$ and $g$ have a unique common fixed point.*

We use the following definitions in our subsequent discussion.

**Definition 1.3.** [4] Let $f$ and $g$ be selfmaps of a metric space $(X, d)$. The pair $(f, g)$ is said to be a compatible pair on $X$, if $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$ whenever $\{x_n\}$ is a sequence in $X$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$, for some $t \in X$.

**Definition 1.4.** [5] Let $f$ and $g$ be selfmaps of a metric space $(X, d)$. The pair $(f, g)$ is said to be weakly compatible, if they commute at their coincidence points. i.e., $fgx = gfx$ whenever $fx = gx, x \in X$.

Every compatible pair of maps is weakly compatible, but its converse need not true [5].

**Definition 1.5.** [6] Let $f$ and $g$ be selfmaps of a metric space $(X, d)$. Then $f$ and $g$ are said to be reciprocally continuous, if $\lim_{n\to\infty} fgx_n = ft$ and $\lim_{n\to\infty} gfx_n = gt$, whenever $\{x_n\}$ is a sequence in $X$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$, for some $t \in X$.

In 2011, Pant, Bisht and Arora [8] introduced the concept of weakly reciprocally continuous as follows.

**Definition 1.6.** [8] Let $f$ and $g$ be selfmaps of a metric space $(X, d)$. Then $f$ and $g$ are said to be weakly reciprocally continuous, if $\lim_{n\to\infty} fgx_n = ft$ or $\lim_{n\to\infty} gfx_n = gt$, whenever $\{x_n\}$ is a sequence in $X$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$, for some $t \in X$.

Clearly, every reciprocal continuous is weakly reciprocally continuous, but its converse need not be true [8].

In 1994, Pant [7] introduced the concept of $R-$weakly commuting maps as follows.

**Definition 1.7.** [7] Let $f$ and $g$ be selfmaps of a metric space $(X, d)$. Then $f$ and $g$ are said to be $R-$weakly commuting, if there exists an $R > 0$ such that
$d(fgx, gfx) \leq Rd(fx, gx)$, for all $x \in X$.

**Definition 1.8.** [9] Let $f$ and $g$ be selfmaps of a metric space $(X, d)$. Then $f$ and $g$ are said to be $R-$weakly commuting of type $(A_g)$, if there exists an $R > 0$ such that $d(ffx, gfx) \leq Rd(fx, gx)$, for all $x \in X$.

**Definition 1.9.** [9] Let $f$ and $g$ be selfmaps of a metric space $(X, d)$. Then $f$ and $g$ are said to be $R-$weakly commuting of type $(A_f)$, if there exists an $R > 0$ such that $d(fgx, ggx) \leq Rd(fx, gx)$, for all $x \in X$.

In 2003, Singh and Tomar [11] did a nice comparative study of various weaker forms of commuting maps. Clearly $R-$weakly commuting maps of both types $(A_g)$ and $(A_f)$

commute at their coincidence points. The notations of $R-$weakly commuting and $R-$weakly commuting of type $(A_f)$ are independent [11].

In 2002, Aamri and Moutawaki [1] introduced the idea of the property (E.A) for a pair of selfmappings defined on a metric space.

**Definition 1.10.** [1] Two selfmappings $f$ and $g$ of a metric space $(X, d)$ are said to satisfy the property (E.A), if there exists a sequence $\{x_n\}$ in $X$ such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$, for some $t \in X$.

In 2014, Phaneendra and Prasad [10] proved the existence of common fixed points for a pair of compatible maps as follows.

**Theorem 1.11.** [10] *Let $(X, d)$ be a complete metric space. Let $f$ and $g$ be compatible selfmaps of $X$. Assume that*
(i) $f(X) \subseteq g(X)$
(ii) *there exists $k \in [0, 1)$ such that*
$$d(fx, fy) \le k \max\{d(gx, gy), d(gx, fx), d(gy, fy), d(gx, fy), d(gy, fx)\}, \text{ and}$$
(iii) $\min\{d(gx, gy), d(fx, gy), d(y, gy), d(gy, fy)\} \le d(y, gx) + d(y, fx)$, *for all $x, y \in X$ except for those $x, y$ with $gx = fx = y$. Then $f$ and $g$ have a unique common fixed point.*

In Section 2, we prove the existence of common fixed points of a pair of selfmaps with reciprocal continuity in a complete metric space by relaxing the inequality (iii) of Theorem 1.11, but by imposing reciprocal continuity. Also, we prove the existence of common fixed points of two pairs of selfmaps on a metric space in which either one of the pairs satisfies the property (E.A) and restricting the completeness of $X$ to its subspace. We provide examples in support of our results.

## 2. Main Results

The following is the main result of this paper.

**Theorem 2.1.** *Let $(X, d)$ be a complete metric space. Let $f$ and $g$ be selfmapings of $X$.*

*Assume that*
(i) $f(X) \subseteq g(X)$
(ii) *there exists $k \in [0, 1)$ such that*
$$d(fx, fy) \le k \max\{d(gx, gy), d(fx, gx), d(fy, gy), d(fx, gy), d(fy, gx)\} \quad (2.1.1)$$
*for all $x, y \in X$ and*
(iii) *the pair $(f, g)$ is reciprocally continuous on $X$.*
*If $f$ and $g$ are either $R$-weakly commuting of type $(A_g)$ (or) $R$-weakly commuting of type $(A_f)$ then $f$ and $g$ have a unique common fixed point.*

*Proof.* Let $x_0$ be any point in $X$.
Since $f(X) \subseteq g(X)$, there exists $x_1 \in X$ such that $y_0 = fx_0 = gx_1$.
In general, $y_n = fx_n = gx_{n+1}, n = 0, 1, 2, \dots$.
Then $\{y_n\}$ is a Cauchy sequence in $X$, as shown in [2].
Since $X$ is complete, there exists a point $t \in X$ such that $\lim_{n\to\infty} y_n = t$.
Moreover, $y_n = fx_n = gx_{n+1} \to t$ as $n \to \infty$.
**Case (i):** Suppose that $f$ and $g$ are $R$-weakly commuting of type $(A_g)$.
Now, reciprocal continuity of $f$ and $g$ implies that $fgx_n \to ft$ and $gfx_n \to gt$ as $n \to \infty$.
$\Rightarrow \lim_{n\to\infty} fgx_n = \lim_{n\to\infty} ffx_n = ft$ and $\lim_{n\to\infty} gfx_n = \lim_{n\to\infty} ggx_{n+1} = gt$
Then $R$-weakly commuting of type $(A_g)$ of $f$ and $g$ gives that
$d(ffx_n, gfx_n) \le Rd(fx_n, gx_n), R > 0$.

On letting $n \to \infty$, we get $d(ft, gt) \leq 0$ which implies that $ft = gt$.
Again by $R$-weakly commuting of type $(A_g)$ of $f$ and $g$, we have
$d(fft, gft) \leq Rd(ft, gt)$ implies that $fft = gft = ggt = fgt$.
Now, we show that $ft = fft$.
Suppose that $ft \neq fft$.
Then by the inequality (2.1.1), we obtain

$$\begin{aligned} d(ft, fft) &\leq k \max\{d(gt, gft), d(ft, gt), d(fft, gft), d(ft, gft), d(fft, gt)\} \\ &= k \max\{d(ft, fft), 0, 0, d(ft, fft), d(fft, ft)\} \end{aligned}$$

implies that $d(ft, fft) \leq kd(ft, fft) < d(ft, fft)$,
a contradiction.
Therefore $d(ft, fft) = 0$ and hence $ft = fft = gft$. Hence $ft$ is a common fixed point of $f$ and $g$.
**Case (ii):** Suppose that $f$ and $g$ are $R$-weakly commuting of type $(A_f)$.
Now, reciprocal continuity of $f$ and $g$ implies that $fgx_n \to ft$ and $gfx_n \to gt$ as $n \to \infty$.
$\Rightarrow \lim\limits_{n\to\infty} fgx_n = \lim\limits_{n\to\infty} ffx_n = ft$ and $\lim\limits_{n\to\infty} gfx_n = \lim\limits_{n\to\infty} ggx_{n+1} = gt$
Then $R$-weakly commuting of type $(A_f)$ of $f$ and $g$ gives that
$d(fgx_n, ggx_n) \leq Rd(fx_n, gx_n), R > 0$.
On letting $n \to \infty$, we get $d(ft, gt) \leq 0$ which implies that $ft = gt$.
Again by $R$-weakly commuting of type $(A_f)$ of $f$ and $g$, we have
$d(fgt, ggt) \leq Rd(ft, gt)$ implies that $fgt = ggt = gft = fft$.
We now prove that $ft = fft$.
Suppose that $ft \neq fft$.
Using the inequality (2.1.1), we get

$$\begin{aligned} d(ft, fft) &\leq k \max\{d(gt, gft), d(ft, gt), d(fft, gft), d(ft, gft), d(fft, gt)\} \\ &= k \max\{d(ft, fft), 0, 0, d(ft, fft), d(fft, ft)\} \end{aligned}$$

implies that $d(ft, fft) \leq kd(ft, fft) < d(ft, fft)$,
a contradiction.

Therefore $d(ft, fft) \leq 0$ and hence $ft = fft = gft$. Hence $ft$ is a common fixed point of $f$ and $g$.

Uniqueness of the common fixed point of $f$ and $g$ follows trivially from the inequality (2.1.1). ∎

**Theorem 2.2.** *Let $(X, d)$ be a complete metric space. Let $f$ and $g$ be selfmapings of $X$. Assume that*

(i) $f(X) \subseteq g(X)$ *and satisfy the inequality* (2.1.1)

(ii) *the pair $(f, g)$ is weakly reciprocally continuous on $X$. If $f$ and $g$ are compatible then $f$ and $g$ have a unique common fixed point.*

*Proof.* Let $x_0$ be any point in $X$.
Since $f(X) \subseteq g(X)$, there exists $x_1 \in X$ such that $y_0 = fx_0 = gx_1$.
In general, $y_n = fx_n = gx_{n+1}, n = 0, 1, 2, \ldots$.
Then $\{y_n\}$ is a Cauchy sequence in $X$, as shown in [2].
Since $X$ is complete, there exists a point $t \in X$ such that $\lim\limits_{n\to\infty} y_n = t$.
Moreover, $y_n = fx_n = gx_{n+1} \to t$ as $n \to \infty$.
Suppose that $f$ and $g$ are compatible.
By the weakly reciprocal continuity of $f$ and $g$, we have

$fgx_n \to ft$ or $gfx_n \to gt$ as $n \to \infty$.
**Case (i):** Suppose that $\lim_{n\to\infty} gfx_n = gt$.
Then by the compatibility of $f$ and $g$, we have $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$.
i.e., $\lim_{n\to\infty} fgx_n = \lim_{n\to\infty} gfx_n$.
Therefore $\lim_{n\to\infty} fgx_n = \lim_{n\to\infty} gfx_n = gt$
implies that $\lim_{n\to\infty} ffx_{n+1} = gt$.
Using the inequality (2.1.1), we get
$d(ft, ffx_n) \leq k \max\{d(gt, gfx_n), d(ft, gt), d(ffx_n, gfx_n), d(ffx_n, gt), d(ft, gfx_n)\}$.
On letting $n \to \infty$, we get
$d(ft, gt) \leq k \max\{d(gt, gt), d(ft, gt), d(gt, gt), d(gt, gt), d(ft, gt)\}$
which implies that $ft = gt$.
Again compatibility of $f$ and $g$ implies that commute at their coincidence point.
Hence $fgt = gft = ggt = fft$.
Now, we show that $ft = fft$.
Suppose that $ft \neq fft$.
Then by the inequality (2.1.1), we obtain

$$\begin{aligned} d(ft, fft) &\leq k \max\{d(gt, gft), d(ft, gt), d(fft, gft), d(ft, gft), d(fft, gt)\} \\ &= k \max\{d(ft, fft), 0, 0, d(ft, fft), d(fft, ft)\}, \end{aligned}$$

which implies that $d(ft, fft) \leq kd(ft, fft) < d(ft, fft)$,
a contradiction.
Therefore $d(ft, fft) = 0$ and hence $ft = fft = gft$. Hence $ft$ is a common fixed point of $f$ and $g$.
**Case (ii):** Suppose that $\lim_{n\to\infty} fgx_n = ft$.
Since $f(X) \subseteq g(X)$, there exists $u \in X$ such that $ft = gu$.
Therefore $\lim_{n\to\infty} fgx_n = gu \Rightarrow \lim_{n\to\infty} ffx_{n+1} = gu$.
Then by the compatibility of $f$ and $g$, we have $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$.
i.e., $\lim_{n\to\infty} fgx_n = \lim_{n\to\infty} gfx_n = gu$.
Using the inequality (2.1.1), we get
$d(fu, ffx_n) \leq k \max\{d(gu, gfx_n), d(fu, gu), d(ffx_n, gfx_n), d(ffx_n, gu), d(fu, gfx_n)\}$.
On letting $n \to \infty$, we get
$d(fu, gu) \leq k \max\{d(gu, gu), d(fu, gu), d(gu, gu), d(gu, gu), d(fu, gu)\}$
which implies that $fu = gu$.
Again compatibility of $f$ and $g$ implies that commute at their coincidence point.
Hence $fgu = gfu = ggu = ffu$.
Now, we show that $fu = ffu$.
Suppose that $fu \neq ffu$.
Then by the inequality (2.1.1), we obtain

$$\begin{aligned} d(fu, ffu) &\leq k \max\{d(gu, gfu), d(fu, gu), d(ffu, gfu), d(fu, gfu), d(ffu, gu)\} \\ &= k \max\{d(fu, ffu), 0, 0, d(fu, ffu), d(ffu, fu)\} \end{aligned}$$

implies that $d(fu, ffu) \leq kd(fu, ffu) < d(fu, ffu)$,
a contradiction.

Therefore $d(fu, ffu) = 0$ and hence $fu = ffu = gfu$. Hence $fu$ is a common fixed point of $f$ and $g$. ∎

In the following, we prove the existence of common fixed points for four selfmappings.

**Theorem 2.3.** *Let* $(X, d)$ *be a metric space. Let* $A, B, S$ *and* $T$ *be selfmapings of* $X$*. Assume that*

(i) $A(X) \subseteq T(X), B(X) \subseteq S(X)$ (ii) *there exists* $k \in [0, 1)$ *such that*

$$d(Ax, By) \leq k \max\{d(Sx, Ty), d(Ax, Sx), d(By, Ty), \tfrac{1}{2}[d(Sx, By)+d(Ax, Ty)]\} \quad (2.3.1)$$

*for all* $x, y \in X$ *and* (iii) *the pairs* $(A, S)$ *and* $(B, T)$ *are weakly compatible.*

*If either* $(A, S)$ *(or)* $(B, T)$ *satisfies the property (E.A) and either* $S(X)$ *(or)* $T(X)$ *is a closed subspace of* $X$*, then* $A, B, S$ *and* $T$ *have a unique common fixed point.*

*Proof.* First suppose that the pair $(B, T)$ satisfy the property (E.A) and $S(X)$ is closed. Then there exists a sequence $\{x_n\}$ in $X$ such that $\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = p$, for some $p \in X$.

Since $B(X) \subseteq S(X)$, there exists a sequence $\{y_n\}$ in $X$ such that $Bx_n = Sy_n$.

Therefore $\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sy_n = p$.

Since $S(X)$ is closed, we have $p \in S(X)$. Then there exists $r \in X$ such that $Sr = p$.

Hence $\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Sy_n = \lim_{n\to\infty} Tx_n = p = Sr$.

Now, we prove that $Ar = Sr$.

Suppose that $d(Ar, Sr) > 0$.

By the inequality (2.3.1), we get

$d(Ar, Bx_n) \leq k \max\{d(Sr, Tx_n), d(Ar, Sr), d(Bx_n, Tx_n), \frac{1}{2}[d(Sr, Bx_n) + d(Ar, Tx_n)]\}$.

Letting $n \to \infty$, we obtain

$d(Ar, Bx_n) \leq k \max\{d(Sr, Sr), d(Ar, Sr), d(Sr, Sr), \frac{1}{2}[d(Sr, Sr) + d(Ar, Sr)]\}$

implies that

$d(Ar, Sr) \leq d(Ar, Sr) < d(Ar, Sr)$,

a contradiction.

Therefore $d(Ar, Sr) \leq 0$ implies that $Ar = Sr = p$.

Further, since $A(X) \subseteq T(X)$ there exists $u \in X$ such that $Ar = Tu = p$.

Therefore $Ar = Sr = Tu = p$.

Now, we prove that $Bu = Tu$.

On the contrary suppose that $Bu \neq Tu$.

Using the inequality (2.3.1), we obtain

$d(Ar, Bu) \leq k \max\{d(Sr, Tu), d(Ar, Sr), d(Bu, Tu), \frac{1}{2}[d(Sr, Bu) + d(Ar, Tu)]\}$

which implies that

$d(Ar, Bu) \leq k \max\{0, 0, d(Bu, Ar), \frac{1}{2}[d(Ar, Bu) + 0]\}$

implies that

$d(Ar, Bu) \leq kd(Ar, Bu) < d(Ar, Bu)$,

a contradiction.

Therefore $Ar = Bu = Sr = Tu = p$.

Suppose that the pairs $(A, S)$ and $(B, T)$ are weakly compatible and $Ar = Sr = p$, we have $ASr = SAr$ which implies that $Ap = Sp$. We now show that $Ap = p$.

Suppose that $d(Ap, p) > 0$.

By the inequality (2.3.1), we obtain

$$\begin{aligned} d(Ap, p) &= d(Ap, Bu) \\ &\leq k \max\{d(Sp, Tu), d(Ap, Sp), d(Bu, Tu), \tfrac{1}{2}[d(Sp, Bu) + d(Ap, Tu)]\} \\ &= k \max\{d(Ap, p), 0, 0, \tfrac{1}{2}[d(Ap, p) + d(Ap, p)]\} \\ &= kd(Ap, p) < d(Ap, p), \end{aligned}$$

a contradiction.

Therefore $Ap = Sp = p$.
Now, weakly compatibility of $B$ and $T$ and $Bu = Tu = p$, we have
$BTu = TBu$ which implies that $Bp = Tp$.
We now show that $Bp = p$.
Suppose that $d(Bp, p) > 0$.
By the inequality (2.3.1), we obtain

$$\begin{aligned} d(p, Bp) &= d(Ap, Bp) \\ &\leq k \max\{d(Sp, Tp), d(Ap, Sp), d(Bp, Tp), \tfrac{1}{2}[d(Sp, Bp) + d(Ap, Tp)]\} \\ &= k \max\{d(p, Bp), 0, 0, \tfrac{1}{2}[d(p, Bp) + d(p, Bp)]\} \\ &= kd(p, Bp) < d(p, Bp), \end{aligned}$$

a contradiction.
Therefore $Ap = Bp = Sp = Tp = p$.
Hence $p$ is a common fixed point of $A, B, S$ and $T$.
Now, we suppose that the pair $(A, S)$ satisfy the property (E.A) and $T(X)$ is closed.
Then there exists a sequence $\{x_n\}$ in $X$ such that $\lim\limits_{n\to\infty} Ax_n = \lim\limits_{n\to\infty} Sx_n = p$, for some $p \in X$.
Since $A(X) \subseteq T(X)$, there exists a sequence $\{y_n\}$ in $X$ such that $Ax_n = Ty_n$.
Therefore $\lim\limits_{n\to\infty} Ax_n = \lim\limits_{n\to\infty} Sx_n = \lim\limits_{n\to\infty} Ty_n = p$.
Since $T(X)$ is closed, we have $p \in T(X)$. Then there exists $r \in X$ such that $Tr = p$.
Hence $\lim\limits_{n\to\infty} Ax_n = \lim\limits_{n\to\infty} Sx_n = \lim\limits_{n\to\infty} Ty_n = p = Tr$.
We now prove that $Br = Tr$.
Suppose that $d(Br, Tr) > 0$.
Using the inequality (2.3.1), we obtain
$d(Ax_n, Br) \leq k \max\{d(Sx_n, Tr), d(Ax_n, Sx_n), d(Br, Tr), \frac{1}{2}[d(Sx_n, Br) + d(Ax_n, Tr)]\}$.
On letting $n \to \infty$, we get
$d(Tr, Br) \leq k \max\{d(Tr, Tr), d(Tr, Tr), d(Br, Tr), \frac{1}{2}[d(Tr, Br) + d(Tr, Tr)]\}$
which implies that
$d(Tr, Br) \leq k \max\{0, 0, d(Tr, Br), \frac{1}{2}d(Tr, Br)\}$
implies that
$d(Tr, Br) \leq kd(Tr, Br) < d(Tr, Br)$,
a contradiction.
Therefore $d(Tr, Br) = 0$ implies that $Br = Tr = p$.
Further, since $B(X) \subseteq S(X)$ there exists $u \in X$ such that $Br = Su = p$.
Therefore $Br = Su = Tr = p$.
We now show that $Au = Su$.
On the contrary suppose that $Au \neq Su$.
By the inequality (2.3.1), we obtain
$d(Au, Su) = d(Au, Br) \leq k \max\{d(Su, Tr), d(Au, Su), d(Br, Tr), \frac{1}{2}[d(Su, Br) + d(Au, Tr)]\}$
which implies that
$d(Au, Su) \leq k \max\{0, d(Au, Su), 0, \frac{1}{2}[0 + d(Au, Su)]\}$
implies that
$d(Au, Su) \leq kd(Au, Su) < d(Au, Su)$,
a contradiction.
Therefore $Au = Br = Su = Tr = p$.
Suppose that the pairs $(A, S)$ and $(B, T)$ are weakly compatible and $Au = Su = p$, we have

$ASu = SAu$ which implies that $Ap = Sp$.
We now show that $Ap = p$.
Suppose that $d(Ap, p) > 0$.
By the inequality (2.3.1), we obtain

$$\begin{aligned} d(Ap,p) &= d(Ap,Bu) \\ &\leq k\max\{d(Sp,Tu), d(Ap,Sp), d(Bu,Tu), \tfrac{1}{2}[d(Sp,Bu)+d(Ap,Tu)]\} \\ &= k\max\{d(Ap,p), 0, 0, \tfrac{1}{2}[d(Ap,p)+d(Ap,p)]\} \\ &= kd(Ap,p) < d(Ap,p), \end{aligned}$$

a contradiction.
Therefore $Ap = Sp = p$.
Now, weakly compatibility of $B$ and $T$ and $Br = Tr = p$, we have
$BTr = TBr$ which implies that $Bp = Tp$.
We now show that $Bp = p$.
Suppose that $d(Bp, p) > 0$.
By the inequality (2.3.1), we obtain

$$\begin{aligned} d(p,Bp) &= d(Ap,Bp) \\ &\leq k\max\{d(Sp,Tp), d(Ap,Sp), d(Bp,Tp), \tfrac{1}{2}[d(Sp,Bp)+d(Ap,Tp)]\} \\ &= k\max\{d(p,Bp), 0, 0, \tfrac{1}{2}[d(p,Bp)+d(p,Bp)]\} \\ &= kd(p,Bp) < d(p,Bp), \end{aligned}$$

a contradiction.
Therefore $Ap = Bp = Sp = Tp = p$.
Hence $p$ is a common fixed point of $A, B, S$ and $T$.

Similarly, we can prove the result when the pair $(B, T)$ satisfies the property (E.A) and $T(X)$ is closed. Also, it can be proved when the pair $(A, S)$ satisfies the property (E.A) and $S(X)$ is closed. ∎

**Theorem 2.4.** *Let $(X, d)$ be a metric space. Let $A, B, S$ and $T$ be selfmapings of $X$ satisfy the inequality (2.3.1). Assume that*
*(i) $A(X) \subseteq T(X), B(X) \subseteq S(X)$ and*
*(ii) either $(A, S)$ and $(B, T)$ are R-weakly commuting of type $(A_f)$ (or) of type $(A_g)$.*
*If either the pair $(A, S)$ (or) $(B, T)$ satisfies the property (E.A) and either $S(X)$ (or) $T(X)$ is a closed subspace of $X$, then $A, B, S$ and $T$ have a unique common fixed point.*

*Proof.* First suppose that the pair $(B, T)$ satisfy the property (E.A) and $S(X)$ is closed. Then there exists a sequence $\{x_n\}$ in $X$ such that $\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = p$, for some $p \in X$.
Since $B(X) \subseteq S(X)$, there exists a sequence $\{y_n\}$ in $X$ such that $Bx_n = Sy_n$.
Therefore $\lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} Sy_n = p$.
We now prove that $\lim_{n\to\infty} Ay_n = p$.
Suppose that $\lim_{n\to\infty} Ay_n = q \neq p$.
Using the inequality (2.3.1), we obtain
$d(Ay_n, Bx_n) \leq k\max\{d(Sy_n, Tx_n), d(Ay_n, Sy_n), d(Bx_n, Tx_n), \frac{1}{2}[d(Sy_n, Bx_n)+d(Ay_n, Tx_n)]\}$.
On letting $n \to \infty$, we get
$d(q, p) \leq k\max\{d(p, p), d(q, p), d(p, p), \frac{1}{2}[d(p, p) + d(q, p)]\}$
which implies that $d(q, p) \leq kd(q, p) < d(q.p)$,
a contradiction.
Therefore $\lim_{n\to\infty} Ay_n = \lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Sy_n = \lim_{n\to\infty} Tx_n = p$.

Since $S(X)$ is closed, we have $p \in S(X)$. Then there exists $r \in X$ such that $Sr = p$.
Hence $\lim_{n\to\infty} Ay_n = \lim_{n\to\infty} Bx_n = \lim_{n\to\infty} Sy_n = \lim_{n\to\infty} Tx_n = p = Sr$.
Now, we prove that $Ar = Sr$.
Suppose that $d(Ar, Sr) > 0$.
By the inequality (2.3.1), we get
$d(Ar, Bx_n) \leq k \max\{d(Sr, Tx_n), d(Ar, Sr), d(Bx_n, Tx_n), \frac{1}{2}[d(Sr, Bx_n) + d(Ar, Tx_n)]\}$.
Letting $n \to \infty$, we obtain
$d(Ar, Bx_n) \leq k \max\{d(Sr, Sr), d(Ar, Sr), d(Sr, Sr), \frac{1}{2}[d(Sr, Sr) + d(Ar, Sr)]\}$
implies that
$d(Ar, Sr) \leq d(Ar, Sr) < d(Ar, Sr)$,
a contradiction.
Therefore $d(Ar, Sr) \leq 0$ implies that $Ar = Sr = p$.
Further, since $A(X) \subseteq T(X)$ there exists $u \in X$ such that $Ar = Tu = p$.
Therefore $Ar = Sr = Tu = p$.
Now, we prove that $Bu = Tu$.
On the contrary suppose that $Bu \neq Tu$.
Using the inequality (2.3.1), we obtain
$d(Ar, Bu) \leq k \max\{d(Sr, Tu), d(Ar, Sr), d(Bu, Tu), \frac{1}{2}[d(Sr, Bu) + d(Ar, Tu)]\}$
which implies that
$d(Ar, Bu) \leq k \max\{0, 0, d(Bu, Ar), \frac{1}{2}[d(Ar, Bu) + 0]\}$
implies that
$d(Ar, Bu) \leq kd(Ar, Bu) < d(Ar, Bu)$,
a contradiction.
Therefore $Ar = Bu = Sr = Tu = p$.
Suppose that the pairs $(A, S)$ and $(B, T)$ are $R$-weakly commuting of type $(A_f)$.
Then $d(ASr, SSr) \leq Rd(Ar, Sr)$
which implies that
$d(ASr, SSr) = 0$
implies that $ASr = SSr$ which implies that $Ap = Sp$.
We now show that $Ap = p$.
Suppose that $d(Ap, p) > 0$.
By the inequality (2.3.1), we obtain

$$\begin{aligned} d(Ap, p) &= d(Ap, Bu) \\ &\leq k \max\{d(Sp, Tu), d(Ap, Sp), d(Bu, Tu), \tfrac{1}{2}[d(Sp, Bu) + d(Ap, Tu)]\} \\ &= k \max\{d(Ap, p), 0, 0, \tfrac{1}{2}[d(Ap, p) + d(Ap, p)]\} \\ &= kd(Ap, p) < d(Ap, p), \end{aligned}$$

a contradiction.
Therefore $Ap = Sp = p$.
Since the pair $(B, T)$ is an $R-$weakly commuting of type $(A_f)$, we have
$d(BTu, TTu) \leq Rd(Bu, Tu)$
which implies that
$d(BTu, TTu) = 0$
implies that $BTu = TTu$ which implies that $Bp = Tp$.
We now show that $Bp = p$.
Suppose that $d(Bp, p) > 0$.
By the inequality (2.3.1), we obtain

$$d(p, Bp) = d(Ap, Bp)$$

$$\leq k \max\{d(Sp,Tp), d(Ap,Sp), d(Bp,Tp), \tfrac{1}{2}[d(Sp,Bp)+d(Ap,Tp)]\}$$
$$= k \max\{d(p,Bp), 0, 0, \tfrac{1}{2}[d(p,Bp)+d(p,Bp)]\}$$
$$= kd(p,Bp) < d(p,Bp),$$
a contradiction.
Therefore $Ap = Bp = Sp = Tp = p$.
Hence $p$ is a common fixed point of $A, B, S$ and $T$.
Similarly we can prove that the pairs $(A,S)$ and $(B,T)$ are $R$-weakly commuting of type $(A_g)$.
Now, suppose that the pair $(A,S)$ satisfy the property (E.A) and $T(X)$ is closed.
Then there exists a sequence $\{x_n\}$ in $X$ such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = p$, for some $p \in X$.
Since $A(X) \subseteq T(X)$, there exists a sequence $\{y_n\}$ in $X$ such that $Ax_n = Ty_n$.
Therefore $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = p$.
We now prove that $\lim_{n\to\infty} By_n = p$.
Suppose that $\lim_{n\to\infty} By_n = q \neq p$.
Using the inequality (2.3.1), we obtain
$d(Ax_n, By_n) \leq k \max\{d(Sx_n,Ty_n), d(Ax_n,Sx_n), d(By_n,Ty_n), \frac{1}{2}[d(Sx_n,By_n)+d(Ax_n,Ty_n)]\}$.
Letting $n \to \infty$, we get
$d(p,q) \leq k \max\{d(p,p), d(p,q), d(p,p), \frac{1}{2}[d(p,p)+d(p,q)]\}$
which implies that $d(p,q) \leq kd(p,q) < d(p.q)$,
a contradiction.
Therefore $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = p$.
Since $T(X)$ is closed, we have $p \in T(X)$.
Then there exists $r \in X$ such that $Tr = p$.
Hence $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Ty_n = p = Tr$.
We now prove that $Br = Tr$.
Suppose that $d(Br,Tr) > 0$.
Using the inequality (2.3.1), we obtain
$d(Ax_n, Br) \leq k \max\{d(Sx_n,Tr), d(Ax_n,Sx_n), d(Br,Tr), \frac{1}{2}[d(Sx_n,Br)+d(Ax_n,Tr)]\}$.
On letting $n \to \infty$, we get
$d(Tr,Br) \leq k \max\{d(Tr,Tr), d(Tr,Tr), d(Br,Tr), \frac{1}{2}[d(Tr,Br)+d(Tr,Tr)]\}$
which implies that
$d(Tr,Br) \leq k \max\{0, 0, d(Tr,Br), \frac{1}{2}d(Tr,Br)\}$
implies that
$d(Tr,Br) \leq kd(Tr,Br) < d(Tr,Br)$,
a contradiction.
Therefore $d(Tr,Br) = 0$ implies that $Br = Tr = p$.
Further, since $B(X) \subseteq S(X)$ there exists $u \in X$ such that $Br = Su = p$.
Therefore $Br = Su = Tr = p$.
We now show that $Au = Su$.
On the contrary, suppose that $Au \neq Su$.
By the inequality (2.3.1), we obtain
$d(Au,Su) = d(Au,Br) \leq k \max\{d(Su,Tr), d(Au,Su), d(Br,Tr), \frac{1}{2}[d(Su,Br)+d(Au,Tr)]\}$
which implies that
$d(Au,Su) \leq k \max\{0, d(Au,Su), 0, \frac{1}{2}[0+d(Au,Su)]\}$

implies that
$d(Au, Su) \leq kd(Au, Su) < d(Au, Su)$,
a contradiction.
Therefore $Au = Br = Su = Tr = p$.
Now, we suppose that the pairs $(A, S)$ and $(B, T)$ are $R$-weakly commuting of type $(A_g)$.
Then $d(AAu, SAu) \leq Rd(Au, Su) \Rightarrow d(AAu, SAu) \leq 0$
implies that $AAu = SAu$ which implies that $Ap = Sp$.
We now prove that $Ap = p$. Suppose that $d(Ap, p) > 0$.
Using the inequality (2.3.1), we get

$$\begin{aligned} d(Ap, p) &= d(Ap, Br) \\ &\leq k \max\{d(Sp, Tr), d(Ap, Sp), d(Br, Tr), \tfrac{1}{2}[d(Sp, Br) + d(Ap, Tr)]\} \\ &= k \max\{d(Ap, p), 0, 0, \tfrac{1}{2}[d(Ap, p) + d(Ap, p)]\} \\ &= kd(Ap, p) < d(Ap, p), \end{aligned}$$

a contradiction.
Therefore $Ap = Sp = p$.
By $R-$weakly commuting of type $(A_g)$ of $B$ and $T$, we have
$d(BBr, TBr) \leq Rd(Br, Tr) \Rightarrow d(BBr, TBr) \leq 0$
implies that $BBr = TBr$ which implies that $Bp = Tp$.
We prove now that $Bp = p$.
Suppose that $d(Bp, p) > 0$.
By the inequality (2.3.1), we obtain

$$\begin{aligned} d(p, Bp) &= d(Ap, Bp) \\ &\leq k \max\{d(Sp, Tp), d(Ap, Sp), d(Bp, Tp), \tfrac{1}{2}[d(Sp, Bp) + d(Ap, Tp)]\} \\ &= k \max\{d(p, Bp), 0, 0, \tfrac{1}{2}[d(p, Bp) + d(p, Bp)]\} \\ &= kd(p, Bp) < d(p, Bp), \end{aligned}$$

a contradiction.
Therefore $Ap = Bp = Sp = Tp = p$.
Hence $p$ is a common fixed point of $A, B, S$ and $T$.

Similarly, we can prove the result when the pairs $(A, S)$ and $(B, T)$ are $R$-weakly commuting of type $(A_f)$. ∎

## 3. Corollaries and Examples

In this section, we draw some corollaries from the main results of Section 2 and provide examples in support of our results.

The following is an example in support of Theorem 2.1.

**Example 3.1.** Let $X = [0, 2]$ with the usual metric. We define selfmaps $f, g$ on $X$ by
$f(x) = \begin{cases} \frac{2}{3} & \text{if } 0 \leq x \leq 1 \\ \frac{1}{2} & \text{if } 1 < x \leq 2, \end{cases}$ $g(x) = \begin{cases} \frac{4}{3} - x & \text{if } 0 \leq x \leq 1 \\ 0 & \text{if } 1 < x \leq 2. \end{cases}$
Let $\{x_n\} \subseteq [0, 1]$.
Then $\lim\limits_{n\to\infty} fx_n = \lim\limits_{n\to\infty} gx_n$.
$\Rightarrow \frac{2}{3} = \lim\limits_{n\to\infty} (\frac{4}{3} - x_n)$.
$\Rightarrow \lim\limits_{n\to\infty} x_n = \frac{2}{3}$.
Therefore for any $\{x_n\} \subseteq [0, 1]$ with $\lim\limits_{n\to\infty} x_n = \frac{2}{3}$, we have
$\lim\limits_{n\to\infty} fx_n = \lim\limits_{n\to\infty} gx_n = \frac{2}{3}$.

Now, $\lim\limits_{n\to\infty} gfx_n = \lim\limits_{n\to\infty} g(\frac{2}{3}) = \frac{2}{3} = g(\frac{2}{3})$
and $\lim\limits_{n\to\infty} fgx_n = \lim\limits_{n\to\infty} f(\frac{4}{3} - x_n), 0 \leq x_n \leq 1$.
Case (i): $\frac{1}{3} \leq \frac{4}{3} - x_n \leq 1$.
In this case, $f(\frac{4}{3} - x_n) = \frac{2}{3}$.
Therefore $\lim\limits_{n\to\infty} fgx_n = \lim\limits_{n\to\infty} f(\frac{4}{3} - x_n) = \frac{2}{3} = f(\frac{2}{3})$.
Case (ii): $1 \leq x_n \leq \frac{4}{3}$.
This case doesn't arise, since we are considering the sequence $\{x_n\}$ with $\lim\limits_{n\to\infty} x_n = \frac{2}{3}$.
Therefore the pair $(f, g)$ is reciprocally continuous. Clearly the pair $(f, g)$ is weakly compatible.
We now show that the maps $f$ and $g$ are $R-$weakly commuting of type $(A_f)$ and type $(A_g)$.
Let $x \in [0, 1]$.
Then $d(fgx, ggx) = d(f(\frac{4}{3} - x), g(\frac{4}{3} - x))$
If $x \in [\frac{1}{3}, 1], \frac{4}{3} - x \in [\frac{1}{3}, 1]$
then $d(f(\frac{4}{3} - x), g(\frac{4}{3} - x)) = d(\frac{2}{3}, \frac{4}{3} - x) = |\frac{2}{3} - \frac{4}{3} + x| = | - \frac{2}{3} + x|$
and $d(fx, gx) = | - \frac{2}{3} + x|$. Clearly $d(fgx, ggx) \leq Rd(fx, gx)$
If $x \in [0, \frac{1}{3}), \frac{4}{3} - x \in (1, \frac{4}{3}]$ then $d(f(\frac{4}{3} - x), g(\frac{4}{3} - x)) = d(\frac{1}{2}, 0) = \frac{1}{2}$ and $d(fx, gx) = | - \frac{2}{3} + x|$. Clearly $d(fgx, ggx) \leq Rd(fx, gx)$
Suppose $x \in (1, 2]$
Then $d(fgx, ggx) = d(f(0), g(0)) = d(\frac{2}{3}, \frac{4}{3} - x) = | - \frac{2}{3} + x|$ and $d(fx, gx) = \frac{1}{2}$.
In this case, $d(fgx, ggx) \leq Rd(fx, gx)$.
Therefore $f$ and $g$ are $R-$weakly commuting of type $(A_f)$ with $R = 3$.
Let $x \in [0, 1]$.
Then $d(ffx, gfx) = d(f(\frac{2}{3}), g(\frac{2}{3})) = 0 \leq Rd(fx, gx)$
Suppose $x \in (1, 2]$
Then $d(ffx, gfx) = d(f(\frac{1}{2}), g(\frac{1}{2})) = d(\frac{2}{3}, \frac{5}{6}) = \frac{1}{6}$ and $d(fx, gx) = d(\frac{1}{2}, 0) = \frac{1}{2}$.
Clearly $d(ffx, gfx) \leq Rd(fx, gx)$
Therefore $f$ and $g$ are $R-$weakly commuting of type $(A_g)$ with $R = 3$.
We now verify the inequality (2.1.1).
Case (i): $x, y \in [0, 1]$
$d(fx, fy) = 0 \leq k \max\{d(gx, gy), d(fx, gx), d(fy, gy), d(fx, gy), d(fy, gx)\}$
In this case, the inequality (2.1.1) trivially holds.
Case (ii): $x, y \in (1, 2]$
$d(fx, fy) = 0 \leq k \max\{d(gx, gy), d(fx, gx), d(fy, gy), d(fx, gy), d(fy, gx)\}$
In this case the inequality (2.1.1) trivially holds.
Case (iii): $x \in [0, 1], y \in (1, 2]$
$d(fx, fy) = \frac{1}{6}, d(gx, gy) = \frac{4}{3} - x, d(fx, gx) = | - \frac{2}{3} + x|, d(fy, gy) = \frac{1}{2}, d(fx, gy) = \frac{2}{3}$ and $d(fy, gx) = | - \frac{5}{6} + x|$

$$\begin{aligned} d(fx, fy) = \frac{1}{6} \leq \frac{1}{2}\frac{1}{2} &= \frac{1}{2} d(fy, gx) \\ &\leq \frac{1}{2} \max\{d(gx, gy), d(fx, gx), d(fy, gy), d(fx, gy), d(fy, gx)\} \end{aligned}$$

Case (iv): $x \in (1, 2], y \in [0, 1]$
$d(fx, fy) = \frac{1}{6}, d(gx, gy) = \frac{4}{3} - y, d(fx, gx) = \frac{1}{2}, d(fy, gy) = | - \frac{2}{3} + y|, d(fx, gy) = | - \frac{5}{6} + y|$

and $d(fy, gx) = \frac{2}{3}$

$$\begin{aligned} d(fx, fy) = \frac{1}{6} \leq \frac{1}{2}\frac{1}{2} &= \frac{1}{2} d(fx, gx) \\ &\leq \frac{1}{2} \max\{d(gx, gy), d(fx, gx), d(fy, gy), d(fx, gy), d(fy, gx)\} \end{aligned}$$

Hence from the above cases we choose $k = \frac{1}{2}$ and inequality (2.1.1) holds with $k = \frac{1}{2}$. Therefore $f$ and $g$ satisfy all the hypotheses of Theorem 2.1 and $\frac{2}{3}$ is a unique common fixed point of $f$ and $g$.
In the following, we observe that condition (iii) of Theorem 1.11 fails to hold.
Let $x \in [0, 1], y \in (1, 2]$.
$d(gx, gy) = \frac{4}{3} - x, d(fy, gy) = \frac{1}{2}, d(fx, gy) = \frac{2}{3}, d(y, gy) = y$
and $d(y, gx) + d(y, fx) = |(x + y) - \frac{4}{3}| + (y - \frac{2}{3})$
If $(x + y) < \frac{4}{3}$ then $d(y, gx) + d(y, fx) = \frac{2}{3} - x$.
Thus by choosing $x = \frac{1}{5}, y = \frac{16}{15}$, we have $x + y < \frac{4}{3}$ and
$d(gx, gy) = \frac{17}{15}, d(fy, gy) = \frac{1}{2}, d(fx, gy) = \frac{2}{3}, d(y, gy) = y$
and $d(y, gx) + d(y, fx) = \frac{7}{15}$.
Therefore

$$\begin{aligned} \min\{d(gx, gy), d(fx, gy), d(y, gy), d(gy, fy)\} &= \min\{\tfrac{17}{15}, \tfrac{2}{3}, y, \tfrac{1}{2}\} \\ &= \tfrac{1}{2} \\ &\not\leq \tfrac{7}{15} = d(y, gx) + d(y, fx). \end{aligned}$$

Hence condition (iii) of Theorem 1.11 does not hold.

**Remark 3.2.** Theorem 2.1 and Example 3.1 suggest that condition (iii) of Theorem 1.11 is redundant in proving Theorem 1.11.

**Example 3.3.** Let $X = [1, 35]$ with the usual metric. We define selfmaps $f, g$ on $X$ by

$$f(x) = \begin{cases} 6 & \text{if } 1 < x \leq 5 \\ 1 & \text{if otherwise,} \end{cases} \quad g(x) = \begin{cases} 1 & \text{if } x = 1 \\ 14 & \text{if } 1 < x \leq 5 \\ \frac{1+x}{6} & \text{if } 5 < x \leq 35. \end{cases}$$

Here $f(X) \subseteq g(X)$. The pair $(f, g)$ satisfies all the hypotheses of Theorem 2.2 and 1 is the unique common fixed point of $f$ and $g$ in $X$.

**Example 3.4.** Let $X = [0, 1]$ with the usual metric. We define selfmaps $A, B, S,$ T on $X$ by

$$A(x) = \begin{cases} \frac{x^2}{2} & \text{if } 0 \leq x \leq \frac{1}{2} \\ 0 & \text{if } \frac{1}{2} < x \leq 1, \end{cases} \quad B(x) = \begin{cases} \frac{x^2}{4} & \text{if } 0 \leq x \leq \frac{1}{2} \\ 0 & \text{if } \frac{1}{2} < x \leq 1, \end{cases}$$

$$S(x) = \begin{cases} x^2 & \text{if } 0 \leq x \leq \frac{1}{2} \\ 1 & \text{if } \frac{1}{2} < x \leq 1 \end{cases} \text{ and } T(x) = \begin{cases} \frac{x^2}{2} & \text{if } 0 \leq x \leq \frac{1}{2} \\ 0 & \text{if } \frac{1}{2} < x \leq 1. \end{cases}$$

Here $A(X) = [0, \frac{1}{8}], B(X) = [0, \frac{1}{16}], S(X) = [0, \frac{1}{4}] \cup \{1\}$ and $T(X) = [0, \frac{1}{8}]$.
Clearly $A(X) \subseteq T(X)$ and $B(X) \subseteq S(X)$.
Let $\{x_n\} = \frac{1}{2n}, n \geq 1$.
Then $\lim\limits_{n\to\infty} Ax_n = \lim\limits_{n\to\infty} Sx_n = 0$ and $\lim\limits_{n\to\infty} Bx_n = \lim\limits_{n\to\infty} Tx_n = 0$.
Therefore the pairs $(A, S)$ and $(B, T)$ satisfy the property (E.A). Clearly the pairs $(A, S)$ and $(B, T)$ are weakly compatible and $R-$weakly commuting pair of type $A_g$ with $R = \frac{1}{2}$.
Therefore $A, B, S$ and $T$ satisfy all the hypotheses of Theorem 2.3 and 0 is the unique common fixed point of $A, B, S$ and $T$.

**Corollary 3.5.** *Let $A, S$ and $T$ be selfmaps on $X$ satisfying the inequality* $d(Ax, Ty) \leq q \max\{d(Sx, Sy), d(Ax, Sx), d(Ty, Sy), \frac{1}{2}[d(Ax, Sy) + d(Ty, Sx)]\}$, *for all $x, y \in X$, where $0 \leq q < 1$. Suppose either $(A, S)$ (or) $(T, S)$ satisfies the property (E.A) and $S(X)$ is a closed subspace of $X$. If either $(A, S)$ (or) $(T, S)$ is an $R-$weakly commuting pair of type $(A_f)$ (or) of type $(A_g)$, then $A, T$ and $S$ have a unique common fixed point.*

*Proof.* By choosing $B = T$ and $T = S$ in Theorem 2.4, the conclusion follows. ∎

**Corollary 3.6.** *Let $(X, d)$ be a metric space. Let $A$ and $S$ be selfmaps of $X$. Assume that $A(X) \subseteq S(X)$ and there exists $k \in [0, 1)$ such that*

$$d(Ax, Ay) \leq k \max\{d(Sx, Sy), d(Ax, Sx), d(Ay, Sy), \tfrac{1}{2}[d(Sx, Ay)+d(Ax, Sy)]\} \qquad (3.6.1)$$

*for all $x, y \in X$. Assume that the pair $(A, S)$ is weakly compatible and satisfies property (E.A). If $S(X)$ is closed in $X$ then $A$ and $S$ have a unique common fixed point in $X$.*

*Proof.* By choosing $B = A$ and $T = S$ in Theorem 2.3, the conclusion follows. ∎

The following is an example in support of Corollary 3.5.

**Example 3.7.** Let $X = [0, 1]$ with the usual metric. We define selfmaps $A, S$ on $X$ by

$$A(x) = \begin{cases} \frac{1}{2} & \text{if } 0 \leq x \leq \frac{1}{2} \\ \frac{2}{3} & \text{if } \frac{1}{2} < x \leq 1 \end{cases}, \quad S(x) = \begin{cases} 1-x & \text{if } 0 \leq x \leq \frac{1}{2} \\ \frac{1}{3} & \text{if } \frac{1}{2} < x \leq 1. \end{cases}$$

Clearly $A$ and $S$ satisfy all the hypotheses of Corollary 3.5 and $\frac{1}{2}$ is the unique common fixed point.

**Remark 3.8.** In Corollary 3.5, if we relax the condition '$S(X)$ is closed' then $A$ and $S$ may not have a common fixed point in $X$.

**Example 3.9.** Let $X = (0, 1]$ with the usual metric. We define selfmaps $A, S$ on $X$ by

$$A(x) = \begin{cases} \frac{1}{2} & \text{if } 0 < x < \frac{1}{2} \\ \frac{2}{3} & \text{if } x = \frac{1}{2} \\ \frac{3}{4} & \text{if } \frac{1}{2} < x \leq 1 \end{cases}, \quad S(x) = \begin{cases} 1-x & \text{if } 0 < x < \frac{1}{2} \\ \frac{8}{9} & \text{if } x = \frac{1}{2} \\ \frac{1}{3} & \text{if } \frac{1}{2} < x \leq 1. \end{cases}$$

Clearly $S(X)$ is not closed, the pair $(A, S)$ satisfies all the hypotheses of Corollary 3.5 but $A, S$ have no common fixed points in $X$.

**Remark 3.10.** In Corollary 3.5, if we relax the condition 'the pair $(A, S)$ satisfies property (E.A)' then $A$ and $S$ may not have a common fixed point in $X$.

**Example 3.11.** Let $X = [0, 1]$ with the usual metric. We define selfmaps $A, S$ on $X$ by

$$A(x) = \begin{cases} \frac{2}{3} & \text{if } 0 \leq x \leq \frac{2}{3} \\ \frac{1}{2} & \text{if } \frac{2}{3} \leq x \leq 1, \end{cases} \quad S(x) = \begin{cases} 1 & \text{if } 0 \leq x < \frac{2}{3} \\ \frac{1}{2} & \text{if } \frac{2}{3} = x \\ \frac{2}{3} & \text{if } \frac{2}{3} < x \leq 1. \end{cases}$$

Here the pair $(A, S)$ satisfies all the hypotheses of Corollary 3.5 but fails to satisfies property (E.A) and $A, S$ have no common fixed points in $X$.

**Corollary 3.12.** *Let $(X, d)$ be a metric space. Let $A$ and $S$ be selfmaps of $X$. Assume that $A(X) \subseteq S(X)$ and satisfy the inequality (3.6.1). Assume that the pair $(A, S)$ is $R$-weakly commuting of type $(A_f)$ (or) of type $(A_g)$ and satisfies (E.A) property. If $S(X)$ is closed in $X$ then $A$ and $S$ have a unique common fixed point in $X$.*

*Proof.* By choosing $B = A$ and $T = S$ in Theorem 2.4, the conclusion follows. ∎

The following is an example in support of Corollary 3.12.

**Example 3.13.** Let $X = [0,1]$ with the usual metric. We define selfmaps $A, S$ on $X$ by
$A(x) = \begin{cases} \frac{x^2}{4} & \text{if } 0 \le x \le \frac{1}{2} \\ 0 & \text{if } \frac{1}{2} < x \le 1, \end{cases}$ $S(x) = \begin{cases} x^2 & \text{if } 0 \le x \le \frac{1}{2} \\ 1 & \text{if } \frac{1}{2} < x \le 1. \end{cases}$
Clearly $A$ and $S$ satisfy all the hypotheses of Corollary 3.12 and 0 is the unique common fixed point.

If we relax the condition pair $(A, S)$ is $R-$weakly commuting of type $(A_f)$ (or) of type $(A_g)$ in Corollary 3.12 then the conclusion may fails to hold due to the following example.

**Example 3.14.** Let $X = [0,1]$ with the usual metric. We define selfmaps $A, S$ on $X$ by
$A(x) = \begin{cases} \frac{3}{4} & \text{if } 0 \le x \le \frac{1}{2} \\ \frac{2}{3} & \text{if } \frac{1}{2} < x \le 1, \end{cases}$ $S(x) = \begin{cases} \frac{3}{4} & \text{if } 0 \le x < \frac{1}{2} \\ \frac{2}{3} & \text{if } x = \frac{1}{2} \\ \frac{1}{3} & \text{if } \frac{1}{2} < x \le 1. \end{cases}$
Clearly the pair $(A, S)$ is neither $R-$weakly commuting of type $(A_f)$ nor $R-$weakly commuting of type $(A_g)$, but $A$ and $S$ satisfy all the remaining hypotheses of Corollary 3.12 and have no common fixed points in $X$.

**Conflict of Interests**

*The author declare that there is no conflict of interests regarding the publication of this paper.*

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**J**ournal of **M**athematical **C**omputational **S**cience
ISSN: 2408-154X
**Bangmod-JMCS** Online @ http://bangmod-jmcs.kmutt.ac.th/


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Email: tacs@kmutt.ac.th