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APPROXIMATIING METHODS FOR MONOTONE INCLUSION AND TWO VARIATIONAL INEQUALITY PROBLEMS

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Abstract In this work, we consider a common solution of the monotone inclusion problems and variational inequality problems in framework of Hilbert space. For solving this problem, we propose a new inertial forwardbackward algorithm involving an extrapolation factor and describe a projection-type method and also prove the strong convergence theorem of proposed algorithm under some mild conditions. For supporting our main results, we provide some numerical experiments in the real line.

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1. Introduction

The classical inclusion problem is to find a element x in a Hilbert space H such that

$$x \in (A+B)^{-1}0. \tag{1.1}$$

where A is an operator on H and B is a set-valued operator on H. This problem includes, as special cases, variational inequality problem, convex programming problems, split feasibility problem, linear inverse problem and minimization problem. The forward-backward splitting method is a classical method for solving problem (1.1) defined by the

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iteration following: for arbitrarily $x_0 \in H$,

$$x_{n+1} = (I + rB)^{-1}(x_n - rAx_n) \ \forall n \geq 0. \tag{1.2}$$

Also, the following splitting iterative methods in a real Hilbert space introduced by Lions and Mercier [1] which is the nonlinear Peaceman-Rachford algorithm expressed as follows: for arbitrarily $x_0 \in H$,

$$x_{n+1} = (2J_r^A - I)(2J_r^B - I)x_n \ \forall n \geq 0, \tag{1.3}$$

and the nonlinear Douglas-Rachford algorithm wriiten as,

$$x_{n+1} = [J_r^A(2J_r^B - I) + (I - J_r^B)]x_n \ \forall n \geq 0. \tag{1.4}$$

where A is a operter and $J_r^A = (I + rA)^{-1}$ is the resolvent of A with $r > 0$. Lions and Mercier [1] show that both algorithms is weakly convergent to a solution. In 2001, Alvarez and Attouch [2] interoduced the inertial proximal point algorithm of a general maximal monotone operator which is translated from the heavy ball method in the framework of the proximal point algorithm. The inertial proximal point algorithm is expressed as follows: for arbitrarily $x_0, x_1 \in H$,

$$\begin{aligned} y_n &= x_n + \phi_n(x_n - x_{n-1}) \\ x_{n+1} &= (I + r_nB)^{-1}(x_n) \ \forall n \geq 0, \end{aligned} \tag{1.5}$$

where r_n is non-decreasing and $\phi_n \in [0, 1)$ with

$$\sum_{n=1}^{\infty} \phi_n \|x_n - x_{n-1}\|^2 < \infty. \tag{1.6}$$

It is shown that under certain conditions (1.6), the algorithm (1.5) is weakly convergent to a zero point of a maximal monotone operator B. In 2003, Moudafi and Oliny [3] proposed the inertial proximal point algorithm by adding a Lipschitz continuous operator A as follows: for arbitrarily $x_0, x_1 \in H$,

$$\begin{aligned} y_n &= x_n + \phi_n(x_n - x_{n-1}) \\ x_{n+1} &= (I + r_nB)^{-1}(y_n - r_nAx_n) \ \forall n \geq 0, \end{aligned} \tag{1.7}$$

where A is an operator on H and B is a set-valued operator on H. Under the same condition (1.6) and $r_n < \frac{2}{L}$, this algorithm is weakly convergent to a solution, where L is the Lipschitz constant of A. Recently, Cholamjiak and Suantai [4] studied a generalized inertial forward-backward for monotone operators A; for arbitrarily $x_0, x_1 \in H$,

$$\begin{aligned} y_n &= x_n + \phi_n(x_n - x_{n-1}) \\ x_{n+1} &= \alpha_n u + \beta_n y_n + \gamma_n J_r^B(y_n - r_nAy_n) \ \forall n \geq 0. \end{aligned} \tag{1.8}$$

They presented a sequence $\{x_n\}$ generated by the algorithm (1.8) strongly converges to a zero point of sum of two monotone operators (1.1). Stampacchia [5, 6] initially studied following a classical variational inequality denoted by $VI(C, E)$ and ever since have been widely studied that is to find $x \in C$ such that

$$\langle Ex, y - x \rangle \geq 0, \tag{1.9}$$

with a constant $\lambda > 0$. For solving a solution of a classical variational inequality, the projection algorithm is proposed according to the formula:

$$x_{n+1} = P_C(x_n - \alpha_n Ex_n) \ \forall n \geq 1, \tag{1.10}$$

with a real sequence $\{\alpha_n\} \subseteq (0,\infty)$. Iiduka, Takahashi and Toyoda [7] presented the sequence $\{x_n\}$ generated by (1.10) weakly converges to a solution of $VI(C,E)$ where E is α-inverse strongly monotone. In 2004, Iiduka, Takahashi and Toyoda [7] studied the following a hybrid projection iterative scheme: for arbitrarily $x_0 \in H$ and $n \geq 1$,

$$\begin{cases} u_n = P_C(x_n - \lambda_n E x_n), \\ C_n = \{v \in C : \|u_n - v\| \leq \|x_n - v\|\}, \\ Q_n = \{v \in C : \langle x_n - v, x_n - x_0 \rangle \leq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n}(x_0), \end{cases} \tag{1.11}$$

where $\{\lambda_n\}$ is a positive real sequence. They proved that the sequence $\{x_n\}$ generated by (1.11) converges strongly to $P_{VI(C,E)}(x_0)$. In 2012, the modified the set of two variational inequality problems is proposed by Kangtunyakarn [8] as follows:

$$VI(C, \alpha E + (1-\alpha)F) = \{x \in C : \langle z - x, (\alpha E + (1-\alpha)F)x \rangle \geq 0, \forall z \in C\}, \tag{1.12}$$

where $\alpha \in (0,1)$, E and F are the mappings of C into H. He also introduced a new iterative scheme for finding a solution of two sets of variational inequality problem $VI(C, \alpha E + (1-\alpha)F)$ and proved the strong convergence theorem of two sets of variational inequality problem, fixed point problems of infinite family of pseudo contractive mappings and the equilibrium problem. In this paper, inspired by [4, 8, 9] we modify an algorithm for solving a common solution of two operator monotone inclusions problems and two variational inequality problems and also prove the strong convergence theorem of proposed algorithm. Moreover, we provide some numerical experiments for supporting our main results.

2. Preliminaries

Let H be a real Hillbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\|\cdot\|$, and let C be a nonempty closed convex subset of H, We write $x_n \to x$ to indicate that the sequence $\{x_n\}$ converge strongly to x. Next we present several properties of operators and set-valued mappings which will be useful later on. In Hillbert space, it is well known that,

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2,$$

for all x, y in H and $\lambda \in [0,1]$. For any point $x \in H$, there exists a unique nearest point of C, denoted by $P_C(x)$ such that $\|x - P_C(x)\| \leq \|x - y\|$, for all $y \in C$. The operator P_C denotes the metric projection from H onto C. It is known that P_C is a firmly nonexpansive mapping [10], that is

$$\|P_C(x) - P_C(y)\|^2 \leq \langle P_C(x) - P_C(y), x - y \rangle, \text{ for all } x, y \in H.$$

Furthermore, for any $x \in H$ and $z \in C$, we note that $z = P_C(x)$ if and only if

$$\langle x - z, z - y \rangle \geq 0,$$

$$\|x-y\|^2 \geq \|x-z\|^2 + \|y-z\|^2 \text{ for all } y \in C.$$

Proposition 2.1. *Let H be a real Hilbert space and C be a nonempty closed convex subset of H. Let $T : C \to H$ be a mapping.*

(1) T is called nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \ \forall x, y \in C \tag{2.1}$$

(2) T is firmly nonexpansive if $2T - I$ is nonexpansive. In other direction, T is firmly nonexpansive if and only if

$$\langle Tx - Ty, x - y\rangle \geq \|Tx - Ty\|^2 \text{ for all } x, y \in C.$$

We denote $F(T)$ by the fixed point set of T, that is $F(T) = \{x \in C : Tx = x\}$. It is known that T is firmly nonexpansive if and only if $I - T$ is firmly nonexpansive.

Recall that H satisfies Opial's condition [15],i.e., for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for every $y \in H$with $y \neq x$.

Definition 2.2. [9] Let $S : C \to H$ be a mapping.

(1) S is called monotone if

$$\langle x - y, Sx - Sy\rangle \geq 0, \ \forall x, y \in C.$$

(2) S is said to be $\alpha -$ *inverse strongly monotone* if there exists a positive real number α such that

$$\langle x - y, Sx - Sy\rangle \geq \alpha\|Sx - Sy\|^2, \forall x, y \in C.$$

(3) S is said to be $\rho -$ *strongly monotone* if there exists a positive real number ρ such that

$$\langle Sx - Sy, x - y\rangle \geq \rho\|x - y\|^2, \forall x, y \in C.$$

(4) S is said to be $\mu -$ *Lipschitz continuous* if there exists a nonnegative real number $\mu \geq 0$ such that

$$\|Sx - Sy\| \leq \mu\|x - y\|, \forall x, y \in C.$$

A set-valued mapping $A : H \to 2^H$ is called monotone if for all $x, y \in H$, $f \in Ax$ and $g \in Ay$ imply $\langle x-y, f-g\rangle \geq 0$. A monotone mapping $A : H \to 2^H$ is maximal if the graph of $G(A)$ of A is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping A is maximal if and only if for $(x, f) \in H \times H$, $\langle x - y, f - g\rangle \geq 0$ for every $(y, g) \in G(A)$ implies $f \in Ax$. Let $J_r^A = (I + rA)^{-1}$, $r > 0$ be the resolvent of A. It is well known that J_r^A is single-valued, $D(J_r^A) = H$ and J_r^A is firmly nonexpansive for all $r > 0$.

Lemma 2.3. *[11] Let H be a real Hilbert space and let $S : C \to C$ be a nonexpansive mapping with a fixed point. For each fixed $u \in C$ and every $\nu \in (0, 1)$, the unique fixed point $x_\nu \in C$ of the contraction $C \ni x \mapsto \nu u + (1 - \nu)Sx$ converges strongly as $\nu \longrightarrow 0$ to a fixed point of T.*
In what follows, we shall use the following notation:

$$S_r^{A,B} = J_r^B(I - rA) = (I + rB)^{-1}(I - rA), r > 0.$$

Lemma 2.4. *[12] Let H be a real Hilbert space. Let A: $H \to H$ be an $\alpha -$ inverse strongly monotone operator and B: $H \to 2^H$ a maximal monotone operator. Then, we have*

(i) *For $r > 0$, $F(S_r^{A,B}) = (A + B)^{-1}(0)$;*
(ii) *For $0 < s \leq r$ and $x \in H$, $\|x - S_s^{A,B}x\| \leq 2\|x - S_r^{A,B}x\|$.*

Lemma 2.5. *[13] Let H be a real Hilbert space. Then, the following inequality holds:*
(i) $\|x-y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x-y, y\rangle$, *for all* $x, y \in H$;
(ii) $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y\rangle$, *for all* $x, y \in H$;

Proposition 2.6. *[14] Let H be a real Hilbert space. Let $m \in \mathbf{N}$ be fixed. Let $\{x_i\}_{i=1}^m \subseteq H$ and $t_i \geq 0$ for all $i = 1, 2, ..., m$ with $\sum_{t=i}^m t_i \leq 1$. Then, we have*

$$\|\sum_{i=1}^m t_i x_i\|^2 \leq \frac{\sum_{i=1}^m t_i \|x_i\|^2}{2 - \sum_{i=1}^m t_i} \tag{2.2}$$

Lemma 2.7. *[16] Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \leq (1-\alpha_n)s_n + \delta_n, \quad \forall n \geq 0, \tag{2.3}$$

where α_n is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that
(i) $\sum_{n=1}^\infty \alpha_n = \infty$
(ii) $\limsup_{n\to\infty} \dfrac{\delta_n}{\alpha_n} \leq 0$ *or* $\sum_{n=1}^\infty |\delta_n| < \infty$. *Then,* $\lim_{n\to\infty} s_n = 0$.

Lemma 2.8. *[18] Let $\{a_n\}$ and $\{c_n\}$ be a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\delta_n)a_n + b_n + c_n, n \geq 1, \tag{2.4}$$

where $\{\delta_n\}$ is a sequence in (0,1) and $\{b_n\}$ is a real sequence . Assume $\sum_{n=1}^\infty c_n < \infty$. Then, the following results hold:
(i) *It $b_n \leq \delta_n M$ for some $M \geq 0$, then $\{a_n\}$ is a bounded sequence.*
(ii) *It $\sum_{n=1}^\infty \delta_n = \infty$ and $\limsup_{n\to\infty} \dfrac{b_n}{\delta_n} \leq 0$, then $\lim_{n\to\infty} a_n = 0$*

Lemma 2.9. *[19] Assume $\{s_n\}$ is a sequence of nonnegative real numbers such that*

$$s_{n+1} \leq (1-\delta_n)s_n + \delta_n \tau_n, \ n \geq 1 \tag{2.5}$$

and

$$s_{n+1} \leq s_n - \eta_n + \rho_n, \ n \geq 1 \tag{2.6}$$

where $\delta_n \in$ (0,1), $\eta_n \in$ (0,∞) and $\{\tau_n\}$ and $\{\rho_n\}$ are sequence of real numbers such that
(i) $\sum_{n=1}^\infty \delta_n = \infty$,
(ii) $\lim_{n\to\infty} \rho_n = 0$,
(iii) $\lim_{k\to\infty} \eta_{n_k} = 0$ *implies* $\limsup_{k\to\infty} \tau_{n_k} \leq 0$ *for any subsequence of real numbers* $\{n_k\}$ *of* $\{n\}$.
Then $\lim_{n\to\infty} s_n = 0$.

Remark 2.10. [13] Let H be a Hilbert space, let C be a nonempty closed convex subset of H and let E be a mapping of Cinto H. Let $x \in C$. Then, for $\lambda > 0$,

$$x = P_C(I - \lambda E)x \Leftrightarrow x \in VI(C, E)$$

where P_C is the metric projection of H onto C.

One can see that the variational inequality (1.9) is equivalent to a fixed point problem.

Lemma 2.11. *[8] Let C be a nonempty closed convex subset of a real Hilbert space H and let $E, F : C \to H$ be α, β- inverse strongly monotone, respectively with α, $\beta > 0$ and $VI(C,E) \cap VI(C,F) \neq \emptyset$. Then,*

$$VI(C, \lambda E + (1-\lambda)F) = VI(C,E) \cap VI(C,F), \quad \forall \lambda \in (0,1).$$

Furthermore, if $0 < \rho < min\{2\alpha, 2\beta\}$, then we have $I - \rho(\lambda E + (1-\lambda)F)$ is a nonexpansive mapping.

3. Main Results

In this section, we prove the strong convergence theorem for solving common solution of monotone inclusions problems and variational inequality problems that is notation by the set $\Omega := F(T_r^{A,B}) \cap VI(C, \lambda E + (1-\lambda)F)$.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hillbert space H. Suppose that A, E and F are α, β, γ-inverse strongly monotone operator on H and $B : C \to 2^H$ a maximal monotone operator. Assume that Ω is nonempty set and the sequence $\{x_n\}$ is generated by,*

$$\begin{cases} \text{putting } u, x_0, x_1 \in H \\ z_n = x_n + \phi_n(x_n - x_{n-1}), \\ u_n = \alpha_n u + \beta_n z_n + \gamma_n J_{r_n}^B (I - r_n A) z_n, \\ x_{n+1} = P_C(I - \rho(\lambda E + (1-\lambda)F))u_n, \ \forall n \geq 1, \end{cases} \tag{3.1}$$

where $J_{r_n}^B = (I + r_n B)^{-1}$, $0 < r_n \leq 2\alpha$, $0 < \rho < min\{2\beta, 2\gamma\}$, $\alpha = [0,\infty)$, $\lambda \in (0,1)$, $\{\phi_n\} \subset [0, \phi]$ with $\phi \in [0,1)$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ are sequences in $[0,1]$ with $\alpha_n + \beta_n + \gamma_n = 1$. Assume that the following conditions hold:

(i) $\sum_{n=1}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$;

(ii) $\lim_{n\to\infty} \dfrac{\phi_n}{\alpha_n} \|x_n - x_{n-1}\| = 0$;

(iii) $0 < \liminf_{n\to\infty} r_n \leq \limsup_{n\to\infty} r_n < 2\alpha$;

Then, the sequence $\{x_n\}$ strongly converges to $P_\Omega u$.

Proof. To complete the proof, we devide into the five steps.

Step (I): We will show that the sequence $\{x_n\}$ *is bounded*. Putting $T_{r_n} = J_{r_n}^B(I - r_n A)$ *and define the sequnce* $\{s_n\}$ *by* $s_{n+1} = P_C(I - \rho D)(\alpha_n u + \beta_n s_n + \gamma_n T_{r_n} s_n)$, where $D = \lambda E + (1-\lambda)F$. Note that

$$\begin{aligned} \|x_{n+1} - s_{n+1}\| &= \|P_C(I-\rho D)u_n - P_C(I-\rho D)(\alpha_n u + \beta_n s_n + \gamma_n T_{r_n} s_n)\| \\ &\leq \|u_n - (\alpha_n u + \beta_n s_n + \gamma_n T_{r_n} s_n)\| \\ &\leq \beta_n \|z_n - s_n\| + \gamma_n \|T_{r_n} z_n - T_{r_n} s_n\| \\ &\leq \beta_n \|z_n - s_n\| + \gamma_n \|z_n - s_n\| \\ &= (1-\alpha_n)\|z_n - s_n\| \\ &= (1-\alpha_n)\|x_n + \phi_n(x_n - x_{n-1}) - s_n\| \\ &\leq (1-\alpha_n)\|x_n - s_n\| + \phi_n \|x_n - x_{n-1}\| \\ &\leq (1-\alpha_n)\|x_n - s_n\| + \frac{\phi_n}{\alpha_n}\|x_n - x_{n-1}\| \end{aligned} \tag{3.2}$$

By Lemma 2.7 and the condition (ii), we obtain that

$$\lim_{n\to\infty} \|x_n - s_n\| = 0. \tag{3.3}$$

Let $z \in P_\Omega u$, then $z \in VI(C,E) \cup VI(C,F)$ and so $z = P_C(I-\rho D)z$. Consider,

$$\begin{aligned} \|s_{n+1}-z\| &= \|P_C(I-\rho D)(\alpha_n u+\beta_n s_n+\gamma_n T_{r_n}s_n)-z\| \\ &\leq \alpha_n\|u-z\|+\beta_n\|s_n-z\|+\gamma_n\|T_{r_n}s_n-z\| \\ &\leq \alpha_n\|u-z\|+(1-\alpha_n)\|s_n-z\|. \end{aligned} \tag{3.4}$$

By lemma 2.7, we obtain that

$$\lim_{n\to\infty}\|s_n-z\|=0. \tag{3.5}$$

This implies that $\{s_n\}$ is bounded and also $\{x_n\}$, $\{u_n\}$ and $\{z_n\}$ are bounded.
Step (II): In this step, we will show that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$.
Note that

$$\begin{aligned} \|x_{n+1}-x_n\| &= \|P_C(I-\rho D)u_n-P_C(I-\rho D)u_{n-1}\| \\ &\leq \|u_n-u_{n-1}\|. \end{aligned} \tag{3.6}$$

By the definition of $\{u_n\}$, we have

$$\begin{aligned} \|u_n-z\|^2 &= \|\alpha_n u+\beta_n z_n+\gamma_n T_{r_n}z_n-z\|^2 \\ &= \|\alpha_n(u-z)+\beta_n(z_n-z)+\gamma_n(T_{r_n}z_n-z)\|^2 \\ &\leq \|\beta_n(z_n-z)+\gamma_n(T_{r_n}z_n-z)\|^2+2\alpha_n\langle u-z,u_n-z\rangle. \end{aligned} \tag{3.7}$$

By using Proposition 2.6, we conclude that

$$\begin{aligned} \|\beta_n(z_n-z)+\gamma_n(T_{r_n}z_n-z)\|^2 &\leq \frac{\beta_n\|z_n-z\|^2+\gamma_n\|T_{r_n}z_n-z\|^2}{2-(\beta_n+\gamma_n)} \\ &= \frac{\beta_n}{1+\alpha_n}\|z_n-z\|^2+\frac{\gamma_n}{1+\alpha_n}\|T_{r_n}z_n-T_{r_n}z\|^2 \\ &\leq \frac{\beta_n}{1+\alpha_n}\|z_n-z\|^2+\frac{\gamma_n}{1+\alpha_n}\Big(\|z_n-z\|^2 \\ &\quad -r_n(2\alpha-r_n)\|Az_n-Az\|^2 \\ &\quad -\|(I-J_{r_n}^B)(I-r_nA)z_n-(I-J_{r_n}^B)(I-r_nA)z\|^2\Big) \\ &= \frac{\beta_n}{1+\alpha_n}\|z_n-z\|^2+\frac{\gamma_n}{1+\alpha_n}\Big(\|z_n-z\|^2 \\ &\quad -r_n(2\alpha-r_n)\|Az_n-Az\|^2 \\ &\quad -\|z_n-Ar_nz_n-J_{r_n}^B(I-r_nA)z_n+r_nAz\|^2\Big) \\ &= \frac{(1-\alpha_n)}{1+\alpha_n}\|z_n-z\|^2-\frac{\gamma_nr_n(2\alpha-r_n)}{1+\alpha_n}\|Az_n-Az\|^2 \\ &\quad -\frac{\gamma_n}{1+\alpha_n}\|z_n-Ar_nz_n-T_{r_n}z_n+r_nAz\|^2. \end{aligned} \tag{3.8}$$

Sudstitute (3.8) into (3.7), we have

$$\begin{aligned}
\|u_n - z\|^2 &\leq \frac{(1-\alpha_n)}{1+\alpha_n}\|z_n - z\|^2 - \frac{\gamma_n r_n(2\alpha - r_n)}{1+\alpha_n}\|Az_n - Az\|^2 \\
&\quad -\frac{\gamma_n}{1+\alpha_n}\|z_n - Ar_n z_n - T_{r_n} z_n + r_n Az\|^2 + 2\alpha_n\langle u - z, u_n - z\rangle \\
&\leq \frac{(1-\alpha_n)}{1+\alpha_n}\Big(\|x_n - z\|^2 + 2\phi_n\langle x_n - x_{n-1}, z_n - z\rangle\Big) \\
&\quad -\frac{\gamma_n r_n(2\alpha - r_n)}{1+\alpha_n}\|Az_n - Az\|^2 \\
&\quad -\frac{\gamma_n}{1+\alpha_n}\|z_n - Ar_n z_n - T_{r_n} z_n + r_n Az\|^2 + 2\alpha_n\langle u - z, u_n - z\rangle \\
&= (1 - \frac{2\alpha_n}{1+\alpha_n})\|x_n - z\|^2 + (1 - \frac{2\alpha_n}{1+\alpha_n})2\phi_n\langle x_n - x_{n-1}, z_n - z\rangle \\
&\quad -\frac{\gamma_n}{1+\alpha_n} r_n(2\alpha - r_n)\|Az_n - Az\|^2 \\
&\quad -\frac{\gamma_n}{1+\alpha_n}\|z_n - Ar_n z_n - T_{r_n} z_n - r_n Az\|^2 + 2\alpha_n\langle u - z, u_n - z\rangle \\
&= (1 - \frac{2\alpha_n}{1+\alpha_n})\|x_n - z\|^2 + \frac{2\alpha_n}{1+\alpha_n}\Big[(\frac{1-\alpha_n}{\alpha_n})\phi_n\langle x_n - x_{n-1}, z_n - z\rangle \\
&\quad +(1+\alpha_n)\langle u - z, u_n - z\rangle\Big] - \frac{\gamma_n}{1+\alpha_n} r_n(2\alpha - r_n)\|Az_n - Az\|^2 \\
&\quad -\frac{\gamma_n}{1+\alpha_n}\|z_n - Ar_n z_n - T_{r_n} z_n - r_n Az\|^2. \qquad (3.9)
\end{aligned}$$

Then, using (3.9), we get that

$$\begin{aligned}
\|u_n - z\|^2 &\leq (1 - \frac{2\alpha_n}{1+\alpha_n})\|x_n - z\|^2 + \frac{2\alpha_n}{1+\alpha_n}\Big[(\frac{1-\alpha_n}{\alpha_n})\phi_n\langle x_n - x_{n-1}, z_n - z\rangle \\
&\quad +(1+\alpha_n)\langle u - z, u_n - z\rangle\Big]. \qquad (3.10)
\end{aligned}$$

Also, we conclude that

$$\begin{aligned}
\|u_n - z\|^2 &\leq \|x_n - z\|^2 - \frac{\gamma_n}{1+\alpha_n} r_n(2\alpha - r_n)\|Az_n - Az\|^2 \\
&\quad -\frac{\gamma_n}{1+\alpha_n}\|z_n - Ar_n z_n - T_{r_n} z_n - r_n Az\|^2 \\
&\quad +2\alpha_n\langle u - z, u_n - z\rangle + \frac{2(1-\alpha_n)}{1+\alpha_n}\phi_n\langle x_n - x_{n-1}, z_n - z\rangle. \qquad (3.11)
\end{aligned}$$

By the definition of $\{x_n\}$, we note that

$$\begin{aligned}
\|x_n - z\| &= \|P_C(I - \rho(\lambda E + (1-\lambda)F))u_{n-1} - z\| \\
&\leq \|u_{n-1} - z\|. \qquad (3.12)
\end{aligned}$$

From (3.11) and (3.12), we get that

$$\begin{aligned}\|u_n - z\|^2 &\leq \|u_{n-1} - z\|^2 - \frac{\gamma_n r_n(2\alpha - r_n)}{1+\alpha_n}\|Az_n - Az\|^2 \\ &-\frac{\gamma_n}{1+\alpha_n}\|z_n - r_nAz_n - T_{r_n}z_n + r_nAz\| \\ &+2\alpha_n\langle u - z, u_n - z\rangle + \frac{2(1-\alpha_n)}{1+\alpha_n}\phi_n\langle x_n - x_{n-1}, z_n - z\rangle. \quad (3.13)\end{aligned}$$

For each $n \geq 1$, we set fice following sequences as:

$$\begin{aligned} s_n &= \|u_{n-1} - z\|^2 \\ \delta_n &= \frac{2\alpha_n}{1+\alpha_n} \\ \tau_n &= (1+\alpha_n)\langle u - z, u_n - z\rangle + \frac{1-\alpha_n}{\alpha_n}\phi_n\langle x_n - x_{n-1}, z_n - z\rangle \\ \eta_n &= \frac{\gamma_n r_n(2\alpha - r_n)}{1+\alpha_n}\|Az_n - Az\|^2 + \frac{\gamma_n}{1+\alpha_n}\|z_n - r_nAz_n - T_{r_n}z_n + r_nAz\| \\ \text{and} \quad \rho_n &= 2\alpha_n\langle u - z, u_n - z\rangle + \frac{2(1-\alpha_n)}{1+\alpha_n}\phi_n\langle x_n - x_{n-1}, z_n - z\rangle. \quad (3.14)\end{aligned}$$

For $n \geq 1$, the inequalities (3.11) and (3.12) are reduced to the following:

$$s_{n+1} \leq (1-\delta_n)s_n + \delta_n\tau_n, \quad (3.15)$$

and

$$s_{n+1} \leq s_n - \eta_n + \rho_n, \quad (3.16)$$

Next, we will show all conditions in Lemma 2.9 hold. Since $\sum_{n=1}^{\infty}\alpha_n = \infty$ and $\lim_{n\to\infty}\alpha_n = 0$, it follows that $\sum_{n=1}^{\infty}\delta_n = \infty$ and $\lim_{n\to\infty}\rho_n = 0$.

Finally, we will show that the condition (iii) in Lemma 2.9 holds. Let $\{\eta_{n_k}\}$ *of* $\{\eta_n\}$ such that $\lim_{k\to\infty}\eta_{n_k} = 0$, we can deduce that

$$\lim_{k\to\infty}\frac{\gamma_{n_k}r_{n_k}(2\alpha - r_{n_k})}{1+\alpha_{n_k}}\|Az_{n_k} - Az\|^2 = 0. \quad (3.17)$$

This implies that

$$\lim_{k\to\infty}\|Az_{n_k} - Az\| = 0 \quad (3.18)$$

and

$$\lim_{k\to\infty}\frac{\gamma_{n_k}}{1+\alpha_{n_k}}\|z_{n_k} - r_{n_k}Az_{n_k} - T_{r_{n_k}}z_{n_k} + r_{n_k}Az\|. \quad (3.19)$$

Also, we have

$$\lim_{k\to\infty}\|z_{n_k} - r_{n_k}Az_{n_k} - T_{r_{n_k}}z_{n_k} + r_{n_k}Az\| = 0. \quad (3.20)$$

Note that,

$$\|z_{n_k} - T_{r_{n_k}}z_{n_k}\| \leq \|z_{n_k} - r_{n_k}Az_{n_k} - T_{r_{n_k}}z_{n_k} + r_{n_k}Az_{n_k}\| + \|r_{n_k}(Az - Az_{n_k})\| \quad (3.21)$$

By (3.18), (3.20) and (3.21), we have

$$\lim_{k\to\infty}\|z_{n_k} - T_{r_{n_k}}z_{n_k}\| = 0. \quad (3.22)$$

Since $\liminf_{n\to\infty} r_n > 0$, there is a real number $r > 0$ such that $r_n \geq r$ for all $n \geq 1$. In particular, $r_{n_k} \geq r$ for all $k \geq 1$. By Lemma 2.4, (i) yields that

$$\|T_r^{A,B} z_{n_k} - z_{n_k}\| \leq 2\|T_{r_{n_k}}^{A,B} z_{n_k} - z_{n_k}\|. \tag{3.23}$$

From (3.22) and (3.23), we have

$$\limsup_{k\to\infty} \|T_r^{A,B} z_{n_k} - z_{n_k}\| = 0, \tag{3.24}$$

which implies that,

$$\lim_{k\to\infty} \|T_r^{A,B} z_{n_k} - z_{n_k}\| = 0. \tag{3.25}$$

By the condition (ii) and (3.25), we obtain that

$$\begin{aligned}
\|T_r^{A,B} z_{n_k} - x_{n_k}\| &= \|T_r^{A,B} z_{n_k} - z_{n_k} + z_{n_k} - x_{n_k}\| \\
&\leq \|T_r^{A,B} z_{n_k} - z_{n_k}\| + \|z_{n_k} - x_{n_k}\| \\
&= \|T_r^{A,B} z_{n_k} - z_{n_k}\| + \|x_{n_k} + \phi_{n_k}(x_{n_k} - x_{n_k-1}) - x_{n_k}\| \\
&= \|T_r^{A,B} z_{n_k} - z_{n_k}\| + \|\phi_{n_k}(x_{n_k} - x_{n_k-1})\| \\
&\leq \|T_r^{A,B} z_{n_k} - z_{n_k}\| + \frac{\phi_{n_k}}{\alpha_{n_k}}\|x_{n_k} - x_{n_k-1}\|.
\end{aligned} \tag{3.26}$$

Hence,

$$\lim_{k\to\infty} \|T_r^{A,B} z_{n_k} - x_{n_k}\| = 0. \tag{3.27}$$

Let $z_\nu = \nu u + (1-\nu)T_r^{A,B} z_\nu$, $\nu \in (0,1)$. Employing Lemma 2.3, we have $z_\nu \to P_\Omega u = z$ as $\nu \to 0$. So, we obtain

$$\begin{aligned}
\|z_\nu - x_{n_k}\|^2 &= \|\nu u + (1-\nu)T_r^{A,B} z_\nu - x_{n_k}\|^2 \\
&= \|\nu(u - x_{n_k}) + (1-\nu)(T_r^{A,B} z_\nu - x_{n_k})\|^2 \\
&= (1-\nu)^2\|T_r^{A,B} z_\nu - T_r^{A,B} z_{n_k} + T_r^{A,B} z_{n_k} - x_{n_k}\|^2 \\
&\quad +2\nu\langle u - z_\nu, z_\nu - x_{n_k}\rangle + 2\nu\|z_\nu - x_{n_k}\|^2 \\
&\leq (1-\nu)^2\left(\|z_\nu - z_{n_k}\| + \|T_r^{A,B} z_{n_k} - x_{n_k}\|\right)^2 \\
&\quad +2\nu\langle u - z_\nu, z_\nu - x_{n_k}\rangle + 2\nu\|z_\nu - x_{n_k}\|^2 \\
&= (1-\nu)^2\left(\|z_\nu - x_{n_k}\| + \phi_n\|x_{n_k} - x_{n_k-1}\| + \|T_r^{A,B} z_{n_k} - x_{n_k}\|\right)^2 \\
&\quad +2\nu\langle u - z_\nu, z_\nu - x_{n_k}\rangle + 2\nu\|z_\nu - x_{n_k}\|^2.
\end{aligned} \tag{3.28}$$

Form (3.28), we have

$$\begin{aligned}
-2\nu\langle u - z_\nu, z_\nu - x_{n_k}\rangle \leq\ & (1-\nu)^2\Big(\|z_\nu - x_{n_k}\| + \phi_n\|x_{n_k} - x_{n_k-1}\| \\
&+\|T_r^{A,B} z_{n_k} - x_{n_k}\|\Big)^2 + (2\nu - 1)\|z_\nu - x_{n_k}\|^2,
\end{aligned} \tag{3.29}$$

which implies that,

$$\begin{aligned}
\langle z_\nu - u, z_\nu - x_{n_k}\rangle \leq\ & \frac{(1-\nu)^2}{2\nu}\Big(\|z_\nu - x_{n_k}\| + \phi_n\|x_{n_k} - x_{n_k-1}\| \\
&+\|T_r^{A,B} z_{n_k} - x_{n_k}\|\Big)^2 + \frac{(2\nu - 1)}{2\nu}\|z_\nu - x_{n_k}\|^2.
\end{aligned} \tag{3.30}$$

From condition (ii) and (3.30), we obtain

$$\limsup_{k\to\infty}\langle z_\nu - u, z_\nu - x_{n_k}\rangle \;\leq\; \frac{(1-\nu)^2}{2\nu}C^2 + \frac{(2\nu-1)}{2\nu}C^2 = \frac{\nu}{2}C^2 \tag{3.31}$$

for some the positive real number C large enough. Taking $\nu \to 0$ in (3.31), we have

$$\limsup_{k\to\infty}\langle z - u, z - x_{n_k}\rangle \;\leq\; 0. \tag{3.32}$$

On the other hand, we get

$$\begin{aligned}
\|u_{n_k} - x_{n_k}\| &= \|\alpha_{n_k}u + \beta_{n_k}z_{n_k} + \gamma_{n_k}J_{n_k}(I - r_{n_k}A)z_{n_k} - x_{n_k}\| \\
&= \|\alpha_{n_k}u + \beta_{n_k}z_{n_k} + \gamma_{n_k}T_{n_k}z_{n_k} - (\alpha_{n_k} + \beta_{n_k} + \gamma_{n_k})x_{n_k}\| \\
&\leq \alpha_{n_k}\|u - x_{n_k}\| + \beta_{n_k}\|z_{n_k} - x_{n_k}\| + \gamma_{n_k}\|T_{n_k}z_{n_k} - x_{n_k}\| \\
&= \alpha_{n_k}\|u - x_{n_k}\| + (1 - \alpha_{n_k} - \gamma_{n_k})\|z_{n_k} - x_{n_k}\| + \gamma_{n_k}\|T_{n_k}z_{n_k} - x_{n_k}\| \\
&\leq \alpha_{n_k}\|u - x_{n_k}\| + (1 - \alpha_{n_k})\|z_{n_k} - x_{n_k}\| + \gamma_{n_k}\|T_{n_k}z_{n_k} - x_{n_k}\| \\
&\leq \alpha_{n_k}\|u - x_{n_k}\| + \frac{(1 - \alpha_{n_k})}{\alpha_{n_k}}\|\phi_{n_k}(x_{n_k} - x_{n_k-1})\| + \gamma_{n_k}\|T_{n_k}z_{n_k} - x_{n_k}\|
\end{aligned} \tag{3.33}$$

By condition (i), (ii) and (3.27), we see that

$$\lim_{k\to\infty}\|u_{n_k} - x_{n_k}\| = 0. \tag{3.34}$$

Combining (3.32) and (3.34), we have

$$\limsup_{k\to\infty}\langle z - u, z - u_{n_k}\rangle \;\leq\; 0. \tag{3.35}$$

It also follows that $\limsup_{k\to\infty}\tau_{n_k} \leq 0$. By using Lemma 2.9, we conclude that $\lim_{n\to\infty} s_n = 0$ by Lemma 2.9. Therefore,

$$\lim_{n\to\infty}\|u_{n-1} - z\| = 0. \tag{3.36}$$

Since

$$\|u_n - u_{n-1}\| \leq \|u_n - z\| + \|z - u_{n-1}\|, \tag{3.37}$$

so we have

$$\lim_{n\to\infty}\|u_n - u_{n-1}\| = 0 \text{ and } \lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{3.38}$$

Step (III): we will show that $\lim_{n\to\infty}\|Du_n - Dz\| = 0$, $\lim_{n\to\infty}\|u_n - P_C(I-\rho Du_n)\| = 0$ and $\lim_{n\to\infty}\|x_n - u_n\| = 0$. Consider,

$$\begin{aligned}
\|x_n - z\|^2 &= \|P_C(I - \rho D)u_n - z\|^2 \\
&= \|P_C(I - \rho D)u_n - P_C(I - \rho D)z\|^2 \\
&\leq \|(I - \rho D)u_n - (I - \rho D)z\|^2 \\
&= \|u_n - z - \rho(Du_n - Dz)\|^2 \\
&= \|u_n - z\|^2 - 2\rho\langle u_n - z, Du_n - Dz\rangle + \rho^2\|Du_n - Dz\|^2,
\end{aligned} \tag{3.39}$$

and

$$\begin{aligned}\|u-z\|^2 &= \|\alpha_n u+\beta_n z_n+\gamma_n J_{r_n}^B(I-r_nA)z_n-z\|^2\\ &= \|\alpha_n(u-z)+\beta_n(z_n-z)+\gamma_n(T_{r_n}z_n-z)\|^2\\ &\leq \|\alpha_n(u-z)+(1-\alpha_n)(z_n-z)\|^2\\ &\leq \alpha_n\|u-z\|^2+(1-\alpha_n)\|z_n-z\|^2\\ &\leq \alpha_n\|u-z\|^2+\|z_n-z\|^2. \end{aligned}\tag{3.40}$$

by the notation $D:=\lambda E+(1-\lambda)F$, we have

$$\begin{aligned}\langle Du_n-Dz,u_n-z\rangle &= \langle(\lambda E+(1-\lambda)F)u_n-(\lambda E+(1-\lambda)F)z,u_n-z\rangle\\ &= \langle\lambda(Eu_n-Ez)+(1-\lambda)(Fu_n-Fz),u_n-z\rangle\\ &= \lambda\langle Eu_n-Ez,u_n-z\rangle+(1-\lambda)\langle Fu_n-Fz,u_n-z\rangle\\ &\geq \lambda\alpha\|Eu_n-Ez\|^2+(1-\lambda)\beta\|Fu_n-Fz\|^2. \end{aligned}\tag{3.41}$$

Substitute (3.40) and (3.41) into (3.39), we conclude that

$$\begin{aligned}\|x_n-z\|^2 &\leq \alpha_n\|u-z\|^2+\|z_n-z\|^2-2\rho\lambda\alpha\|Eu_n-Ez\|^2\\ &\quad -2\rho(1-\lambda)\beta\|Fu_n-Fz\|^2+\rho^2\|Du_n-Dz\|^2\\ &= \alpha_n\|u-z\|^2+\|x_n-z\|^2+2\phi_n\|x_n-x_{n-1}\|\|z_n-z\|\\ &\quad -2\rho\lambda\alpha\|Eu_n-Ez\|^2-2\rho(1-\lambda)\beta\|Fu_n-Fz\|^2\\ &\quad +\rho^2\|\lambda(Eu_n-Ez)+(1-\lambda)(Fu_n-Fz)\|^2\\ &\leq \alpha_n\|u-z\|^2+\|x_n-z\|^2+2\phi_n\|x_n-x_{n-1}\|\|z_n-z\|\\ &\quad -2\rho\lambda\alpha\|Eu_n-Ez\|^2-2\rho(1-\lambda)\beta\|Fu_n-Fz\|^2+\rho^2\lambda\|Eu_n-Ez\|^2\\ &\quad +\rho^2(1-\lambda)\|Fu_n-Fz\|^2\\ &= \alpha_n\|u-z\|^2+\|x_n-z\|^2+2\phi_n\|x_n-x_{n-1}\|\|z_n-z\|\\ &\quad -\rho\lambda(2\alpha-\rho)\|Eu_n-Ez\|^2-\rho(1-\lambda)(2\beta-\rho)\|Fu_n-Fz\|^2. \end{aligned}\tag{3.42}$$

By using (3.42) and condition (i) we have,

$$\begin{aligned}\rho\lambda(2\alpha-\rho)\|Eu_n-Ez\|^2 &\leq \alpha_n\|u-z\|^2+\|x_n-z\|^2+2\phi_n\|x_n-x_{n-1}\|\|z_n-z\|\\ &\quad -\|x_{n+1}-z\|^2\\ &\leq \alpha_n\|u-z\|^2+2\phi_n\|x_n-x_{n-1}\|\|z_n-z\|\\ &\quad +\Big(\|x_n-z\|-\|x_{n+1}-z\|\Big)\Big(\|x_n-z\|+\|x_{n+1}-z\|\Big)\\ &\leq \alpha_n\|u-z\|^2+\frac{2\phi_n}{\alpha_n}\|x_n-x_{n-1}\|\|z_n-z\|\\ &\quad +\Big(\|x_n-z\|+\|x_{n+1}-z\|\Big)\Big(\|x_{n+1}-x_n\|\Big). \end{aligned}\tag{3.43}$$

From condition (i) and (3.38), it implies that

$$\lim_{n\to\infty}\|Eu_n-Ez\|=0. \tag{3.44}$$

By using the same method as (3.43), we have

$$\lim_{n\to\infty}\|Fu_n-Fz\|=0. \tag{3.45}$$

Note that

$$\|Du_n - Dz\| \quad \leq \quad \lambda\|Eu_n - Ez\| + (1-\lambda)\|Fu_n - Fz\|, \tag{3.46}$$

then, from (3.44) and (3.45), we have

$$\lim_{n\to\infty} \|Du_n - Dz\| = 0. \tag{3.47}$$

Since

$$\begin{aligned} \|x_{n+1} - z\|^2 &= \|P_C(I - \rho(\lambda E + (1-\lambda)F)u_n - z\|^2 \\ &= \|P_C(I - \rho D)u_n - z\|^2, \end{aligned} \tag{3.48}$$

and by the properties of $P_C(I - \rho D)$, we have

$$\begin{aligned} \|P_C(I-\rho D)u_n - z\|^2 &= \|P_C(I-\rho D)u_n - P_C(I-\rho D)z\|^2 \\ &\leq \langle (I-\rho D)u_n - (I-\rho D)z, P_C(I-\rho D)u_n - z\rangle \\ &= \frac{1}{2}\Big(\|(I-\rho D)u_n - (I-\rho D)z\|^2 + \|P_C(I-\rho D)u_n - z\|^2 \\ &\quad -\|(I-\rho D)u_n - (I-\rho D)z - P_C(I-\rho D)u_n + z\|^2\Big) \\ &\leq \frac{1}{2}\Big(\|u_n - z\|^2 + \|P_C(I-\rho D)u_n - z\|^2 \\ &\quad -\|u_n - P_C(I-\rho D)u_n - \rho(Du_n - Dz)\|^2\Big). \end{aligned} \tag{3.49}$$

Note that

$$\begin{aligned} \|u_n - z\|^2 &= \|\alpha_n u + \beta_n z_n + \gamma_n T_{r_n} z_n - z\|^2 \\ &= \|\alpha_n(u-z) + \beta_n(z_n - z) + \gamma_n(T_{r_n} z_n - z)\|^2 \\ &\leq \alpha_n\|u-z\|^2 + \beta_n\|z_n - z\|^2 + \gamma_n\|T_{r_n} z_n - z\|^2 \\ &\leq \alpha_n\|u-z\|^2 + \beta_n\|z_n - z\|^2 + \gamma_n\|z_n - z\|^2 \\ &\leq \alpha_n\|u-z\|^2 + (1+\alpha_n)\|z_n - z\|^2 \\ &= \alpha_n\|u-z\|^2 + (1+\alpha_n)\|x_n + \phi_n(x_n - x_{n-1}) - z\|^2 \\ &= \alpha_n\|u-z\|^2 + (1+\alpha_n)\|(x_n - z) + \phi_n(x_n - x_{n-1})\|^2 \\ &= \alpha_n\|u-z\|^2 + (1-\alpha_n)\Big[\|x_n - z\|^2 \\ &\quad +2\phi_n\langle x_n - z, x_n - x_{n-1}\rangle + \phi_n^2\|x_n - x_{n-1}\|^2\Big] \\ &\leq \alpha_n\|u-z\|^2 + \|x_n - z\|^2 + 2\phi_n\langle x_n - z, x_n - x_{n-1}\rangle \\ &\quad +\phi_n^2\|x_n - x_{n-1}\|^2. \end{aligned} \tag{3.50}$$

Sutstitute (3.50) into (3.49), we have

$$\begin{aligned}\|P_C(I-\rho D)u_n - z\|^2 &\leq \frac{1}{2}\Big\{\|x_n - z\|^2 + \alpha_n\|u - z\|^2 + 2\phi_n\|x_n - z\|\|x_n - x_{n-1}\| \\ &\quad +\phi_n^2\|x_n - x_{n-1}\|^2 + \|P_C(I-\rho D)u_n - z\|^2 \\ &\quad -\|u_n - P_C(I-\rho D)u_n - \rho(Du_n - Dz)\|^2\Big\} \\ &= \frac{1}{2}\Big\{\|x_n - z\|^2 + \alpha_n\|u - z\|^2 + 2\phi_n\|x_n - z\|\|x_n - x_{n-1}\| \\ &\quad +\phi_n^2\|x_n - x_{n-1}\|^2 + \|P_C(I-\rho D)u_n - z\|^2 \\ &\quad -\|u_n - P_C(I-\rho D)u_n\|^2 - \|\rho(Du_n - Dz)\|^2 \\ &\quad +2\rho\langle u_n - P_C(I-\rho D)u_n, Du_n - Dz\rangle\Big\} \\ &\leq \frac{1}{2}\Big\{\|x_n - z\|^2 + \alpha_n\|u - z\|^2 + 2\phi_n\|x_n - z\|\|x_n - x_{n-1}\| \\ &\quad +\phi_n^2\|x_n - x_{n-1}\|^2 + \|P_C(I-\rho D)u_n - z\|^2 \\ &\quad -\|u_n - P_C(I-\rho D)u_n\|^2 - \|\rho(Du_n - Dz)\|^2 + \\ &\quad 2\rho\|u_n - P_C(I-\rho D)u_n\|\|Du_n - Dz\|\Big\} \\ &\leq \|x_n - z\|^2 + \alpha_n\|u - z\|^2 + 2\phi_n\|x_n - z\|\|x_n - x_{n-1}\| \\ &\quad +\phi_n^2\|x_n - x_{n-1}\|^2 - \|u_n - P_C(I-\rho D)u_n\|^2 \\ &\quad -\|\rho(Du_n - Dz)\|^2 \\ &\quad +2\rho\|u_n - P_C(I-\rho D)u_n\|\|Du_n - Dz\| \end{aligned} \tag{3.51}$$

Sutstitute (3.51) into (3.48), we obtain that

$$\begin{aligned}\|x_{n+1} - z\|^2 &\leq \|x_n - z\|^2 + \alpha_n\|u - z\|^2 + 2\phi_n\|x_n - z\|\|x_n - x_{n-1}\| + \phi_n^2\|x_n - x_{n-1}\|^2 \\ &\quad -\|u_n - P_C(I-\rho D)u_n\|^2 - \|\rho(Du_n - Dz)\|^2 \\ &\quad +2\rho\|u_n - P_C(I-\rho D)u_n\|\|Du_n - Dz\|, \end{aligned} \tag{3.52}$$

it follows from (3.52) that

$$\begin{aligned}\|u_n - P_C(I-\rho D)u_n\|^2 &\leq \|x_n - z\|^2 - \|x_{n+1} - z\|^2 + \alpha_n\|u - z\|^2 \\ &\quad +2\phi_n\|x_n - z\|\|x_n - x_{n-1}\| \\ &\quad +\phi_n^2\|x_n - x_{n-1}\|^2 - \|\rho(Du_n - Dz)\|^2 \\ &\quad +2\rho\|u_n - P_C(I-\rho D)u_n\|\|Du_n - Dz\| \\ &\leq \Big(\|x_n - z\| + \|x_{n+1} - z\|\Big)\|x_{n+1} - x_n\| + \alpha_n\|u - z\|^2 \\ &\quad +\frac{2\phi_n}{\alpha_n}\|x_n - z\|\|x_n - x_{n-1}\| \\ &\quad +\frac{\phi_n^2}{\alpha_n^2}\|x_n - x_{n-1}\|^2 - \|\rho(Du_n - Dz)\|^2 \\ &\quad +2\rho\|u_n - P_C(I-\rho D)u_n\|\|Du_n - Dz\|. \end{aligned} \tag{3.53}$$

By using condition (i) and (3.47), it implies that

$$\lim_{n\to\infty} \|u_n - P_C(I-\rho D)u_n\| = 0. \tag{3.54}$$

Consider,

$$\begin{aligned}\|x_n - u_n\| &= \|x_n - P_C(I-\rho D)u_n + P_C(I-\rho D)u_n - u_n\| \\ &\leq \|x_n - P_C(I-\rho D)u_n\| + \|P_C(I-\rho D)u_n - u_n\| \\ &= \|x_n - x_{n+1}\| + \|P_C(I-\rho D)u_n - u_n\|, \qquad (3.55)\end{aligned}$$

it follows from (3.38) and (3.54), we have

$$\lim_{n\to\infty} \|x_n - u_n\| = 0. \qquad (3.56)$$

Step (IV): Now, we will show that $\limsup_{n\to\infty}\langle u - v^*, x_n - v^*\rangle \leq 0$, where $v^* = P_\Omega u$ so we this take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle u - v^*, x_n - v^*\rangle = \lim_{k\to\infty}\langle u - v^*, x_{n_k} - v^*\rangle, \qquad (3.57)$$

without loss of generality, we may assume that $x_{n_k} \rightharpoonup \omega$ as $k \to \infty$ where $\omega \in C$. Since $\lim_{k\to\infty}\|u_{n_k} - x_{n_k}\| = 0$, we have $u_{n_k} \rightharpoonup \omega \ as \ k \to \infty$. Assume that $\omega \neq P_C(I-\rho D)\omega$, by the nonexpansiveness of $P_C(I-\rho D)$ and opial's property, we have

$$\begin{aligned}\liminf_{k\to\infty}\|u_{n_k} - \omega\| &< \liminf_{k\to\infty}\|u_{n_k} - P_C(I-\rho D)\omega\| \\ &\leq \liminf_{k\to\infty}\Big[\|u_{n_k} - P_C(I-\rho D)u_{n_k}\| + \|P_C(I-\rho D)u_{n_k} - P_C(I-\rho D)\omega\|\Big] \\ &\leq \liminf_{k\to\infty}\Big[\|u_{n_k} - P_C(I-\rho D)u_{n_k}\| + \|u_{n_k} - \omega\|\Big] \\ &\leq \liminf_{k\to\infty}\|u_{n_k} - \omega\|. \qquad (3.58)\end{aligned}$$

This is a contradiction, then we have $\omega \in (P_C(I-\rho D))$. Since, by Remark 2.10 and Lemma 2.11 we get that $\omega \in VI(C,E)\cap VI(C,F)$. Next, we will show that $\omega \in F(T_r^{A,B})$, we may assume that $x_{n_k} \rightharpoonup \omega \in C$. Since $\lim_{k\to\infty}\|z_{n_k} - x_{n_k}\| = 0$, we have $z_{n_k} \rightharpoonup \omega \ as \ k \to \infty$. Assume that $\omega \neq T_r^{A,B}\omega$, By opial's property, we have

$$\begin{aligned}\liminf_{k\to\infty}\|z_{n_k} - \omega\| &< \liminf_{k\to\infty}\|z_{n_k} - T_r^{A,B}\omega\| \\ &\leq \liminf_{k\to\infty}\|z_{n_k} - T_r^{A,B}z_{n_k}\| + \|T_r^{A,B}z_{n_k} - T_r^{A,B}\omega\| \\ &\leq \liminf\|z_{n_k} - \omega\|. \qquad (3.59)\end{aligned}$$

This is a contradiction, then we have $\omega \in F(T_r^{A,B})$, and so $\omega \in \Omega$. Since $x_{n_k} \rightharpoonup \omega$ and $\omega \in \Omega$, we have

$$\begin{aligned}\limsup_{n\to\infty}\langle u - v^*, x_n - v^*\rangle &= \lim_{k\to\infty}\langle u - v^*, x_{n_k} - v^*\rangle \\ &= \langle u - v^*, \omega - v^*\rangle \\ &\leq 0. \qquad (3.60)\end{aligned}$$

It implies that.

$$\limsup_{n\to\infty}\langle u - v^*, x_n - v^*\rangle \leq 0. \qquad (3.61)$$

Finally, we will show that $\lim_{n\to\infty} x_n = v^*$, where $v^* = P_\Omega u$. Note that,

$$\begin{aligned}
\|x_{n+1} - v^*\|^2 &= \|P_C(I-\rho D)u_n - v^*\|^2 \\
&\leq \|u_n - v^*\|^2 \\
&= \|\alpha_n u + \beta_n z_n + \gamma_n T_{r_n} z_n - v^*\|^2 \\
&= \|\alpha_n(u - v^*) + \beta_n(z_n - v^*) + \gamma_n(T_{r_n} z_n - v^*)\|^2 \\
&\leq \|\alpha_n(u - v^*) + \beta_n(z_n - v^*) + \gamma_n(z_n - v^*)\|^2 \\
&= \|\alpha_n(u - v^*) + (1-\alpha_n)(z_n - v^*)\|^2 \\
&\leq (1-\alpha_n)\|z_n - v^*\|^2 + 2\alpha_n\langle x_{n+1} - v^*, u - v^*\rangle, \qquad (3.62)
\end{aligned}$$

and

$$\begin{aligned}
\|z_n - v^*\|^2 &= \|x_n + \phi_n(x_n - x_{n-1}) - v^*\|^2 \\
&= \|x_n - v^* + \phi_n(x_n - x_{n-1})\|^2 \\
&\leq \|x_n - v^*\|^2 + 2\phi_n\langle x_n - x_{n-1}, z_n - v^*\rangle. \qquad (3.63)
\end{aligned}$$

Substitute (3.63) into (3.62), we have

$$\begin{aligned}
\|x_{n+1} - v^*\|^2 &\leq (1-\alpha_n)\Big[\|x_n - v^*\|^2 + 2\phi_n\langle x_n - x_{n-1}, z_n - v^*\rangle\Big] \\
&\quad +2\alpha_n\langle x_{n+1} - v^*, u - v^*\rangle \\
&\leq (1-\alpha_n)\|x_n - v^*\|^2 + 2\alpha_n\langle x_{n+1} - v^*, u - v^*\rangle \\
&\quad +(1-\alpha_n)2\phi_n\|x_{n-1} - x_n\|\|z_n - v^*\| \\
&\leq (1-\alpha_n)\|x_n - v^*\|^2 + \frac{2\phi_n}{\alpha_n}\|x_{n-1} - x_n\|\|z_n - v^*\| \\
&\quad +2\alpha_n\langle x_{n+1} - v^*, u - v^*\rangle. \qquad (3.64)
\end{aligned}$$

By Lemma 2.8, condition(i) and (3.61), we can conclude that $\{x_n\}$ converges strongly to $v^* = P_\Omega u$ This completes the proof of Theorem 3.1.

4. Numerical example

Example 4.1. Let $\mathbb{R}$ be the set of real number and let the mappings A,B,E and F : $\mathbb{R} \to \mathbb{R}$ defined by $A(x) = \frac{x-1}{2}$, $B(x) = 2(x-1)$, $E(x) = \frac{x-1}{3}$ and $F(x) = \frac{x-1}{5}$, $\forall x \in \mathbb{R}$.

Algorithm 1. we choose $\lambda \in (0,1)$, $0 < r_n \leq 2\alpha$, $0 < \rho < min\{2\beta, 2\gamma\}$, $\{\phi_n\} \subset [0,1)$.

Step 1: Let $u, x_0, x_1 \in H$. Set $n := 1$.

Step 2: Compute

$$z_n = x_n + \phi_n(x_n - x_{n-1}),$$

Step 3: Compute

$$u_n = \tfrac{1}{n}u + \tfrac{3n-3}{5n}z_n + \tfrac{2n-2}{5n}J^B_{r_n}(I - r_n A)z_n,$$

Step 4: Compute

$$x_{n+1} = u_n - \rho\lambda(\tfrac{u_n-1}{3}) + (1-\lambda)(\tfrac{u_n-1}{5}).$$

where

$$\phi_n = \begin{cases} min\{\frac{1}{(n^2+1)||x_n - x_{n-1}||}, \ 0.5\} & \text{if } x_n \neq x_{n-1}, \\ 0.5 & \text{otherwise.} \end{cases}$$

Step 5:. Set $n = n + 1$, and go to Step 2.

It is easy to see that the sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ satisfy all conditions in Theorem 3.1. We can conclude that the sequences $\{x_n\}$ converge strongly to $\{1\}$.

In figure 1, we choose three difference initial points x_0 and x_1 with the same parameters and observe a behavior of the sequence $\{x_n\}$. We see that even if we choose the difference initial point, the sequence $\{x_n\}$ always converges to the solution, in this case is $\{1\}$.

Next figure 2, we take initial points random fixed same point. For the conparison three parameter $\rho = 0.01, 0.50, 0.99$. We see that the sequence $\{x_n\}$ always converges fastest to a solution, where $\rho = 0.99$.

Finally, we test the to rate of convergence by choosing $r_n = 0.99\frac{n^2}{n^2+1}, \frac{n}{n^3+1}, \frac{n^2}{n^3+1}$ whene the initial point $x_0 = 7,\ x_1 = 5$, the parameter $\rho = 0.99$ and $u = 2$ are fixed. In this test, the result is shown in figure 3.

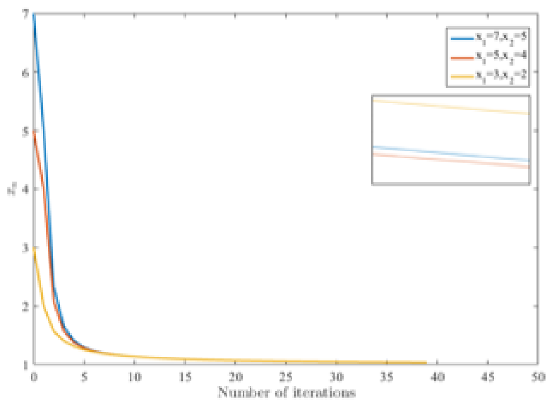

FIGURE 1. The comparison of convergent rate form 3 initial points.

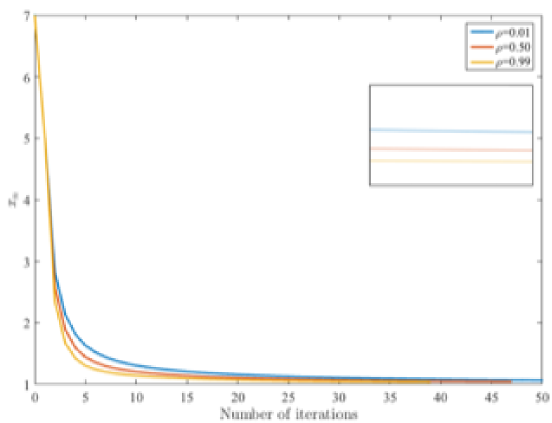

FIGURE 2. The comparison of convergent rate form 3 the parameter ρ.

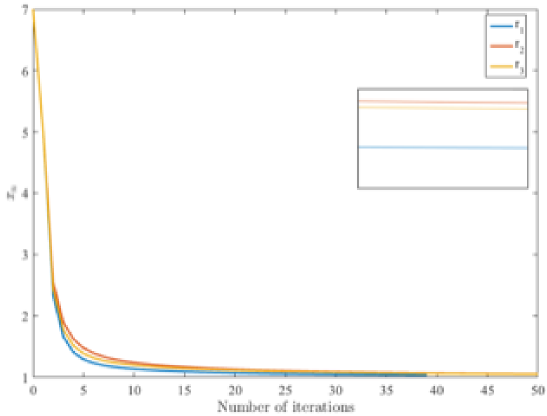

Figure 3. The comparition of convergent rate form 3 different sequences $\{r_n\}$.

Conflict of Interests

The author declare that there is no conflict of interests regarding the publication of this paper.

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