BANGMOD INTERNATIONAL JOURNAL OF
MATHEMATICAL & COMPUTATIONAL SCIENCE
ISSN: 2408-154X
https://bangmodjmcs.com/
Volume 8 (2022)
Pages 1–16

BEST PROXIMITY POINTS OF GENERALIZED p-CYCLIC WEAK φ-CONTRACTIONS

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Received: 28 March 2022 / Accepted: 25 September 2022

Abstract In this manuscript, we extend the notion of generalized cyclic weak φ-contractions to p sets, $p \geq 2$. We investigate the convergence of best proximity points of such maps in p-cyclic complete metric spaces. We also give an example to support our main results. Our works generalize and improve the related results in the literature.

MSC: 47H10; 54H25
Keywords: p-cyclic contractions; strict contractions; best proximity points; generalized weak φ-contractions; p-cyclic metric space

1. Introduction

In 1968, Bryant [1] constructed a remarkable result in fixed point theory and proved that, in a complete metric space, if for some positive integer $n \geq 2$, the nth iteration of

Published online: 3 October 2022

Please cite this article as: B. Nuntadilok et al., Best proximity points of generalized p-cyclic weak ψ-contractions, Bangmod J-MCS., Vol. 8 (2022) 1–16.

the given mapping forms a contraction, then it possess a unique fixed point. Another outstanding approach was proposed by Kirk, Srinivasan and Veeramani [13] by introducing the notion of cyclic contraction. More precisely, every cyclic contraction in a complete metric space possess a unique fixed point. This statement is plain but significant when we compare with the results of Bryant. Later, the concept of the cyclic contractions has been investigated immensely by a considerable large number of authors who brought several brilliant notions and derived a number of interesting results (see, e.g.[2–6, 10, 11, 14–18, 20, 21, 24, 29] and the references therein). Let T be a self-mapping on a metric space (X, ρ). Suppose that E and F are non-empty subsets of X such that $X = E \cup F$. A self-mapping $T : E \cup F \to E \cup F$ is called a cyclic contraction [13] if

1). $T(E) \subseteq F \quad$ and $\quad T(F) \subseteq E$.
2). If there is a $k \in (0, 1)$ such that the following inequality is satisfied

$$d(Tx, Ty) \leq kd(x, y), \quad \text{for all } x \in E, y \in F.$$

After this initial construction, several extensions of cyclic mappings and cyclic contractions have been introduced. In this paper, we mainly follow the notations defined in [19, 23]. In [19], a notion of p-cyclic map was introduced. Let $D_1, D_2, \dots, D_p (p \geq 2)$ be non-empty sets. A p-cyclic map $T : \cup_{i=1}^{p} D_i \to \cup_{i=1}^{p} D_i$ is defined such that $T(D_i) \subseteq D_{i+1}, \forall i \in \{1, 2, \dots, p\}$, $x = x_0 \in D_i$, defines a sequence $\{x_n\} \subset \cup_{i=1}^{p} D_i$ as $x_n = Tx_{n-1}$. Then, $\{x_{pn}\}$ is a subsequence in D_i, $\{x_{pn+1}\}$ is a subsequence in D_{i+1} and so on. From the arrangement of such a sequence formed by a p-cyclic map, Karapinar et al. in [23] introduced a notion of p-cyclic sequence (Definition 2.1(1)). If $D_i s$ are subsets of a metric space (X, ρ), then, to obtain a best proximity point of T under various contractive conditions (some of them given in the literature), it is enough to prove that: given $\varepsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that

$$\rho(x_{pn}, x_{pm+1}) < dist(D_i, D_{i+1}) + \varepsilon, \forall n, m \geq N_0.$$

This observation motivated the authors [23] to introduce a concept of p-cyclic Cauchy sequence and p-cyclic complete metric space (Definition 2.1). In addition, while investigating the behavior of such p-cyclic maps, it is often the case that, if $\rho(x, y) > dist(D_i, D_{i+1})$, then $\rho(Tx, Ty) < \rho(x, y)$ and, if $\rho(x, y) = dist(D_i, D_{i+1})$, then $\rho(Tx, Ty) = \rho(x, y), x \in D_i, y \in D_{i+1}$. They call a p-cyclic map with this property as *p-cyclic strict contraction map* (Definition 3.1). Note that, if the distances between the adjacent sets are zero, then a p-cyclic strict contraction map is a strict contraction map in the usual sense. All such maps invariably satisfy the condition: $x, y \in D_i, \rho(T^{pn}x, T^{pn+1}y) \to dist(D_i, D_{i+1})$ as $n \to \infty$. In the paper [23], all p-cyclic maps which satisfy the above two properties are said to belong to class Ω (Definition 3.4). Finally, the authors proved the existence and convergence of best proximity points of Ω class of mappings in a p-cyclic complete metric space.

Now we recollect some essential definitions.

Definition 1.1 (see [4]). A continuous function $F : [0, \infty)^2 \to \mathbb{R}$ is called a C-class function, if for any $s, t \in [0, \infty)$, the following conditions hold:

(1) $F(s, t) \leq s$;
(2) $F(s, t) = s$ implies that either $s = 0$ or $t = 0$.

Remark 1.2. We denote the class of all C-class functions as $\mathbb{C}$.

Example 1.3 (see [4]). Following examples show that the class $\mathbb{C}$ of C-class functions is nonempty:

(1) $F(s,t) = s - t$.
(2) $F(s,t) = ms, 0 < m < 1$
(3) $F(s,t) = \frac{s}{(1+t)^r}$ for some $r \in (0,\infty)$.
(4) $F(s,t) = \log(t + a^s)/(1+t)$, for some $a > 1$.
(5) $F(s,t) = \ln(1 + a^s)/2$, for $a > e$. Indeed $F(s,1) = s$ implies that $s = 0$.
(6) $F(s,t) = (s + l)^{(1/(1+t)^r)} - l$, $l > 1$, for $r \in (0,\infty)$.
(7) $F(s,t) = s \log_{t+a} a$, for $a > 1$.
(8) $F(s,t) = s - (\frac{1+s}{2+s})(\frac{t}{1+t})$.
(9) $F(s,t) = s\beta(s)$, where $\beta : [0,\infty) \to [0,1)$. etc.

More examples of C-class functions can be found in [4].

Definition 1.4 (see [22]). A function $\psi : [0,\infty) \to [0,\infty)$ is called an altering distance function, if the following properties are satisfied:
(i) ψ is non-decreasing and continuous,
(ii) $\psi(t) = 0$ if and only if $t = 0$.

We denote the class of ***altering distance functions*** as Ψ.

Definition 1.5 (see [4]). An ultra altering distance function is a continuous, non-decreasing mapping $\varphi : [0,\infty) \to [0,\infty)$ such that $\varphi(t) > 0$, $t > 0$ and $\varphi(0) \geq 0$

We denote the class of ***ultra altering distance functions*** as Ψ_u.

In what follows, we recollect some definitions and fundamental results which are crucial to prove our main results.

Definition 1.6. ([19], Definitions 3.1). For a non-empty set X, suppose $\rho : X \times X \to [0,\infty)$ forms a metric and $D_1, D_2, \ldots, D_p, (p \geq 2)$ are non-empty subsets of X. Define $D_{p+i} := D_i$, for all $i \in \{1,2,\ldots,p\}$. A map $T : \cup_{i=1}^p D_i \to \cup_{i=1}^p D_i$ is called **a p-cyclic map**, if $T(D_i) \subseteq D_{i+1}, \forall i \in \{1,2,\ldots,p\}$. If $p = 2$, then T is called **a cyclic map**. A point $x \in D_i$ is said to be a best proximity point of T in D_i, if $\rho(x,Tx) = dist(D_i, D_{i+1})$, where $dist(D_i, D_{i+1}) := \inf\{\rho(x,y) : x \in D_i, y \in D_{i+1}\}$.

In [23], the authors introduced the conditions for the underlying space and for the subsets of the space, to have a unique best proximity point under a p-cyclic map, if it exists, irrespective of the contraction condition imposed on the map.

Proposition 1.7 ([23]). *Let $D_1, D_2, \ldots, D_p, (p \geq 2)$ be non-empty convex subsets of a strictly convex norm linear space X such that $dist(D_i, D_{i+1}) > 0, i \in \{1,2,\ldots,p\}$. Let $T : \cup_{i=1}^p D_i \to \cup_{i=1}^p D_i$ be a p-cyclic map. Then, T has at most one best proximity point in $D_i, 1 \leq i \leq p$.*

Let T be a p-cyclic map as given in Definition 1.6. T is said to be p-cyclic non expansive map if for all $x \in D_i, y \in D_{i+1}$, the following holds:

$$\rho(Tx,Ty) \leq \rho(x,y), \forall i \in \{1,2,\ldots,p\}.$$

The following lemma naturally follows for a p-cyclic non-expansive map.

Lemma 1.8. ([19], Lemma 3.3). *For a non-empty set X, suppose $\rho : X \times X \to [0,\infty)$ forms a metric and $D_1, D_2, \ldots, D_p, (p \geq 2)$ are non-empty subsets of X. If $T : \cup_{i=1}^p D_i \to \cup_{i=1}^p D_i$ is a p-cyclic non-expansive map, then*

$$dist(D_i, D_{i+1}) = dist(D_{i+1}, D_{i+2}) = dist(D_1, D_2), \forall i \in \{1,2,\ldots,p\}. \tag{1.1}$$

In addition, if $\nu \in D_i \cap \mathbf{D}(T)_i \neq \emptyset$, then $T^j\nu \in D_{i+1} \cap \mathbf{D}(T)_{i+j} \neq \emptyset$, for all $j = 1, 2, \ldots, (p\text{-}1)$, where $\mathbf{D}(T)_k$ is the set of best proximity point of the mapping T in D_k.

The following lemma (see [11, 23]) is essential to prove that a given sequence is Cauchy.

Lemma 1.9. ([11], Lemma 3.7). *For a uniformly convex Banach space $(X, \|.\|)$, we suppose that E, F are non-empty closed subsets of X and $\{a_n\}, \{b_n\} \subset E$ and $\{d_n\} \subset F$. If E is convex such that*

(i) $\|b_n - d_n\| \to dist(E, F)$; and

(ii) for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $m > n > N$,

$$\|a_m - d_n\| \leq dist(E, F) + \varepsilon,$$

then for all $\varepsilon > 0$, there exists $N_1 \in \mathbb{N}$ such that for all $m > n > N_1$, $\|a_m - b_n\| \leq \varepsilon$.

Next, we recall a few p-cyclic maps with some contraction conditions imposed on them, which are defined in [2, 3, 9, 12, 19].

Definition 1.10. ([2], Definition 3.1). For a non-empty set X, suppose $\rho : X \times X \to [0, \infty)$ forms a metric and $D_1, D_2, \ldots, D_p, (p \geq 2)$ are non-empty subsets of X. Let $T : \cup_{i=1}^{p} D_i \to \cup_{i=1}^{p} D_i$ be **a p-cyclic map**, T is said to be **a p-cyclic contraction**, if there exists $k \in (0, 1)$ such that for all $x \in D_i$ and $y \in D_{i+1}$, we have

$$\rho(Tx, Ty) \leq k\rho(x, y) + (1 - k)dist(D_i, D_{i+1}), \forall i \in \{1, 2, \ldots, p\}.$$

Definition 1.11. ([3], Definition 2.1). For a non-empty set X, suppose $\rho : X \times X \to [0, \infty)$ forms a metric, E and F are non-empty subsets of X. A cyclic map $T : E \cup F \to E \cup F$ is said to be a cyclic φ-contraction if

$$\rho(Tx, Ty) \leq \rho(x, y) - \varphi(\rho(x, y)) + \varphi(dist(E, F)), \forall x \in E, y \in F,$$

where $\varphi : [0, \infty) \to [0, \infty)$ is a strictly increasing map.

Definition 1.12. ([9], Definition 2.1). Let E and F be nonempty subsets of a metric space (X, ρ). Suppose that $\varphi : [0, \infty) \to [0, \infty)$ is a strictly increasing map. A cyclic map $T : E \cup F \to E \cup F$ is said to be a generalized cyclic weak φ-contraction, if for any $x \in E, y \in F$

$$\rho(Tx, Ty) \leq m(x, y) - \varphi(m(x, y)) + \varphi(dist(E, F)) \tag{1.2}$$

where $m(x, y) = \max\{\rho(x, y), \rho(x, Tx), \rho(y, Ty), \frac{1}{2}[\rho(x, Ty) + \rho(y, Tx)]\}$.

Definition 1.13. ([6], Definition 2.1). Let E and F be nonempty subsets of a metric space(X, ρ). Suppose that $\varphi, \psi : [0, \infty) \to [0, \infty)$ and φ is a strictly increasing map. A cyclic map $T : E \bigcup F \to E \bigcup F$ is called a generalized cyclic weak (F, ψ, φ)-contraction, if for any $x \in E$ and $y \in F$,

$$\begin{aligned} \psi(\rho(Tx, Ty)) \leq F\Big(&\psi(m(x, y)) - \psi(dist(E, F)), \\ &\varphi(m(x, y)) - \varphi(dist(E, F))\Big) + \psi\big(dist(E, F)\big) \end{aligned} \tag{1.3}$$

where $F \in \mathbb{C}, \psi \in \Psi$ with $\psi(s + t) \leq \psi(s) + \psi(t)$, $\varphi \in \Psi_u$ and

$$m(x, y) = \max\{\rho(x, y), \rho(x, Tx), \rho(y, Ty), \frac{1}{2}[\rho(x, Ty) + \rho(y, Tx)]\}.$$

Remark 1.14. If we take $F(s, t) = s - t$ and $\psi(t) = t$ in Definition 1.13, the we obtain Definition 1.12 above.

2. p-Cyclic Sequences and p-Cyclic Complete Metric Spaces

Throughout this article, let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. In [23], Karapinar et al. introduced the notion of p-cyclic sequence as follows:

Definition 2.1 ([23])**.** For a non-empty set X, suppose $\rho : X \times X \to [0, \infty)$ forms a metric and $D_1, D_2, \ldots, D_p, (p \geq 2)$ are non-empty subsets of X.

1 . A sequence $\{x_n\}_{n=1}^{\infty} \subset \cup_{i=1}^{p} D_i$ is called a p-cyclic sequence if $x_{pn+i} \in D_i$, for all $n \in \mathbb{N}_0$ and $i = 1, 2, \ldots, p$.

2 . We say that $\{x_n\}_{n=1}^{\infty}$ is a p-cyclic Cauchy sequence, if for given $\varepsilon > 0$ there exists an $N_0 \in \mathbb{N}$ such that for some $i \in \{1, 2, \ldots, p\}$, we have

$$\rho(x_{pn+i}, x_{pm+i+1}) < dist(D_i, D_{i+1}) + \varepsilon, \forall m, n \geq N_0. \tag{2.1}$$

3 . A p-cyclic sequence $\{x_n\}_{n=1}^{\infty}$ in $\cup_{i=1}^{p} D_i$ is said to be p-cyclic bounded, if $\{x_{pn+i}\}_{n=1}^{\infty}$ is bounded in D_i for some $i \in \{1, 2, ..., p\}$.

4 . Let $\{x_n\}_{n=1}^{\infty}$ be a p-cyclic sequence in $\cup_{i=1}^{p} D_i$. If for some $j \in \{1, 2, ..., p\}$ the subsequence $\{x_{pn+j}\}$ of $\{x_n\}_{n=1}^{\infty}$ converges in D_j, then we say that $\{x_n\}_{n=1}^{\infty}$ is p-cyclic convergent.

5 . Under the assumption that $D_1, D_2, \ldots, D_p, (p \geq 2)$ are non-empty closed subsets of a metric space (X, ρ), we say that $\cup_{i=1}^{p} D_i$ is p-cyclic complete if every p-cyclic Cauchy sequence in $\cup_{i=1}^{p} D_i$ is p-cyclic convergent.

6 . If there are subsets $D_1, D_2, \ldots, D_p, (p \geq 2)$ of (X, ρ) such that $X = \cup_{i=1}^{p} D_i$ and $\cup_{i=1}^{p} D_i$ is p-cyclic complete, then we call (X, ρ) a p-cyclic complete metric space.

Remark 2.2. Note that a p-cyclic sequence which is a Cauchy sequence in the usual sense is a p-cyclic Cauchy sequence. On the other hand, p-cyclic Cauchy sequences need not be Cauchy sequences in the usual sense, even if $dist(D_i, D_{i+1}) = 0, \ \forall i \in \{1, 2, ..., p\}$.

Examples which illustrate the notion of p-cyclic sequence and p-cyclic Cauchy sequence can be found in ([23], Example 1 and 2). And a complete metric space need not be p-cyclic complete, (see [23], Remark 2, for example).

The following proposition shows that a p-cyclic Cauchy sequence is p-cyclic bounded.

Proposition 2.3 ([23])**.** *For a non-empty set X, suppose $\rho : X \times X \to [0, \infty)$ forms a metric and $D_1, D_2, \ldots, D_p, (p \geq 2)$ are non-empty subsets of X. Then, every p-cyclic Cauchy sequence in $\cup_{i=1}^{p} D_i$ is p-cyclic bounded.*

The following proposition is an example of two-cyclic complete metric space.

Proposition 2.4 ([23])**.** *Let E and F be subsets of a uniformly convex Banach space X, which are non-empty and closed. If either E or F is convex, then $E \cup F$ is two-cyclic complete.*

3. p-Cyclic Strict Contraction Maps

In [23], Karapinar et al. introduced a notion of p-cyclic strict contraction, which is a generalization of strict contraction in the usual sense.

Definition 3.1 ([23])**.** For a non-empty set X, suppose $\rho : X \times X \to [0, \infty)$ forms a metric and $D_1, D_2, \ldots, D_p, (p \geq 2)$ are non-empty subsets of X. A p-cyclic map T is said to be p-cyclic strict contraction if, for all $x \in D_i, y \in D_{i+1}, 1 \leq i \leq p$:

(i) $\rho(x, y) > dist(D_i, D_{i+1}) \Rightarrow \rho(Tx, Ty) < \rho(x, y)$, and

(ii) $\rho(x,y) = dist(D_i, D_{i+1}) \Rightarrow \rho(Tx,Ty) = \rho(x,y)$.

Remark 3.2. Note that, if $D_i = A$, for all $i = 1, 2, \ldots, p$, then p-cyclic strict contraction is a strict contraction in the usual sense. It is clear that the p-cyclic strict contraction also forms a p-cyclic non-expansive map.

The following proposition proves an important property of p-cyclic strict contraction map

Proposition 3.3 ([23]). *For a non-empty set X, suppose $\rho : X \times X \to [0,\infty)$ forms a metric and $D_1, D_2, \ldots, D_p, (p \geq 2)$ are non-empty subsets of X. Let $x \in D_i (1 \leq i \leq p)$. Suppose that $T : \cup_{i=1}^{p} D_i \to \cup_{i=1}^{p} D_i$ is a p-cyclic strict contraction map and if for all $\varepsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that*

$$\rho(T^{pn}x, T^{pm+1}x) < dist(D_i, D_{i+1}) + \varepsilon, n, m \geq n_0, \tag{3.1}$$

then for a given $\varepsilon > 0$, there exists an $n_1 \in \mathbb{N}$ such that

$$\rho(T^{pn+k}x, T^{pm+k+1}x) < dist(D_{i+k}, D_{i+k+1}) + \varepsilon, n, m \geq n_1, k \in \{1, 2, \ldots, p\}.$$

In [23], Karapinar et al. introduced the notion of p-cyclic maps with various contractive conditions and possed some common properties. They also introduced a notion of class Ω, a certain class of mappings.

Definition 3.4 ([23]). For a non-empty set X, suppose $\rho : X \times X \to [0,\infty)$ forms a metric and $D_1, D_2, \ldots, D_p, (p \geq 2)$ are non-empty subsets of X. A p-cyclic map $T : \cup_{i=1}^{p} D_i \to \cup_{i=1}^{p} D_i$ is said to belong to the class Ω if

(1) T is p-cyclic strict contraction.
(2) If $x, y \in D_i$, then $\lim_{n\to\infty} \rho(T^{pn}x, T^{pn+1}y) = dist(D_i, D_{i+1}), 1 \leq i \leq p$.

In this manuscript, we list some p-cyclic maps different from those given in [23] which belong to the class Ω. First, we prove that a p-cyclic contraction map, which is defined via the notion of C-class functions, belongs to the class Ω. We give the following new definition via C-class functions.

Definition 3.5. Let $D_1, D_2, \ldots, D_p$ be non-empty subsets of a metric space (X, ρ). Let $T : \cup_{i=1}^{p} D_i \to \cup_{i=1}^{p} D_i$ is called **a p-cyclic (F, ψ, φ)-contraction map**, if it satisfies

$$\psi\big(\rho(Tx,Ty)\big) \leq F\Big(\psi(\rho(x,y)) - \psi(dist(D_i, D_{i+1})), \varphi(\rho(x,y)) - \varphi(dist(D_i, D_{i+1}))\Big) + \psi\big(dist(D_i, D_{i+1})\big),$$

for all $i \in \{1, 2, \ldots, p\}$, where $F \in \mathbb{C}$, $\psi \in \Psi$ and $\varphi : [0,\infty) \to [0,\infty)$ is a strictly increasing map.

Remark 3.6. If we take $F(s,t) = s - t$, $\psi(t) = t$ and $p = 2$ in Definion 3.5, then we obtain Definition 1.11 (Definition 2.1, defined in [3]).

Next we prove that a p-cyclic (F, ψ, φ)-contraction map belongs to the class Ω.

Example 3.7. Let $D_1, D_2, \ldots, D_p$ be non-empty subsets of a metric space (X, ρ). Let $T : \cup_{i=1}^{p} D_i \to \cup_{i=1}^{p} D_i$ be a p-cyclic (F, ψ, φ)-contraction map. Then, $T \in \Omega$.

https://doi.org/10.58715/bangmodjmcs.2022.8.1

Proof. We first show that T is a p-cyclic strict contraction. Because the the map T is a p-cyclic (F,ψ,φ)-contraction, we have

$$\begin{aligned}\psi\big(\rho(Tx,Ty)\big) \leq & F\Big(\psi(\rho(x,y)) - \psi(dist(D_i,D_{i+1})), \varphi(\rho(x,y)) - \varphi(dist(D_i,D_{i+1}))\Big), \\ & + \psi(dist(D_i,D_{i+1})),\end{aligned}$$

for all $i \in \{1,2,\dots,p\}$, where $F \in \mathbb{C}$, $\psi \in \Psi$. Taking $F(s,t) = s-t$, we have

$$\psi\big(\rho(Tx,Ty)\big) \leq \psi\big(\rho(x,y)\big) - \varphi(\rho(x,y)) + \varphi\big(dist(D_i,D_{i+1})\big).$$

If $\rho(x,y) = dist(D_i,D_{i+1})$, we have

$$\rho(Tx,Ty) \leq \rho(x,y).$$

Since $\rho(x,y) = dist(D_i,D_{i+1}) \leq \rho(Tx,Ty)$, we then have

$$\rho(Tx,Ty) = \rho(x,y).$$

In addition, if $\rho(x,y) > dist(D_i,D_{i+1})$, then

$$\begin{aligned}\psi\big(\rho(Tx,Ty)\big) &\leq F\big(\psi(\rho(x,y)) - \psi(dist(D_i,D_{i+1})), \varphi(\rho(x,y)) - \varphi(dist(D_i,D_{i+1}))\big) \\ &\quad + \psi(dist(D_i,D_{i+1})), \\ &\leq \psi(\rho(x,y) - \varphi(\rho(x,y)) + \varphi(dist(D_i,D_{i+1})) \\ &< \psi(\rho(x,y)) - \varphi(\rho(x,y)) + \varphi(\rho(x,y)).\end{aligned}$$

Therefore

$$\rho(Tx,Ty) < \rho(x,y).$$

Therefore, T is a p-cyclic strict contraction. The second condition of Definition 3.4 follows from Lemma 3.3 in [2]. Hence, $T \in \Omega$. ∎

Remark 3.8. Karapinar et al.[23] showed that the p-cyclic Meir-Keeler map (p-cyclic MK-map) introduced in [19] belongs to the class Ω. See Example 4 in [23].

Next, we establish an example of p-cyclic map satisfying a contraction condition of Geraghtys type [7] and show that it belongs to the class Ω. Here, we use the notion of C-class functions introduced in [4] combining with a class of functions S introduced by Geraghty [7], where S is the class of all functions $\vartheta : [0,\infty) \to [0,1)$ that satisfies $\vartheta(t_n) \to 1$, then $t_n \to 0, t_n \in [0,\infty)$ for $n \in \mathbb{N}$.

Example 3.9. For a non-empty set X, suppose $\rho : X \times X \to [0,\infty)$ forms a metric and $D_1, D_2, \dots, D_p, (p \geq 2)$ are non-empty subsets of X. Let $T : \cup_{i=1}^p D_i \to \cup_{i=1}^p D_i$ be **a p-cyclic** $(F,\psi,\varphi,\vartheta)$**-map** such that

$$\begin{aligned}\rho(Tx,Ty) \leq F\Big(&\psi\big(\vartheta(\rho(x,y))\big)\rho(x,y) - \psi\big(\vartheta(\rho(x,y))\big)dist(D_i,D_{i+1}), \\ &\varphi\big(\vartheta(\rho(x,y))\rho(x,y)\big) - \varphi\big(\vartheta(\rho(x,y))dist(D_i,D_{i+1})\big)\Big) \\ &+\psi\big(\vartheta(\rho(x,y))\big)dist(D_i,D_{i+1}),\end{aligned}$$

for all $i \in \{1,2,\dots,p\}$, where $F \in \mathbb{C}$, $\psi \in \Psi$ where $\psi(t) < t$ and $\vartheta \in S$. Then

(a) T is a p-cyclic strict contraction.
(b) $\lim_{n\to\infty} \rho(T^{pn}x, T^{pn+1}y) = dist(D_i,D_{i+1}), x \in D_i, y \in D_{i+1}$.

Proof. (a) Let $x \in D_i, y \in D_{i+1}$.

Case (1): If $\rho(x,y) > dist(D_i, D_{i+1})$, by the definition of F, we have

$$\begin{aligned}\rho(Tx,Ty) \leq & F\Big(\psi\big(\vartheta(\rho(x,y))\big)\rho(x,y) - \psi\big(\vartheta(\rho(x,y))\big)dist(D_i,D_{i+1}),\\ & \varphi\big(\vartheta(\rho(x,y))\big)\rho(x,y) - \varphi\big(\vartheta(\rho(x,y))\big)dist(D_i,D_{i+1})\Big),\\ & + \psi\big(\vartheta(\rho(x,y))\big)dist(D_i,D_{i+1})\\ & \leq \psi\big(\vartheta(\rho(x,y))\big)\Big[\rho(x,y) - dist(D_i,D_{i+1}) + dist(D_i,D_{i+1})\Big](*)\\ & \leq \psi\big(\vartheta(\rho(x,y))\big)\rho(x,y).\end{aligned}$$

Therefore

$$\rho(Tx,Ty) < \rho(x,y).$$

Case (2): If $\rho(x,y) = dist(D_i, D_{i+1})$, then from (*), we have $\rho(Tx,Ty) \leq \rho(x,y)$. By equation (1.1),

$$\rho(x,y) = dist(D_i,D_{i+1}) = dist(D_{i+1},D_{i+2}) \leq \rho(Tx,Ty) \leq \rho(x,y),$$

therefore

$$\rho(Tx,Ty) = \rho(x,y).$$

Hence, T is p-cyclic strict contraction.

(b) Let $x, y \in D_i$. Since T is p-cyclic non-expansive, $\{\rho(T^{pn}x, T^{pn+1}y)\}$ is a decreasing sequence and is bounded below by $dist(D_i, D_{i+1})$. Therefore,

$$\rho(T^{pn}x, T^{pn+1}y) \to r \text{ as } n \to \infty \text{ and } r \geq dist(D_i, D_{i+1}),$$

where $r = \inf_{n\geq 1} \rho(T^{pn}x, T^{pn+1}y)$.

Claim: $r = dist(D_i, D_{i+1})$.

If $\rho(T^{pn}x, T^{pn+1}y) = dist(D_i, D_{i+1})$ for some n, then by the p-cyclic non-expansiveness of T,

$$\rho(T^{pn+k}x, T^{pn+k+1}y) = \rho(T^{pn}x, T^{pn+1}y), k = 1, 2, \ldots.$$

Hence, we have

$$\rho(T^{pn}x, T^{pn+1}y) \to dist(D_i, D_{i+1}) \text{ as } n \to \infty.$$

Let us assume that $\rho(T^{pn}x, T^{pn+1}y) > dist(D_i, D_{i+1})$, $n \in \mathbb{N}$. Suppose that $r > dist(D_i, D_{i+1})$. Since T is p-cyclic non expansive,

$$\begin{aligned}&\rho(T^{p(n+1)}x, T^{p(n+1)+1}y) \leq \rho(T^{pn+1}x, T^{pn+2}y)\\ &\leq F\Big(\psi\big(\vartheta(\rho(T^{pn}x,T^{pn+1}y))\big)\rho(T^{pn}x,T^{pn+1}y) - \psi(\vartheta(\rho(T^{pn}x,T^{pn+1}y)))dist(D_i,D_{i+1})\big),\\ &\varphi\big(\vartheta(\rho(T^{pn}x,T^{pn+1}y))\rho(T^{pn}x,T^{pn+1}y)\big) - \varphi\big(\vartheta(\rho(T^{pn}x,T^{pn+1}y))dist(D_i,D_{i+1})\big)\Big)\\ &+ \psi(\vartheta(\rho(T^{pn}x,T^{pn+1}y)))dist(D_i,D_{i+1})\\ &\leq \psi(\vartheta(\rho(T^{pn}x,T^{pn+1}y))\Big[\rho(T^{pn}x,T^{pn+1}y) - dist(D_i,D_{i+1}) + dist(D_i,D_{i+1})\Big].\end{aligned}$$

Then

$$\begin{aligned}\rho(T^{p(n+1)}x, & T^{p(n+1)+1}y)\\ & \leq \psi(\vartheta(\rho(T^{pn}x,T^{pn+1}y))[\rho(T^{pn}x,T^{pn+1}y)].\end{aligned}$$

https://doi.org/10.58715/bangmodjmcs.2022.8.1

Since $\vartheta \in S$ and $\psi(t) < t$,

$$\frac{\rho(T^{p(n+1)}x, T^{p(n+1)+1}y)}{\rho(T^{pn}x, T^{pn+1}y)} \leq \vartheta(\rho(T^{pn}x, T^{pn+1}y)) < 1. \tag{3.2}$$

Since $r = \lim_{n\to\infty} \rho((T^{p(n+1)}x, T^{p(n+1)+1}y)) > dist(D_i, D_{i+1})$ by our assumption, letting $n \to \infty$ in equation (3.2), we get

$$1 \leq \lim_{n\to} \vartheta(\rho(T^{pn}x, T^{pn+1}y)) \leq 1,$$

that is,

$$\lim_{n\to\infty} \vartheta(\rho(T^{pn}x, T^{pn+1}y)) = 1$$

However, $\lim_{n\to\infty} \rho(T^{pn}x, T^{pn+1}y) = r > 0$, which contradicts $\vartheta \in S$. Hence, $r = dist(D_i, D_{i+1})$. This proves Part (b). ∎

Next, we recall some essential definitions and some known results as follows.

Definition 3.10. ([9], Definition 2.1) Let E and F be nonempty subsets of a metric space (X, ρ). Suppose that $\varphi : [0, \infty) \to [0, \infty)$ is a strictly increasing map. A cyclic map $T : E \cup F \to E \cup F$ is said to be **a generalized cyclic weak φ-contraction**, if for any $x \in E, y \in F$

$$\rho(Tx, Ty) \leq m(x, y) - \varphi\big(m(x, y)\big) + \varphi\big(dist(E, F)\big) \tag{3.3}$$

where $m(x, y) = \max\{\rho(x, y), \rho(x, Tx), \rho(y, Ty), \frac{1}{2}[\rho(x, Ty) + \rho(y, Tx)]\}$.

Definition 3.11. [25, 26] Let (X, ρ) be a metric space with a mapping $T : X \to X$, if $\lim_{n\to\infty} T^{n_i}(y) = z \Rightarrow \lim_{n\to\infty} T(T^{n_i}(y)) = Tz$, we call mapping T to be orbitally continuous.

The following are known results in [9].

Theorem 3.12. *[9] Let E and F be nonempty subsets of a metric space (X, ρ). Suppose $T : E \cup F \to E \cup F$ is a generalized cyclic weak φ-contraction and there exists $y_0 \in E$. Define $y_{n+1} = Ty_n$ for any $n \in \mathbb{N}$. Then $\rho(y_n, y_{n+1}) \to \rho(E, F)$ as $n \to \infty$.*

Theorem 3.13. *[9] Let E and F be nonempty subsets of a metric space (X, ρ). Suppose $T : E\cup F \to E\cup F$ is a generalized cyclic weak φ-contraction and T is orbitally continuous. Assume E is closed and there exists $y_0 \in E$. Define $y_{n+1} = Ty_n$ for any $n \in \mathbb{N}$. If $\{y_{2n}\}$ has a convergent subsequence in E, then there exixts $p \in E$ such that $\rho(p, Tp) \to \rho(E, F)$.*

4. Best proximity points of generalized p-cyclic weak φ-contractions

In this section we extend and generalize the results by Cheng and Su in [9]. We introduce the following definitions and main results.

Definition 4.1. Let $D_1, D_2, \dots, D_p, (p \geq 2)$ be nonempty subsets of a metric space (X, ρ). Suppose that $\varphi : [0, \infty) \to [0, \infty)$ is a strictly increasing map. A map $T : \cup_{i=1}^{p} D_i \to \cup_{i=1}^{p} D_i$ is said to be **a generalized p-cyclic weak φ-contraction**, if for any $x \in D_i, y \in D_{i+1}$

$$\rho(Tx, Ty) \leq m(x, y) - \varphi(m(x, y)) + \varphi(dist(D_i, D_{i+1})) \tag{4.1}$$

where $m(x, y) = \max\{\rho(x, y), \rho(x, Tx), \rho(y, Ty), \frac{1}{2}[\rho(x, Ty) + \rho(y, Tx)]\}$.

Remark 4.2. If we let $p = 2$ in definition 4.1, then we obtain the definition 3.10 (see [9]).

Theorem 4.3. *For a non-empty set X, suppose $\rho : X \times X \to [0, \infty)$ forms a metric and $D_1, D_2, \dots, D_p, (p \geq 2)$ are non-empty subsets of X. Suppose $T : \cup_{i=1}^{p} D_i \to \cup_{i=1}^{p} D_i$ is p-cyclic maps satisfying (3.3) and there exists $y_0 \in D_i$. Define $y_{n+1} = Ty_n$ for any $n \in \mathbb{N}$. Then $\rho(y_n, y_{n+1}) \to dist(D_i, D_{i+1})$ as $n \to \infty$.*

Proof. Let $\rho_n = \rho(y_n, y_{n+1})$. We first claim that the sequence $\{\rho_n\}$ is non-increasing. By our assumption, we have

$$\begin{aligned}\rho_{n+1} &= \rho(y_{n+1}, y_{n+2}) \\ &= \rho(Ty_n, Ty_{n+1}) \\ &\leq m(y_n, y_{n+1}) - \varphi\big(m(y_n, y_{n+1})\big) + \varphi\big(dist(D_i, D_{i+1})\big),\end{aligned}$$

where

$$\begin{aligned}m(y_n, y_{n+1}) &= \max\{\rho(y_n, y_{n+1}), \rho(y_n, Ty_n), \rho(y_{n+1}, Ty_{n+1}), \\ &\quad \frac{1}{2}[\rho(y_n, Ty_{n+1}) + \rho(y_{n+1}, Ty_n)]\} \\ &= \max\{\rho(y_n, y_{n+1}), \rho(y_{n+1}, y_{n+2})\}. \ (*)\end{aligned}$$

Assume that there exists $n_0 \in \mathbb{N}$ such that $m(y_{n_0}, y_{n_0+1}) = \rho(y_{n_0+1}, y_{n_0+2})$. From $\rho(y_{n_0+1}, y_{n_0+2}) > \rho(y_{n_0}, y_{n_0+1})$, we have

$$\rho(y_{n_0+1}, y_{n_0+2}) \leq \rho(y_{n_0+1}, y_{n_0+2}) - \varphi\big(\rho(y_{n_0+1}, y_{n_0+2})\big) + \varphi\big(dist(D_i, D_{i+1})\big).$$

Then

$$\varphi\big(\rho(y_{n_0+1}, y_{n_0+2})\big) \leq \varphi\big(dist(D_i, D_{i+1})\big).$$

Since φ is a strictly increasing map, we have

$$\rho(y_{n_0+1}, y_{n_0+2})) \leq dist(D_i, D_{i+1}) \leq \rho(y_{n_0+1}, y_{n_0+2}).$$

Obviously, $\rho(y_{n_0+1}, y_{n_0+2})) = dist(D_i, D_{i+1}) \leq \rho(y_{n_0}, y_{n_0+1})$, which is a contradiction. Hence, for all $n \in \mathbb{N}$,

$$m(y_n, y_{n+1}) = \rho(y_n, y_{n+1}). \ (**)$$

From $(*)$ and $(**)$ we conclude that $\rho(y_{n+1}, y_{n+2}) \leq \rho(y_n, y_{n+1})$. This shows the sequence $\{\rho_n\}$ is non-increasing, and by Proposition 2.3 it is bounded below. Therefore $\lim_{n\to\infty} \rho_n$ exists. If $\rho_{n_0} = 0$, for some $n_0 \in \mathbb{N}$, so $\rho_n \to 0$ and $dist(D_i, D_{i+1}) = 0$, that is $\rho_n \to dist(D_i, D_{i+1})$. If $\rho_n \neq 0$ for all $n \in \mathbb{N}$. Put $\rho_n \to \gamma$, then

$$\gamma \geq dist(D_i, D_{i+1}).$$

Since φ is a strictly increasing map, we have

$$\varphi(\gamma) \geq \varphi\big(dist(D_i, D_{i+1})\big). \tag{4.2}$$

From $(*)$ and $(**)$ and (3.3) we can write

$$\rho(y_{n+1}, y_{n+2}) \leq \rho(y_n, y_{n+1}) - \varphi\big(\rho(y_n, y_{n+1})\big) + \varphi\big(dist(D_i, D_{i+1})\big),$$

equivalently,

$$\varphi\big(\rho(y_n, y_{n+1})\big) \leq \rho(y_n, y_{n+1}) - \rho(y_{n+1}, y_{n+2}) + \varphi\big(dist(D_i, D_{i+1})\big)$$

Taking the limit as $n \to \infty$, we get

$$\varphi(\gamma) \leq \varphi\big(dist(D_i, D_{i+1})\big). \tag{4.3}$$

From (4.2) and (4.3), we obtain

$$\gamma = dist(D_i, D_{i+1}).$$

That is $\rho_n \to dist(D_i, D_{i+1})$. Our proof is complete. ∎

Lemma 4.4. *Let $D_1, D_2, \ldots, D_p$ be non-empty subsets of a metric space (X, ρ). Let $T : \cup_{i=1}^{p} D_i \to \cup_{i=1}^{p} D_i$ is a p-cyclic maps satisfying (3.3). Then, $T \in \Omega$.*

Proof. (a). We first show that T is a p-cyclic strict contraction. Since the map T is a generalized p-cyclic weak φ- contraction, then

$$\rho(Ty_n, Ty_{n+1}) \leq m(y_n, y_{n+1}) - \varphi\big(m(y_n, y_{n+1})\big) + \varphi\big(dist(D_i, D_{i+1})\big),$$

for all $i = 1, 2, \ldots, p, (p \geq 2)$, where

$$\begin{aligned} m(y_n, y_{n+1}) =& \max\{\rho(y_n, y_{n+1}), \rho(y_n, Ty_n), \rho(y_{n+1}, Ty_{n+1}), \\ & \frac{1}{2}[\rho(y_n, Ty_{n+1}) + \rho(y_{n+1}, Ty_n)]\} \\ =& \max\{\rho(y_n, y_{n+1}), \rho(y_{n+1}, y_{n+2})\}. \end{aligned}$$

Similar to the proof of above theorem, we have that $m(y_n, y_{n+1}) = \rho(y_n, y_{n+1})$ for all $n \in \mathbb{N}$. That is $m(x, y) = \rho(x, y)$ for all $x, y \in D_i$.

If $\rho(x, y) = dist(D_i, D_{i+1})$, then we have

$$\begin{aligned} \rho(Tx, Ty) &\leq \rho(x, y) - \varphi(\rho(x, y)) + \varphi(\rho(x, y)) \\ &= \rho(x, y). \end{aligned}$$

That is

$$\rho(Tx, Ty) \leq \rho(x, y). \tag{4.4}$$

Therefore, we get $\rho(x, y) = dist(D_i, D_{i+1}) \leq \rho(Tx, Ty) \leq \rho(x, y)$. It yields that

$$\rho(Tx, Ty) = \rho(x, y).$$

In addition, if $\rho(x, y) > dist(D_i, D_{i+1})$, then

$$\begin{aligned} \rho(Tx, Ty) &\leq m(x, y) - \varphi(m(x, y)) + \varphi(dist(D_i, D_{i+1})) \\ &< \rho(x, y) - \varphi(\rho(x, y)) + \varphi(\rho(x, y)) \\ &= \rho(x, y). \end{aligned}$$

That is

$$\rho(Tx, Ty) < \rho(x, y). \tag{4.5}$$

Thus, from (4.4) and (4.5) we conclude that T is a p-cyclic strict contraction.

(b). We next prove the condition (2) of Definition 3.4. Let $x, y \in D_i$. Note that

$$\rho(T^{p(n+1)}x, T^{p(n+1)+1}y) \leq \rho(T^{pn}x, T^{pn+1}y), n \in \mathbb{N}.$$

Then, the sequence $\{\rho(T^{pn}x, T^{pn+1}y)\}_{n=1}^{\infty}$ is bounded below by $dist(D_i, D_{i+1})$ and is a non-increasing sequence. Hence, $\rho(T^{pn}x, T^{pn+1}y) \to r$ as $n \to \infty$ and $r \geq dist(D_i, D_{i+1})$, where $r = \inf_{n \geq 1} \rho(T^{pn}x, T^{pn+1}y)$.

Claim: $r = dist(D_i, D_{i+1})$.

Case 1. If $\rho(T^{pn}x, T^{pn+1}y) = dist(D_i, D_{i+1})$ for some $n \in \mathbb{N}$. Then, by the p-cyclic non-expansiveness of T,

$$\rho(T^{pn+k}x, T^{pn+k+1}y) = dist(D_i, D_{i+1}), k = 1, 2, \dots.$$

Thus, $\rho(T^{pn}x, T^{pn+1}y) \to dist(D_i, D_{i+1})$, as $n \to \infty$.

Case 2. If $\rho(T^{pn}x, T^{pn+1}y) > dist(D_i, D_{i+1})$ for all $n \in \mathbb{N}$.

Since T is p-cyclic non-expansive, we have

$$\begin{aligned}\rho(T^{p(n+1)}x, T^{p(n+1)+1}y) &\leq \rho(T^{pn+1}x, T^{pn+2}y)\\ &\leq m(T^{pn}x, T^{pn+1}y) - \varphi\big(m(T^{pn}x, T^{pn+1}y)\big) + \varphi(dist(D_i, D_{i+1}))\\ &= \rho(T^{pn}x, T^{pn+1}y) - \varphi\big(\rho(T^{pn}x, T^{pn+1}y)\big) + \varphi(dist(D_i, D_{i+1})).\end{aligned}$$

So

$$\rho(T^{p(n+1)}x, T^{p(n+1)+1}y) - \varphi(\rho(T^{pn}x, T^{pn+1}y)) < \rho(T^{pn}x, T^{pn+1}y) - \varphi(\rho(T^{pn}x, T^{pn+1}y).$$

Taking the limit as $n \to \infty$, it yeilds that

$$\varphi(r) > \varphi(r).$$

Since φ is strictly increasing, $r > r$. This is a contradiction. Therefore, we have $\rho(T^{pn}x, T^{pn+1}y) = dist(D_i, D_{i+1})$. Hence, $r = dist(D_i, D_{i+1})$. From both cases (a) and (b) we conclude that $T \in \Omega$. ∎

Theorem 4.5. *Let $D_1, D_2, \dots, D_p, (p \geq 2)$ are non-empty subsets of a p-cyclic metric space (X, ρ). Suppose a cyclic map $T : \cup_{i=1}^{p} D_i \to \cup_{i=1}^{p} D_i$ satisfies the definition of* **a generalized p-cyclic weak φ-contraction** (3.3) *and T is orbitally continuous. Assume that each $D_i, i = 1, 2, \dots, p, (p \geq 2)$ is closed and there exists $y_0 \in D_i$. Define $y_{n+1} = Ty_n$ for any $n \in \mathbb{N}$. If $\{y_{pn}\}$ has a convergent subsequence in D_i, then there exists $y \in D_i$ such that $\rho(y, Ty) = dist(D_i, D_{i+1})$.*

Proof. By the asumption, we know that the subsequence $\{y_{pn_k}\}$ of sequence $\{y_{pn}\}$ converges to apoint $y \in D_i$. By Theorem 4.3, we have

$$\rho(y_{pn_k}, y_{pn_k+1}) = \rho(y_{pn_k}, Ty_{pn_k}) \to dist(D_i, D_{i+1}).$$

Since T is an orbitally continuous, we have $\rho(y, Ty) = dist(D_i, D_{i+1})$. ∎

Definition 4.6. A metric space (X, ρ) is called **regular** if every bounded monotone sequence of X is convergent.

Corollary 4.7. *Let $D_1, D_2, \dots, D_p, (p \geq 2)$ are non-empty subsets of a regular p-cyclic ordered metric space $(X, \rho, \preceq)$. Suppose that $T : \cup_{i=1}^{p} D_i \to \cup_{i=1}^{p} D_i$ is a decreasing generalized p-cyclic weak φ-contraction* (3.3) *and T is also orbitally continuous. Assume that D_i is closed for each $i, i = 1, 2, \dots, p, (p \geq 2)$ and there exists $y_0 \in D_i$ such that $y_0 \preceq T^2 y_0 \preceq \dots \preceq Ty_0$. Define $y_{n+1} = Ty_n$ for any $n \in \mathbb{N}$. Then there exists $y \in D_i$ such that $\rho(y, Ty) = dist(D_i, D_{i+1})$.*

Proof. By the asumption, we have

$$y_0 \preceq y_2 \preceq \dots \preceq y_1.$$

Since X is regular and D_i is closed for each i, the sequence $\{y_{pn}\}$ converges to a point $y \in D_i$. From Theorem 4.5, we conclude that $\rho(y, Ty) = dist(D_i, D_{i+1})$. ∎

Theorem 4.8. *For a non-empty set X, suppose $\rho : X \times X \to [0, \infty)$ forms a metric and $D_1, D_2, \ldots, D_p, (p \geq 2)$ are non-empty subsets of X. Let $T : \cup_{i=1}^p D_i \to \cup_{i=1}^p D_i$ be a p-cyclic maps and satisfies* **a generalized p-cyclic weak φ-contraction** (3.3). *Assume for some $k \in \mathbb{N}$ and $x \in D_i, (1 \leq i, k \leq p), \{T^{pn+k}x\}$ converges to $\nu \in D_{i+k}$. Then, ν is a best proximity point of T in D_{i+k} (That is $\rho(\nu, T\nu) = dist(D_{i+k}, D_{i+k+1})$).*

Proof. Let $x \in D_i$ be as given in the theorem. By equation (1.1), for each $n \in N$, we have,

$$\begin{aligned} dist(D_{i+k}, D_{i+k+1}) &= dist(D_{i+k-1}, D_{i+k}) \\ &\leq \rho(T^{pn+k-1}x, \nu) \\ &\leq \rho(T^{pn+k-1}x, T^{pn+k}x) + \rho(T^{pn+k}x, \nu). \end{aligned}$$

By lemma 4.4, $T \in \Omega$, so

$$\lim_{n\to\infty} \left(\rho(T^{pn+k-1}x, T^{pn+k}x) + \rho(T^{pn+k}x, \nu)\right) = dist(D_{i+k-1}, D_{i+k}).$$

Therefore,

$$\lim_{n\to\infty} \rho(T^{pn+k-1}x, \nu) = dist(D_{i+k-1}, B_{i+k}) = dist(D_{i+k}, D_{i+k+1}). \tag{4.6}$$

Now,

$$\begin{aligned} dist(D_{i+k}, D_{i+k+1}) &\leq \rho(\nu, T\nu) \\ &= \lim_{n\to\infty} \rho(T^{pn+k}x, T\nu) \\ &\leq \lim_{n\to\infty} \rho(T^{pn+k-1}x, \nu) \\ &= dist(D_{i+k}, D_{i+k+1}), \text{ (by equation(4.6))}. \end{aligned}$$

Hence, $\rho(\nu, T\nu) = dist(D_{i+k}, D_{i+k+1})$. ∎

Theorem 4.9. *For a non-empty set X, suppose $\rho : X \times X \to [0, \infty)$ forms a metric and $X_1, X_2, \ldots, D_p, (p \geq 2)$ are non-empty subsets of X. Suppose that $X = \cup_{i=1}^p D_i$ and $\cup_{i=1}^p D_i$ is p-cyclic complete. Let $T : \cup_{i=1}^p D_i \to \cup_{i=1}^p D_i$ be a p-cyclic mapping which satisties* **a generalized p-cyclic weak φ-contraction** (3.3). *Then, there exists a best proximity point of T in D_j for some $j \in \{1, 2, \ldots, p\}$.*

Proof. Let $x \in D_i, 1 \leq i \leq p$. Define a sequence $\{x_n\}_{n=1}^\infty$ in (X, ρ) by

$$x_n := T^n x \text{ for } n \in \mathbb{N}.$$

Claim: $\{T^n x\}_{n=1}^\infty$ is a p-cyclic Cauchy sequence.
Let $m, n \in \mathbb{N}$ be such that $m > n$,

$$\begin{aligned} \rho(T^{pm}x, T^{pn+1}x) &= \rho(T^{p(n+r)}x, T^{pn+1}x), \text{ where } m = n + r, r \in \mathbb{N} \\ &= \rho(T^{pn}y, T^{pn+1}x), \text{ where } y = T^{pr}x \in D_i \\ &\to dist(D_i, D_{i+1}), \text{as } n \to \infty \text{ (because } T \in \Omega). \end{aligned}$$

This implies that, for all $\varepsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that

$$\rho(T^{pm}x, T^{pn+1}x) < \varepsilon + dist(D_i, D_{i+1}), m, n \geq n_0.$$

By Proposition 3.3, for any given $\varepsilon > 0$, there exists an $n_1 \in \mathbb{N}$ such that

$$\rho(T^{pm+k}x, T^{pn+k+1}x) < \varepsilon + dist(D_{i+k}, D_{i+k+1}), m, n \geq n_1, k \in \{1, 2, \ldots, p\}.$$

Therefore, the sequence $\{T^n x\}$ is a p-cyclic Cauchy sequence in (X, ρ). Since (X, ρ) is p-cyclic complete, there exists $k \in \{1, 2, \ldots, p\}$ such that $\{T^{pn+k}x\}$ converges to $z \in D_{i+k}$. By Theorem 4.8, z is best proximity point of T in D_j, where $j = i + k$. ∎

Example 4.10. Let $X =: \mathbb{R}^2$ be the Euclidean plane equipped with the usual Euclidean metric. Let subsets $D_i, i = 1, 2, 3, 4$ be as follows:

$$D_1 = \{(0, 0.5 + x) : 0 \leq x \leq 0.5\}, D_2 = \{(0.5 + x, 0) : 0 \leq x \leq 0.5\},$$

$D_3 = \{(0, -(0.5 + x)) : 0 \leq x \leq 0.5\}$ and $D_4 = \{-(0.5 + x, 0) : 0 \leq x \leq 0.5\}$. Let $\varphi(t) = \frac{1}{5}t, \forall t \geq 0$. Similarly, define $T : \cup_{i=1}^{4} D_i \to \cup_{i=1}^{4} D_i$ as follows:

$$T(0, 0.5 + x) = (0.5 + \frac{x}{10}, 0);$$

$$T(0.5 + x, 0) = (0, -(0.5 + \frac{x}{10}));$$

$$T(0, -(0.5 + x)) = (-(0.5 + \frac{x}{10}), 0);$$

$$T(-(0.5 + x), 0) = (0, 0.5 + \frac{x}{10}).$$

It is clear that $\rho(D_1, D_2) = \rho(D_2, D_3) = \rho(D_3, D_4) = \rho(D_4, D_1) = \frac{1}{2}\sqrt{2}$. Obviously T is a 4-cyclic map. If $x \in D_i, y \in D_{i+1}, i = 1, 2, 3, 4$. One can easily show that

$$\begin{aligned} m(x,y) - &\varphi(m(x,y)) + \varphi\big(\rho(D_i, D_{i+1})\big) - \rho(Tx, Ty) \\ &= \frac{4}{5}m(x,y) + \frac{\sqrt{2}}{10} - \sqrt{(0.5 + \frac{x}{10})^2 + (0.5 + \frac{y}{10})^2} \geq 0, \end{aligned}$$

for all $x \in D_i, y \in D_{i+1}$, where $D_{4+1} = D_1$ and

$$m(x,y) = \max\{\rho(x,y), \rho(x,Tx), \rho(y,Ty), \frac{1}{2}[\rho(x,Ty) + \rho(y,Tx)]\}.$$

Therefore, T is a generalized p-cyclic weak φ-contraction(3.3), where $p = 4$. All the conditions of Theorem 4.9 hold true, and T has the best proximity point. Let $x = (0, 0.5 + y) \in D_1$, where $y \in [0, 0.5]$. Then $\{T^{4n}x\} = \{(0, 0.5 + \frac{y}{10^{4n}})\}$. Clearly,$\{T^{4n}x\} \to (0, 0.5)$ as $n \to \infty$, which is a best proximity point of T in D_1. Also, $T(0, 0.5) = (0.5, 0)$, so $(0.5, 0)$ is a best proximity point of T in D_2, $T^2(0.0.5) = (0, -0.5)$ and $T^3(0, 0.5) = (-0.5, 0)$ are unique best proximity points of T in D_3 and D_4, respectively.

5. ACKNOWLEDGEMENTS

The authors would like to thank the referees for useful comments and suggestions for the improvement of this manuscripts.

6. FUNDING

Not Applicable.

7. COMPETING INTERESTS

The authors declare that they have no competing inteests.

8. AUTHOR CONTRIBUTIONS

The authors contributed equally to this paper. All authors have read and approved the final version of the manuscript.

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https://doi.org/10.58715/bangmodjmcs.2022.8.1