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The Halpern approximation of three operators splitting method for convex minimization problems with an application to image inpainting

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Abstract The three-operator splitting algorithm is a state-of-art algorithm for finding monotone inclusion problems of the sum of maximally monotone operators, where one of the operators is a cocoercive operator. Since the resolvent operator in the original three-operator splitting algorithm is not available in a closed form, we propose an inexact three-operator splitting algorithm that combines inertial forward backward splitting algorithm with the Halpern approximation method to solve monotone inclusion problem. Under mild assumptions, the theoretical convergence properties of the presented iterative technique are studied on the iterative parameters in general Hilbert spaces. Furthermore, we extend this algorithm to solve image inpainting problem. Performance comparisons show that the presented method is competitive, efficient and practical with the compared ones.

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1. Introduction

The image inpainting is a process of restoring damaged areas of an image. This field of research has been very active, prompted by numerous applications: object removal in a context of editing, loss concealment in a context of impaired image transmission, disocclusion in image–based rendering (IBR) of viewpoints different from those captured by the cameras or restoring images from text overlays or scratches. The inpainting problem appeared in [1] by analogy with a process used in art restoration.

In this article, we consider the following monotone inclusion problem:

$$\text{find } x \in \mathcal{H} \text{ such that } 0 \in Ax + Bx + Cx, \tag{1.1}$$

where $\mathcal{H}$ is a real Hilbert space, A, B are maximally monotone operators mapping from $\mathcal{H}$ onto $2^{\mathcal{H}}$ and $C : \mathcal{H} \to \mathcal{H}$ is the inverse strongly monotone operator. The corresponding convex optimization problem related to the three operator inclusion problem (1.1) is given by

$$\underset{x\in\mathcal{H}}{\text{minimize}} \quad \mathcal{R}(x) + \mathcal{S}(x) + \mathcal{P}(x), \tag{1.2}$$

where $\mathcal{R}, \mathcal{S} : \mathcal{H} \to (-\infty, +\infty]$ and $\mathcal{P} : \mathcal{H} \to \mathbb{R}$ are proper lower-semicontinuous convex and a convex continuous differentiable respectively. The gradient $\nabla\mathcal{P}$ is $L-$Lipschitz continuous for some $L > 0$. Assume that the proximity operators of $\mathcal{R}$ and $\mathcal{S}$ have an explicit closed–form solution, the three operator splitting algorithm [2] can be applied to solve the convex minimization problem (1.2) by setting $A = \partial\mathcal{R}, B = \partial\mathcal{S}$ and $C = \nabla\mathcal{P}$, where $\partial\mathcal{R}$ and $\partial\mathcal{S}$ are subdifferentials of $\mathcal{R}$ and $\mathcal{S}$ respectively. The convex optimization problem involves several specific problems that have emerged in material sciences, medical, image processing and signal processing (Refs. [3, 4]).

In special case, since (1.1) if A and B satisfy Rockafellar's condition in Theorem1 [5] can be represented by

$$\text{find } x \in \mathcal{H} \text{ such that } 0 \in \mathcal{A}x + Cx, \tag{1.3}$$

where $\mathcal{A} = A + B$
thus, convex optimization problem can be (1.2) represented by

$$\underset{x\in\mathcal{H}}{\text{minimize}} \quad \mathcal{Q}(x) + \mathcal{P}(x), \tag{1.4}$$

where $\mathcal{Q} = \mathcal{R} + \mathcal{S}$.

It is important to note that the three-operator splitting algorithm [2] is the new algorithm, as such, very few works specifically connected with it exist. In 2018, Cevher et al. [6] extended the three-operator splitting algorithm [2] from the determinist setting to the stochastic setting for solving the problem (1.1). Similarly, solving the convex minimization of the sum of three convex functions, Yurtsever et al. [7] introduced a stochastic three-composite minimization algorithm. In addition, Pedregosa and Gidel [8] developed a novel adaptive three-operator splitting algorithm, which would update the step-size without a prior knowledge of the gradient operator's Lipschitz constant. However, the Pedregosa and Gidel did not take into account the error.

Recently, an efficient fixed point equation for solving monotone inclusion problems with three operator was developed by Davis and Yin [2]. The developed equation employs resolvent and forward operators. In [2], it was shown that their fixed point equation extends the Douglas–Rachford and forward–backward equation. The Douglas-Rachford and forward-backward equation have the following form

$$T := J_\lambda^A(2J_\lambda^B - Id - \lambda C J_\lambda^B) + Id - J_\lambda^B$$

which is average given that λ is properly bounded, and by now, it is the operator for solving the problem (1.1) without employing lifting techniques. Two special cases are immediate:

(1) If $B = 0$, then

$$T := J_\lambda^A(Id - \lambda C)$$

which is the forward–backward splitting algorithm.

(2) If $C = 0$, then

$$T := J_\lambda^A(2J_\lambda^B - Id) + Id - J_\lambda^B$$

which is also the Douglas-Rachford splitting algorithm.

Now, by following the standard approach in operator-splitting, that is, the Krasnosel' skiǐ-Mann (KM) iteration [9], we can solve $x = Tx$. Given $x_n \in \mathcal{H}$ and $\alpha_n \in (0, 1)$, set

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n.$$

The above scheme can be implemented as follows:

Data: An arbitrary point $x_0 \in \mathcal{H}, \lambda \in (0, 2\beta)$, and $\{\alpha_n\} \in (0, \frac{4\beta-\lambda}{2\beta})$.
Initialization;
for $n = 0, 1, 2, \cdots$, *iterate* **do**
    compute;
    1: $y_n = J_\lambda^B x_n$
    2: $u_n = J_\lambda^A(2y_n - x_n - \lambda C y_n)$ // comment : $u_n = J_\lambda^A(2J_\lambda^B - Id - \lambda C J_\lambda^B)x_n$
    3: $x_{n+1} = x_n + \alpha_n(u_n - y_n)$ // comment: $x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T x_n$
    break when a given stopping criterion is met
end
Result: y_n, u_n and x_{n+1}.

Algorithm 1: A three operator splitting algorithm

We note that provided T has a fixed point, their KM iteration will (weakly) converge to a fixed point of T with rate $\|Tx_n - x_n\|^2 = O((n + 1)^{-1})$.

In this article, our interest is to introduce a Halpern approximation of three operator splitting algorithm solving the monotone inclusion problem (1.1). The corresponding resolvent operators and inverse strongly operator are permitted to be computed. Within mild conditions with the parameters and errors, we examine the convergence behavior of the Halpern three-operator splitting algorithm. Moerover, we recover the Halpern forward–backward splitting algorithm and the Halpern Douglas–Rachford splitting algorithm as corollaries. Finally, we extend the proposed algorithm to solve image inpainting problem.

The article is organized as the following. In Section 2 we review background on convex analysis and monotone operators. In Section 3, we give the Halpern three-operator splitting algorithm and its convergence theorem. In Section 4, we discussed the applications of the the Halpern three-operator splitting algorithm in convex minimization, image inpainting problems and present a numerical experiment in image inpainting. Finally, we conclude the paper in section 5.

Data: For arbitrary $x_0 \in \mathcal{H}$, , choose λ and $\lambda_n + \alpha_n + \beta_n = 1$.
Initialization;
for $n = 1, 2, 3, \cdots$, *iterate* **do**
  compute;
  1: $y_n = J_\lambda^B x_n$
  2: $u_n = J_\lambda^A(2y_n - x_n - \lambda(Cy_n))$
  3: $x_{n+1} = \alpha_n u + (\beta_n + \lambda_n)x_n + \lambda_n(u_n - y_n)$
  break when a given stopping criterion is met
end
Result: x_{n+1}.

Algorithm 2: The Halpern of a three-operator splitting algorithm.

We recall the following bound Young's Inequality as follow:

$$ab \leq \frac{a^2}{2\varepsilon} + \frac{b^2\varepsilon}{2} \text{ such that } \forall a, b \in R \text{ and } \forall \varepsilon < 0.$$

2. Preliminaries

Assume that $\mathbb{H}$ is a real Hilbert space. The inner product and norm of $\mathbb{H}$ are denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$ respectively. We denote the class of proper lower-semicontinuous and convex functions from $\mathbb{H}$ to $(-\infty, +\infty]$ by $\Gamma_0(\mathbb{H})$. Fix(T) is the fixed points set of an operator T.

Definition 2.1. Assume that $A : \mathcal{H} \to 2^{\mathcal{H}}$ is a set-valued operator, where $2^{\mathcal{H}}$ is the power set of $\mathcal{H}$. Suppose that Id is the identity operator on $\mathcal{H}$. Then,
(1) $A^{-1}(0) := \{x \in \mathcal{H} : 0 \in Ax\}$ is the set of zeros of A,
(2) $D(A) := \{x \in \mathcal{H} : Ax \neq \varnothing\}$ is the domain of A,
(3) $R(A) := \{y \in \mathcal{H} : \exists x \in \mathcal{H} : y \in Ax\}$ is the range of A,
(4) $G(A) := \{(x, y) \in \mathcal{H} \times \mathcal{H} : y \in Ax\}$ is the graph of A,
(5) The resolvent of A with parameter $\lambda > 0$ is $J_\lambda^A = (Id + \lambda A)^{-1}$.

Definition 2.2. Assume that $A : \mathcal{H} \to 2^{\mathcal{H}}$ is a set-valued operator. Then A is called monotone if

$$\langle x - y, u - v \rangle \geq 0, \quad \forall (x, u), (y, v) \in G(A). \tag{2.1}$$

The operator A is called maximally monotone if there is no monotone operator $B : \mathcal{H} \to 2^{\mathcal{H}}$ which the graph of B properly contains $G(A)$.

Definition 2.3. The operator $B : \mathcal{H} \to \mathcal{H}$ is called $\beta-$inverse strongly monotone with $\beta > 0$ if

$$\beta\|Bx - By\|^2 \leq \langle Bx - By, x - y\rangle, \quad \forall x, y \in \mathcal{H}. \tag{2.2}$$

Definition 2.4. Assume that $C : \mathcal{H} \to 2^{\mathcal{H}}$ is a set-valued operator. Then C is called cocoercive if there is a constant $\mu > 0$ such that

$$\langle Cx - Cy, x - y\rangle \geq \mu\|Cx - Cy\|^2, \quad \forall x, y \in \mathcal{H}. \tag{2.3}$$

Definition 2.5. Assume that $T : \mathcal{H} \to \mathcal{H}$ is an operator. T is called nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in \mathcal{H}. \tag{2.4}$$

Assume that $\alpha \in (0, 1)$. The T is called $\alpha-$averaged if there is a nonexpansive operator R such that

$$T = (1 - \alpha)Id + \alpha R.$$

If $\alpha = 1/2$, then T is said to be the firmly nonexpansive operator.

Lemma 2.6. *[10] Assume that $T : \mathcal{H} \to \mathcal{H}$ is an operator. The following statement are equivalent:*
(1) $2T - Id$ *is nonexpansive.*
(2) T *is firmly nonexpansive.*
(3) $\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y\rangle$, $\forall x, y \in \mathcal{H}$.

Lemma 2.7. *[10] Assume that $T : \mathcal{H} \to \mathcal{H}$ is a nonexpansive operator, and give $\alpha \in (0, 1)$. The following are equivalent:*
(1) $(1 - \frac{1}{\alpha})Id + \frac{1}{\alpha}T$ *is nonexpansive.*
(2) T *is $\alpha-$averaged.*
(3) $\|Tx - Ty\|^2 \leq \|x - y\|^2 - \frac{1-\alpha}{\alpha}\|(Id - T)x - (Id - T)y\|^2$, $\forall x, y \in \mathcal{H}$.

Lemma 2.8. *[10] Assume that $A : \mathcal{H} \to 2^{\mathcal{H}}$ is a maximally monotone operator and $\lambda \in (0, +\infty)$. Then $J_\lambda^A : \mathcal{H} \to \mathcal{H}$ and $Id - J_\lambda^A : \mathcal{H} \to \mathcal{H}$ are maximally monotone and firmly nonexpansive.*

Lemma 2.9. *[11] [12] Assume that $\{\alpha_n\}$ is a sequence of nonnegative real numbers satisfying the property*

$$\alpha_{n+1} \leq (1 - \gamma_n)\alpha_n + \gamma_n\sigma_n,$$

where $\{\gamma_n\} \subset (0, 1)$ and $\{\sigma_n\}$ such that
(i) $\lim_{n\to\infty} \gamma_n = 0$ *and* $\sum_{n=0}^{\infty} \gamma_n = \infty$
(ii) *either* $\limsup_{n\to\infty} \sigma_n \leq 0$ *or* $\sum_{n=0}^{\infty} |\gamma_n\sigma_n| < \infty$.
Then $\{\alpha_n\}$ converges to zero.

Lemma 2.10. *[13] Assume that X is a real inner product space. Then:*
(i) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \quad \forall\, x, y \in X.$
(ii) $\|\alpha x+\beta y\|^2 = \alpha(\alpha+\beta)\|x\|^2+\beta(\alpha+\beta)\|y\|^2-\alpha\beta\|x-y\|^2, \quad \forall\, x, y \in X, \quad \forall\, \alpha, \beta \in \mathbb{R}.$

Lemma 2.11. *[2] Let $S := U + T_1 \circ V$, where $U, T_1 : \mathcal{H} \to \mathcal{H}$ are firmly nonexpansive. Let $W = Id - (2U + V)$. So we have $\forall x, y \in \mathcal{H}$:*

$$\begin{aligned} \|Sx - Sy\|^2 \leq &\|x - y\|^2 - \|(Id - S)x - (Id - S)y\|^2 \\ &- 2\langle T_1 \circ Vx - T_1 \circ Vy,\, Wx - Wy\rangle. \end{aligned} \tag{2.5}$$

3. Main Result

Lemma 3.1. *The following set equality holds*

$$(A+B+C)^{-1}(0) = J_\lambda^B(\mathrm{Fix}(T)).$$

In addition,

$$\mathrm{Fix}(T) = \{x+\lambda u : 0 \in (A+B+C)x, u \in Bx \cap (-Ax-Cx)\}.$$

Proof. We start by showing that $(A+B+C)^{-1}(0) \subseteq J_\lambda^B(\mathrm{Fix}(T))$.

For the spacial case where $(A+B+C)^{-1}(0) = \phi$, it is obvious that $(A+B+C)^{-1}(0) \subseteq J_\lambda^B(\mathrm{Fix}(T))$. Now, suppose $x \in (A+B+C)^{-1}(0)$ then we have $0 \in Ax+Bx+Cx$. Also, let u_A, u_B be two identities such that $u_A+u_B+Cx = 0$, where $u_A \in Ax$, $u_B \in Bx$ and $z = x+\lambda u_B$, then by using two identities, we will present that z is a fixed point of T. First,

$$\begin{aligned} J_\lambda^B(z) = x \quad \text{and} \quad 2J_\lambda^B(z) - z - \lambda CJ_\lambda^B(z) &= 2x - z - \lambda Cx \\ &= x - \lambda Cx - \lambda u_B \\ &= x + \lambda u_A. \end{aligned}$$

Second,

$$x = J_\lambda^A(x+\lambda u_A) = J_\lambda^A(2J_\lambda^B(z) - z - \lambda CJ_\lambda^B(z)).$$

Combining the u_A and u_B identity, we have
$Tz = T(x+\lambda u_B) = J_\lambda^A(x+\lambda u_A) = J_\lambda^A(2J_\lambda^B(z)-z-\lambda CJ_\lambda^B(z))+(I-J_\lambda^B)(z) = x+z-x = z.$

We next show that $J_\lambda^B(\mathrm{Fix}(T)) \subseteq (A+B+C)^{-1}(0)$.

Suppose $z \in \mathrm{Fix}(T)$. Then there is $u_B \in B(J_\lambda^B(z))$ and $u_A \in A(J_\lambda^A(2J_\lambda^B(z) - z - \lambda CJ_\lambda^B(z)))$ such that

$$z = Tz = z + J_\lambda^A(2J_\lambda^B(z) - z - \lambda CJ_\lambda^B(z)) - J_\lambda^B(z) = z - \lambda(u_A + u_B + CJ_\lambda^B(z)).$$

Thus

$$x = J_\lambda^A(2J_\lambda^B(z) - z - \lambda CJ_\lambda^B(z)) = J_\lambda^B(z) \quad \text{and} \quad u_A + u_B + Cx = 0.$$

Thus, the identity for $\mathrm{Fix}(T)$ immediately following the fixed-point construction process, that is, $\mathrm{Fix}(T)$ is $\{x+\lambda u : 0 \in (A+B+C)x, u \in Bx \cap (-Ax-Cx)\}$. ∎

Proposition 3.2. *Assume that $J_\lambda^A, J_\lambda^B : \mathcal{H} \to H$ are firmly nonexpansive and C is $\beta-$cocoercive operator, $\exists\beta > 0$. Give $\lambda \in (0, 2\beta)$. Then*

$$T := Id - J_\lambda^B + J_\lambda^A(2J_\lambda^B - Id - \lambda CJ_\lambda^B)$$

is $\alpha-$averaged with coefficient $\alpha := \frac{2\beta}{4\beta-\lambda} < 1$. In additional, the following inequality holds $\forall x, y \in \mathcal{H}$

$$\|Tx - Ty\|^2 \leq \|x-y\|^2 - \frac{1-\alpha}{\alpha}\|(Id-T)x - (Id-T)y\|^2. \tag{3.1}$$

Proof. Let $U := Id - J_\lambda^B$, $V := 2J_\lambda^B - Id - \lambda C \circ J_\lambda^B$, and $W := \lambda C \circ J_\lambda^B$. By (2.5),$U$ is firmly nonexpansive. Therefore, we get $W = Id - (2U + V)$ and $S := T = Id - J_\lambda^B + J_\lambda^A \circ V$. Thus, We assess the inner product in Lemma 2.11 as follows:

$$\begin{aligned}
&-2\langle J_\lambda^A \circ Vx - J_\lambda^A \circ Vy, Wx - Wy\rangle \\
&\quad = 2\langle (Id - T)x - (Id - T)y, \lambda C \circ J_\lambda^B x - \lambda C \circ J_\lambda^B y\rangle \\
&\quad\quad - 2\langle J_\lambda^B x - J_\lambda^B y, \lambda C \circ J_\lambda^B x - \lambda C \circ J_\lambda^B y\rangle \\
&\quad \le \epsilon \|(Id - T)x - (Id - T)y\|^2 + \frac{\lambda^2}{\epsilon}\|C \circ J_\lambda^B x - C \circ J_\lambda^B y\|^2 \\
&\quad\quad - 2\lambda\beta\|C \circ J_\lambda^B x - C \circ J_\lambda^B y\|^2 \\
&\quad = \epsilon \|(Id - T)x - (Id - T)y\|^2 - \lambda(2\beta - \frac{\lambda}{\epsilon})\|C \circ J_\lambda^B x - C \circ J_\lambda^B y\|^2,
\end{aligned}$$

where $\epsilon > 0$ and C is $\beta-$cocoercive. For the coefficient of $\lambda(2\beta - \frac{\lambda}{\epsilon}) \ge 0$, we set $0 < \epsilon \le \frac{\lambda}{2\beta} < 1$. By Lemma 2.11 and setting $S = T$, we have

$$\begin{aligned}
\|Tx - Ty\|^2 \le & \|x - y\|^2 - (1 - \epsilon)\|(Id - T)x - (Id - T)y\|^2 \\
& - \lambda(2\beta - \frac{\lambda}{\epsilon})\|C \circ J_\lambda^B x - C \circ J_\lambda^B y\|^2,
\end{aligned}$$

where $\epsilon = \frac{\lambda}{2\beta}$. ∎

Theorem 3.3. *Suppose that $A : \mathcal{H} \to 2^{\mathcal{H}}$ and $B : \mathcal{H} \to 2^{\mathcal{H}}$ are maximally monotone operators. Suppose that $C : \mathcal{H} \to 2^{\mathcal{H}}$ is a β-cocoercive operator and*

$$\Omega := (A + B + C)^{-1}(0) \neq \emptyset.$$

Let $\lambda > 0$ and $T : \mathcal{H} \to \mathcal{H}$ as following,

$$T := Id - J_\lambda^B + J_\lambda^A(2J_\lambda^B - Id - \lambda C J_\lambda^B).$$

Assume that $\lambda \in (0, 2\beta)$ and $\alpha_n + \beta_n + \lambda_n = 1$ such that

(*i*) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^\infty \alpha_n = \infty$;
(*ii*) $0 < \liminf \beta_n < \limsup_{n\to\infty} \beta_n < 1$.

Then the $\{x_n\}$ in Algorithm 2 converges strongly to a point $w \in \mathrm{Fix}(T)$. Moreover $\{x_n\}$ converge strongly to $P_\Omega(u)$

Proof. The iterative sequence $\{x_{n+1}\}$ of Algorithm 2 can be written as follows

$$\begin{aligned}
x_{n+1} &= \alpha_n u + (\beta_n + \lambda_n)x_n + \lambda_n(u_n - y_n) \\
&= \alpha_n u + \beta_n x_n + \lambda_n x_n + \lambda_n(J_\lambda^A(2y_n - x_n - \lambda(Cy_n)) - J_\lambda^B x_n) \\
&= \alpha_n u + \beta_n x_n + \lambda_n\Big[x_n - J_\lambda^B x_n + J_\lambda^A(2J_\lambda^B x_n - x_n - \lambda(CJ_\lambda^B x_n))\Big] \\
&= \alpha_n u + \beta_n x_n + \lambda_n T x_n.
\end{aligned} \tag{3.2}$$

Next, we will to show the $\{x_n\}$ converges strongly to $w \in \mathrm{Fix}(T)$, we will divide the proof into four steps.

Step 1: We will show that the $\{x_n\}$ is bounded.

$$\begin{aligned} \|x_{n+1} - w\| &= \|\alpha_n u + \beta_n x_n + \lambda_n T x_n - w\| \\ &\leq \alpha_n \|u - w\| + \beta_n \|x_n - w\| + \lambda_n \|T x_n - T w\| \\ &\leq (1 - \alpha_n)\|x_n - w\| + \alpha_n \|u - w\| \\ &\leq \max \Big\{ \|u - w\|, \|x_n - w\| \Big\} \qquad (3.3) \\ &\ \ \vdots \\ &\leq \max \Big\{ \|u - w\|, \|x_0 - w\| \Big\}. \end{aligned}$$

Therefore, the $\{x_n\}$ is bounded and also $\{y_n\}$ and $\{u_n\}$ are bounded.

Step 2: We will show that $\lim_{n\to\infty} \|x_n - T x_n\| = 0$. Combining Lemma 2.10 (i) and (3.2), we have

$$\begin{aligned} \|x_{n+1} - w\|^2 &= \|\alpha_n u + \beta_n x_n + \lambda_n T x_n - w\|^2 \\ &\leq \|\beta_n (x_n - w) + \lambda_n (T x_n - w)\|^2 + 2\alpha_n \langle u - w, x_{n+1} - w\rangle. \end{aligned} \tag{3.4}$$

On the other hand, by Lemma 2.10 (ii) , bound Young's Inequality and the Cauchy-schwartz inequality, we obtain

$$\begin{aligned} &\|\beta_n(x_n - w) + \lambda_n(T x_n - w)\|^2 \\ &= \beta_n(\beta_n + \lambda_n)\|x_n - w\|^2 + \lambda_n(\beta_n + \lambda_n)\|T x_n - w\|^2 \\ &\quad - \beta_n\lambda_n \|T x_n - x_n\|^2 \\ &\leq \beta_n(1 - \alpha_n)\|x_n - w\|^2 + \lambda_n(1 - \alpha_n)\Big[\|x_n - w\|^2 \\ &\quad - \frac{1 - \alpha}{\alpha}\|T x_n - x_n\|^2 - \gamma(2\beta - \frac{\gamma}{\epsilon})\|C J_\lambda^B x_n - C J_\lambda^B w\|^2\Big] \\ &\quad - \beta_n\lambda_n \|T x_n - x_n\|^2 \\ &\leq (1 - \alpha_n)\|x_n - w\|^2 - \frac{\lambda_n(1 - \alpha_n)(1 - \alpha) + \alpha\beta_n\lambda_n}{\alpha}\|T x_n - x_n\|^2 \\ &\quad - \lambda_n(1 - \alpha_n)\gamma(2\beta - \frac{\gamma}{\epsilon})\|C J_\lambda^B x_n - C J_\lambda^B w\|^2. \end{aligned} \tag{3.5}$$

Substituting (3.5) into (3.4), we have

$$\begin{aligned} \|x_{n+1} - w\|^2 &\leq (1 - \alpha_n)\|x_n - w\|^2 - \frac{\lambda_n(1 - \alpha_n)(1 - \alpha) + \alpha\beta_n\lambda_n}{\alpha}\|T x_n - x_n\|^2 \\ &\quad - \lambda_n(1 - \alpha_n)\gamma(2\beta - \frac{\gamma}{\epsilon})\|C J_\lambda^B x_n - C J_\lambda^B w\|^2 \\ &\quad + 2\alpha_n\langle U - w, x_{n+1} - w\rangle \end{aligned} \tag{3.6}$$

and also

$$\|x_{n+1}-w\|^2 \le \|x_n-w\|^2 - \frac{\lambda_n(1-\alpha_n)(1-\alpha)+\alpha\beta_n\lambda_n}{\alpha}\|Tx_n-x_n\|^2 - \lambda_n(1-\alpha_n)\gamma(2\beta-\frac{\gamma}{\epsilon})\|CJ_\lambda^B x_n - CJ_\lambda^B w\|^2 + \alpha_n M, \tag{3.7}$$

where $M=\sup_{n\in\mathbb{N}}\{2\langle u-w, x_{n+1}-w\rangle\}$ and hence

$$\|x_{n+1}-w\|^2 \le \|x_n-w\|^2 - \lambda_n(1-\alpha_n)\gamma(2\beta-\frac{\gamma}{\epsilon})\|CJ_\lambda^B x_n - CJ_\lambda^B w\|^2 + \alpha_n M. \tag{3.8}$$

Then we have

$$\lambda_n(1-\alpha_n)\gamma(2\beta-\frac{\gamma}{\epsilon})\|CJ_\lambda^B x_n - CJ_\lambda^B w\|^2 \le \|x_n-w\|^2 - \|x_{n+1}-w\|^2 + \alpha_n M. \tag{3.9}$$

From (3.7), we obtain

$$\frac{\lambda_n(1-\alpha_n)(1-\alpha)+\alpha\beta_n\lambda_n}{\alpha}\|Tx_n-x_n\|^2 \le \|x_n-w\|^2 - \|x_{n+1}-w\|^2 + \alpha_n M, \tag{3.10}$$

for some $M>0$. In fact, by condition (ii) we can assume that there is $\epsilon>0$ such that $\beta_n\lambda_n \ge \epsilon$ for $n\in\mathbb{N}$. Therefore, we obtain from (3.10) and conditions (i),(ii) that

$$\lim_{n\to\infty}\|x_n-Tx_n\|=0 \tag{3.11}$$

and

$$\lim_{n\to\infty}\|CJ_\lambda^B x_n - CJ_\lambda^B w\|^2=0. \tag{3.12}$$

Step 3: We will show that $w\in\Omega$.
Let $u_n^B := \frac{1}{\lambda}(w_n-y_n)\in By_n$ and $u_n^A := \frac{1}{\lambda}(2y_n-w_n-\lambda(Cy_n)-u_n)\in Au_n$.
It follows from the nonexpansiveness of J_λ^B, that

$$\begin{aligned}\|y_n-J_\lambda^B w\| &= \|J_\lambda^B x_n - J_\lambda^B w\| \\ &\le \|x_n-w\|.\end{aligned} \tag{3.13}$$

Notice that if $\lim\limits_{n\to\infty}\|x_n-w\|$ exists, then $\{y_n\}$ is bounded. Let z be a sequential weak cluster point of $\{y_n\}$. There is the $\{y_{n_k}\}$ such that $y_{n_k}\rightharpoonup z$ as $n_k\to\infty$. Let $w^*=J_\lambda^B w$. Then $w^*\in\Omega$. By (3.12), we have $Cy_n\to Cw^*$. Notice that $y_{n_k}\rightharpoonup z$, since C is maximally monotone, it following the weak-to-strong sequential closedness of C that, $Cz=Cw^*$. Then $Cy_n\to Cz$.

Step 4: We will prove that $\lim_{n\to\infty}\|x_n-w\|=0$.
Since $\{x_n\}$ is bounded, we have $\{x_{n_k}\}$ such that

$$\limsup_{n\to\infty}\langle u-w, x_n-w\rangle = \lim_{k\to\infty}\langle u-w, x_{n_k}-w\rangle$$

and $\{x_{n_k}\}$ converges weakly to some element p. Hence, we obtain

$$\begin{aligned}\limsup_{n\to\infty}\langle u-w, x_{n+1}-w\rangle &= \limsup_{n\to\infty}\langle u-w, x_n-w\rangle\\ &= \lim_{k\to\infty}\langle u-w, x_{n_k}-w\rangle\\ &= \langle u-w, p-w\rangle \leq 0.\end{aligned}$$

Now, we have from (3.6) that

$$\begin{aligned}\|x_{n+1}-w\|^2 \leq (1-\alpha_n)\|x_n-w\|^2 - \frac{\lambda_n(1-\alpha_n)(1-\alpha)+\alpha\beta_n\lambda_n}{\alpha}\|Tx_n-x_n\|^2\\ -\lambda_n(1-\alpha_n)\gamma(2\beta-\frac{\gamma}{\epsilon})\|CJ_\lambda^B x_n - CJ_\lambda^B w\|^2+\alpha_n M\end{aligned} \tag{3.14}$$

By using conditions (i), (ii) and Lemma 2.9 in (3.14), we obtain that

$$\lim_{n\to\infty}\|x_n-w\| = 0. \tag{3.15}$$

∎

4. Applications

4.1. General Convex Problems

Here, we interest the problem

$$\underset{x\in\mathcal{H}}{\text{minimize}} \quad \mathcal{S}(x)+\mathcal{R}(x)+\mathcal{P}(x), \tag{4.1}$$

where $\mathcal{S},\mathcal{R}:\mathcal{H}\to(-\infty,\infty]$ are closed proper convex functions, $\mathcal{P}:\mathcal{H}\to(-\infty,\infty)$ is convex and differentiable, and $\nabla\mathcal{P}$ is $\frac{1}{\beta}-$Lipschitz continuous. Obviously, we meet the conditions of problems (1.1) with $A:=\partial\mathcal{S}$, $B:=\partial\mathcal{R}$, and $C:=\nabla\mathcal{P}$. We make the following technical assumption:

Assumption 4.1. The set $\text{zer}(\partial\mathcal{S}+\partial\mathcal{R}+\nabla\mathcal{P})$ is nonempty.

Note that the above assumption is guaranteed if $0\in\text{sri}(D(\mathcal{S})-D(\mathcal{R}))$. With no doubt, any zero of $\partial\mathcal{S}+\partial\mathcal{R}+\nabla\mathcal{P}$ is a solution of (4.1). Specialized to (4.1), Algorithm 2 becomes.

Data: For arbitrary $z_0\in\mathcal{H}$, choose λ and $\lambda_n+\alpha_n+\beta_n=1$.
Initialization;
for $n=0,1,2,\cdots$, *iterate* **do**
  compute;
  1: $y_n=\mathbf{prox}_{\lambda\mathcal{R}}x_n$
  2: $u_n=\mathbf{prox}_{\lambda\mathcal{S}}(2y_n-x_n-\lambda\nabla\mathcal{P}y_n)$
  3: $x_{n+1}=\alpha_n u+(\beta_n+\lambda_n)x_n+\lambda_n(u_n-y_n)$
  break when a given stopping criterion is met
end
Result: x_{n+1}.

Algorithm 3: The halpern approximation of three operator splitting algorithm for minimization problem

4.2. The Inpainting Problem

Image inpainting is an ill–posed inverse problem because it does not well–defined unique solution. It is necessary to introduce image priors. Many methods are guided by the assumption that pixels in the unknown and known parts of the image share the same geometrical structures or statistical properties. This assumption translates into different local or global priors, with the goal of having an inpainted image as physically plausible and as visually pleasing as possible.

The image I mathematically defined as

$$I : \Omega \subset \mathbb{R}^n \to \mathbb{R}^m,$$

where x represents a vector indicating spatial coordinates of a pixel p_x. There are two case of image. In the case of a gray scale image where by the image is two-dimensional (2-D), x is defined as $x = (x, y)$ such that x is row and y is column. In the case of a color image where by the image is three-dimensional (3-D), x is defined as $x = (x, y, z)$such that x is row, y is column and z is color channal.

The image inpainting problem for gray scale image is formulated as follows

$$\underset{x}{\text{minimize}} \quad \omega\|x_{(1)}\|_* + \omega\|x_{(2)}\|_* + \frac{1}{2}\|P_\Omega x - P_\Omega y\|^2, \tag{4.2}$$

where $x_{(1)}$ is the matrix $[x(:,:)]$, $x_{(2)}$ is the matrix $[x(:,:)^T]$, y is the gray scale texture image represented also, where $[y(:,:)]$ represents the gray scale channel of the image. The linear operator P_Ω selects the set of known entries of y $(P_\Omega y)$, $\|\cdot\|_*$ denotes the matrix nuclear norm, and ω is a penalty parameter.

The image inpainting problem for color image is formulated as follows

$$\underset{x}{\text{minimize}} \quad \omega\|x_{(1)}\|_* + \omega\|x_{(2)}\|_* + \frac{1}{2}\|P_\Omega x - P_\Omega y\|^2, \tag{4.3}$$

where x is the 3-way tensor variable, $x_{(1)}$ is the matrix $[x(:,:,1)x(:,:,2)x(:,:,3)]$, $x_{(2)}$ is the matrix $[x(:,:,1)^T x(:,:,2)^T x(:,:,3)^T]$, y is the color texture image represented also in a 3-way tensor, where $y(:,:,1), y(:,:,2), y(:,:,3)$ represents the red, green, and blue channels of the image respectively. The linear operator P_Ω selects the set of known entries of y $(P_\Omega y)$, $\|\cdot\|_*$ denotes the matrix nuclear norm, and ω is a penalty parameter.

Problem (4.3) can be formulated to problem (1.1), so it can be solved. The proximal mapping of the term $\|\cdot\|_*$ can be computed using singular value soft-thresholding and P_Ω for gray scale image is defined by

$$P_\Omega(x) = \begin{cases} x_{ij}, & (i,j) \in \Omega \\ 0, & \text{otherwise.} \end{cases}$$

Similarly, P_Ω for color image is defined by

$$P_\Omega(x) = \begin{cases} x_{ijk}, & (i,j,k) \in \Omega \\ 0, & \text{otherwise.} \end{cases}$$

It is obvious that optimization problem (4.3) is a special case of the optimization problem of the sum of three convex functions (4.1) Actually, let $\mathcal{P}(x) = \frac{1}{2}\|P_\Omega x - P_\Omega y\|_F^2$,

$\mathcal{R}(x) = \omega\|x_{(1)}\|_*$ and $\mathcal{S}(x) = \omega\|x_{(2)}\|_*$. Then $\mathcal{P}(x)$ is convex differentiable and $\nabla\mathcal{P}(x) = P_\Omega x - P_\Omega y$ with $1-$Lipschitz continuous. The proximity operators of $\mathcal{R}(x)$ and $\mathcal{S}(x)$ can be computed by the singular value decomposition (SVD). Thus, the three operator splitting algorithm and the Halpern approximation of three operator splitting algorithm can be employed to solve convex minimization problem (1.2). To evaluate the performance of our method Algorithm 3, we test it against the averger filter and Davis and Yin algorithm [2]. See Figure 1 and 2.

4.3. Evaluation and parameters setting

Evaluating the quality of the restored images, we use the signal-to-noise ratio (SNR) and the structural similarity index method (SSIM), which are assigned by

$$\text{SNR} = 20\log\frac{\|x^*\|}{\|x^* - x_n\|}, \text{ and SSIM} = \frac{(2u_{x^*}u_{x_n} + c_1)(2\sigma_{x^*x_n} + c_2)}{(u_{x^*}^2 + u_{x_n}^2 + c_1)(\sigma_{x^*}^2 + \sigma_{x_n}^2 + c_2)},$$

where x^* is the original image, x_n is the restored image, u_{x^*} and u_{x_n} are the mean values of the original image x^* and restored image x_n respectively. The variances of the original and restored images are $\sigma_{x^*}^2$ and $\sigma_{x_n}^2$ while $\sigma_{x^*x_n}^2$ is the covariance of two images, $c_1 = (K_1L)^2$ and $c_2 = (K_2L)^2$ with $K_1 = 0.01$, $K_2 = 0.03$ and L is the dynamic range of pixel values. The value for the SSIM ranges from 0 to 1, and a SSIM value of 1 means perfect recovery.

The iterative process stops when the relative change between successive iterates falls below stopping criterion, that is

$$\frac{\|x_{n+1} - x_n\|}{\|x_n\|} \le \epsilon, \text{ where } \epsilon \text{ is a given small constant.}$$

5. Conclusions

We have presented an algorithm that combines inertial forward backward splitting algorithm with the Halpern approximation method for finding a solution of the sum of three monotone operators. We show that the proposed algorithm generate the sequence that is strong convergence to solution in problem. Numerical experiments in solving image inpainting problem show that the proposed algorithm is competitive, practical and efficient with the compared ones.

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(a) original image


(b) SSIM=0.5001, PSNR=14.6186, SNR=12.2773


(c) SSIM=0.7005, PSNR=20.7961, SNR=18.5260, time=0.4004s


(d) SSIM=0.9917, PSNR=38.5180, SNR=36.2479, ter=125, time=36.5129s


(e) SSIM=0.9918, PSNR=38.5276, SNR=36.2574, ter=125, time=37.4431s


(f) original image


(g) SSIM=0.6720, PSNR=15.0868, SNR=12.1699


(h) SSIM=0.9144, PSNR=22.2323, SNR=19.3269, time=0.5062s


(i) SSIM=0.9966, PSNR=37.5200, SNR=34.6146, ter=160, time=42.8769s


(j) SSIM=0.9966, PSNR=37.5422, SNR=34.6367, ter=175, time=50.7544s


(k) original image


(l) SSIM=0.5405, PSNR=15.0321, SNR=8.3605


(m) SSIM=0.7723, PSNR=21.8970, SNR=14.5402, time=0.6084s


(n) SSIM=0.9966, PSNR=40.4832, SNR=33.1263, ter=110, time=27.2598s


(o) SSIM=0.9967, PSNR=40.5013, SNR=33.1445, ter=105, time=28.2685s


(p) original image


(q) SSIM=0.2796, PSNR=15.1709, SNR=12.8282


(r) SSIM=0.4397, PSNR=22.4756, SNR=20.1993, time=0.3187s


(s) SSIM=0.9979, PSNR=45.5473, SNR=43.2710, ter=160, time=46.4510s


(t) SSIM=0.9979, PSNR=45.5542, SNR=43.2779, ter=145, time=41.3087s

FIGURE 1. Figure (a), (f), (k), (p) are the original images (Left Column), figure (b), (g), (l), (q) (Middle left column) are the blur images, figure (c), (h), (m), (r) (Middle) are the inpainting images via Averger Filter, figure (d), (i), (n), (s) (Middle right) are the inpainting images via Davis and Vin algorithm and figure (e), (j), (o), (t), (Right) are the inpainting image via our Algorithm .

(a) original image (b) SSIM=0.2805, PSNR=15.2942, SNR=9.8174 (c) SSIM=0.4337, PSNR=22.7553, SNR=16.8945, time=0.2751s (d) SSIM=0.9992, PSNR=50.0282, SNR=44.1673, ter=195, time=54.9499s (e) SSIM=0.9992, PSNR=50.0339, SNR=44.1731, ter=190, time=57.0489s

(f) original image (g) SSIM=0.2635, PSNR=15.5148, SNR=12.2842 (h) SSIM=0.4184, PSNR=23.1662, SNR=19.9103, time=0.2643s (i) SSIM=0.9991, PSNR=52.6563, SNR=49.4004, ter=160, time=35.8549s (j) SSIM=0.9991, PSNR=52.6725, SNR=49.4166, ter=155, time=36.4337s

(k) original image (l) SSIM=0.3392, PSNR=15.4393, SNR=11.5434 (m) SSIM=0.5336, PSNR=22.0742, SNR=18.0666, time=0.2900s (n) SSIM=0.9992, PSNR=51.9985, SNR=47.9909, ter=150, time=34.3021s (o) SSIM=0.9992, PSNR=52.0104, SNR=48.0029, ter=140, time=33.0843s

(p) original image (q) SSIM=0.2388, PSNR=15.1836, SNR=11.4240 (r) SSIM=0.3262, PSNR=22.3627, SNR=18.5101, time=0.3421s (s) SSIM=0.9890, PSNR=37.3092, SNR=33.4566, ter=240, time=63.9005s (t) SSIM=0.9891, PSNR=37.3275, SNR=33.4749, ter=255, time=71.0848s

FIGURE 2. Figure (a), (f), (k), (p) are the original images (Left Column), figure (b), (g), (l), (q) (Middle left column) are the blur images, figure (c), (h), (m), (r) (Middle) are the inpainting images via Averger Filter, figure (d), (i), (n), (s) (Middle right) are the inpainting images via Davis and Vin algorithm and figure (e), (j), (o), (t), (Right) are the inpainting image via our Algorithm .

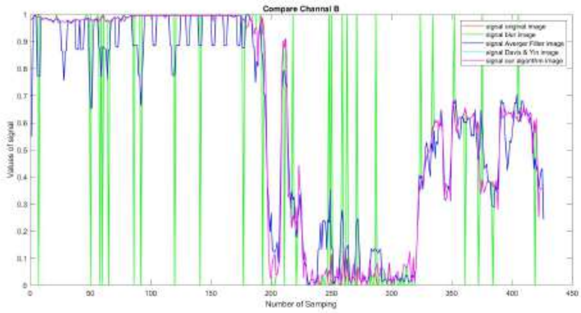

(a) original image


(b) blur image

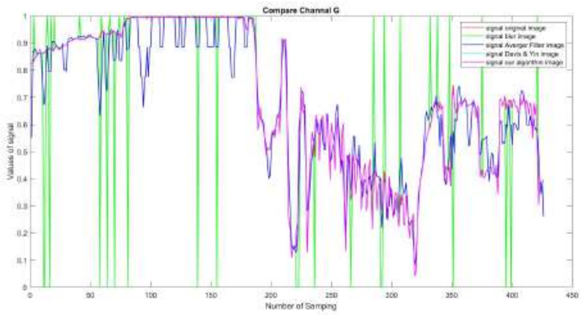

(c) Averger Filter


(d) Davis and Yin

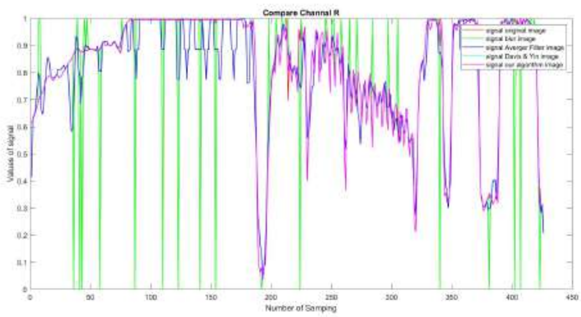

(e) our algorithm



(f) singnal of original & restoration



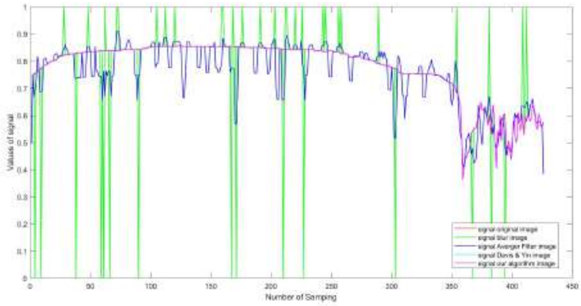
(g) singnal of original & restoration



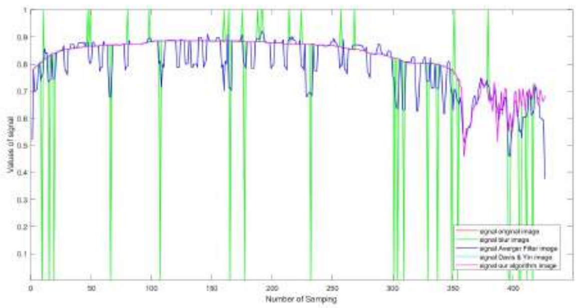
(h) singnal of original & restoration

FIGURE 3. Figure (a), (b), (c), (d), (e) shows the original, blur, Averger Filter, Davis and Yin and our algorithm image respectively, figures (f), (g), (h) shows the signal of original and restoration are the blue, green and red channels of the image, respectively .

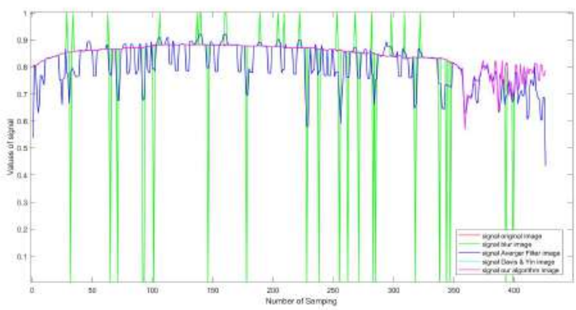
(a) original image (b) blur image (c) Averger Filter (d) Davis and Yin (e) our algorithm



(f) singnal of original & restoration



(g) singnal of original & restoration



(h) singnal of original & restoration

FIGURE 4. Figure (a), (b), (c), (d), (e) shows the original, blur, Averger Filter, Davis and Yin and our algorithm image respectively, figures (f), (g), (h) shows the signal of original and restoration are the blue, green and red channels of the image, respectively .

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