Bangmod International Journal of Mathematical and Computational Science
https://bangmodmcs.wordpress.com
ISSN : 2408-154X
Bangmod Int. J. Math. & Comp. Sci. Vol. 6, No. 1 & 2, 2020, 38 – 54

YAMABE SOLITONS ON PARA-KENMOTSU MANIFOLDS WITH CONFORMAL KILLING VECTOR FIELD

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Abstract The objective of the present paper is to estimate the Yamabe solitons on a para-Kenmotsu manifolds when the Weyl-conformal curvature tensor and projective curvature satisfies some geometric properties with conformal Killing vector filed, like flatness, semi-symmetry, pseudo-symmetry, Ricci-pseudo-symmetry and Einstein semi-symmetry. Finally, we illustrate an example of para-Kenmotsu manifolds admitting a expanding Yamabe soliton.

MSC: 47H09
Keywords: Para-Kenmotsu manifold, Weyl-conformal curvature tensor,scalar curvature, Projective curvature, Pseudosymmetric manifold, Yamabe soliton.

Submission date: 23.01.2020 / Acceptance date: 13.12.2020 / Available online 31.12.2020

1. Introduction

The Yamabe problem in differential geometry concerns the existence of Riemannian metrics with constant scalar curvature, and takes its name from the mathematician Hidehiko Yamabe in 1960. In differential geometry, the Yamabe flow is an intrinsic geometric flow a process which deforms the metric of a Riemannian manifold. The fixed points of the Yamabe flow are metrics of constant scalar curvature in the given conformal class first introduced by R. S. Hamilton [3] by the following equation

$$\frac{\partial}{\partial t}g(t) = -r(t)g(t), \tag{1.1}$$

where $r(t)$ denotes the scalar curvature of the metric $g(t)$. Yamabe soliton corresponds to self-similar solution of the Yamabe flow.

In dimension $n = 2$ the Yamabe flow is equivalent to the Ricci flow define by equation (2.22). However min dimension $n > 2$ the Yamabe and Ricci flow do not agree, since

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Published by Center of Excellence in Theoretical and Computational Science (TaCS-CoE)

the first one preserve the conformal class of the metric but the Ricci flow does not in general. The concept of Yamabe flow as a tool for constructing metrics of constant scalar curvature in a given conformal class of Riemannian matrices on (M^n, g), where $(n \neq 3)$.

A Riemannian manifold (M, g) is a Yamabe soliton if it admits a vector field X such that

$$\mathcal{L}_X g = 2(r - \lambda)g, \tag{1.2}$$

where $\mathcal{L}_X$ denotes the Lie derivative in the direction of the vector field X, r is the scalar curvature of the metric g and λ is a real number. Moreover, a vector filed X is called a *soliton field*. In the following, we denotes the Yamabe soliton satisfying In higher dimensions, Ricci solitons and Yamabe solitons have different behaviors. For instance, since any soliton vector field is a conformal vector field, if the scalar curvature is constant then it must be necessarily zero unless the soliton vector field is Killing [14], (Corollary 2.2(i)]). Yamabe solitons on three dimensional Sasakian manifolds and Kenmotsu manifolds were studied, respectively by R. Sharma ([6], [7]) and Y. Wang [12]. In [5] S. kundu also studied, Yamabe soliton in α-Sasakian manifold. Moreover, in [2] Erken, also, studied Yamabe soliton on three-dimensional normal para-contact metric manifolds.

In 1976, Sato [9] introduced the notion of almost para-contact manifolds. Before Sato, Takahashi [11], defined almost contact manifolds (in particular, Sasakian manifolds) equipped with an associated pseudo-Riemannian metric. In [4] Kaneyuki et al. defined the notion of almost paracontact structure on pseudo-Riemannian manifold of dimension $n = (2m + 1)$. Later Zamkovoy [15] showed that any almost paracontact structure admits a pseudo-Riemannian metric with signature $(n + 1; n)$. The notion of para-Kenmotsu manifold was introduced by Welyczko [13]. This structure is a analogy of Kenmotsu manifold in para-contact geometry. para-Kenmotsu (briefly SP-Kenmotsu) and special para-Kenmotsu (briefly SP-Kenmotsu) manifolds with solitons was studied by Blaga [1] and Siddiqi [8, 10] and others. Motivated by the above studies in this paper, we study Yamabe solitons in para-Kenmotsu manifolds $(n > 2)$ satisfying some geometric properties with conformal Killing vector field, like flatness, semi-symmetry, pseudo-symmetry, Ricci-pseudo-symmetry and Einstein semi-symmetry, using projective curvature and Weyl-conformal curvature tensor.

2. Preliminaries

An smooth manifold (M^n, g) $(n > 2)$ is said to be an almost paracontact metric manifold [5], if it admits a $(1,1)$-tensor field ϕ, a structure vector field ξ, a 1-form η and g is pseudo-Riemannian metric such that

$$\phi^2 X = X - \eta(X)\xi, \tag{2.1}$$

$$\eta(\xi) = 1, \tag{2.2}$$

$$g(\xi, \xi) = 1, \tag{2.3}$$

$$\eta(X) = g(X, \xi), \tag{2.4}$$

$$g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y) \tag{2.5}$$

for all vector fields X, Y on $\chi(M)$.

$$d\eta(X, Y) = g(X, \phi Y) \tag{2.6}$$

for every $X, Y \in \chi(M)$, then we say that $M(\phi, \xi, \eta, g)$ is an almost paracontact metric manifold. Also, we have

$$\phi\xi = 0, \ \ \eta(\phi X) = 0. \tag{2.7}$$

If an almost paracontact metric manifold satisfies

$$(\nabla_X \phi)(Y) = g(\phi X, Y) - \eta(Y)\phi X, \tag{2.8}$$

where ∇ denotes the Levi-Civita connection with respect to g, then M is called a almost para-Kenmotsu manifold [9].

An almost paracontact metric manifold is para-Kenmotsu if and only if

$$\nabla_X \xi = X - \eta(X)\xi. \tag{2.9}$$

Moreover the curvature tensor R, the Ricci tensor S and the Ricci operator Q in a para-Kenmotsu manifold M with respect to the Levi-Civita connection satisfy [13]

$$R(X, Y, Z, W) = [g(X, Z)g(Y, W) - g(Y, Z)g(X, W)], \tag{2.10}$$

$$\eta(R(X, Y)Z) = [g(X, Z)\eta(Y) - g(Y, Z)\eta(X)], \tag{2.11}$$

$$R(X, Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.12}$$

$$R(\xi, X)Y = \eta(Y)X - g(X, Y)\xi, \tag{2.13}$$

$$R(\xi, X)\xi = -R(X, \xi)\xi = X - \eta(X)\xi, \tag{2.14}$$

$$S(X, Y) = -(n-1)g(X, Y), \tag{2.15}$$

$$S(X, \xi) = -(n-1)\eta(X), \tag{2.16}$$

$$QX = -(n-1)(X), \tag{2.17}$$

$$Q\xi = -(n-1)\xi, \tag{2.18}$$

where $g(QX, Y) = S(X, Y)$.

$$S(\phi X, \phi Y) = S(X, Y) + (n-1)\eta(X)\eta(Y). \tag{2.19}$$

Definition 2.1. A para-Kenmotsu manifold M is said to be η-Einstein manifold if its Ricci tensor S is of the form

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \tag{2.20}$$

where a and b are scalar.

Definition 2.2. A vector field V on a Riemannian manifold (M^n, g) $(n > 2)$ is said to be conformal Killing vector field if it satisfies

$$\mathcal{L}_V g = \rho g, \tag{2.21}$$

where ρ is some scalar function.

For an n-dimensional almost contact metric manifold $(n > 2)$ the Weyl-conformal curvature tensor C is given by:

$$\begin{aligned} C(X,Y)Z = R(X,Y)Z &- \frac{1}{(n-2)}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] \\ &+ \frac{r}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y] \end{aligned} \tag{2.22}$$

$$C(\xi,X)Y = \left[1 - \frac{2(n-1)}{(n-2)} - \frac{r}{(n-1)(n-2)}\right][\eta(Y)X - g(X,Y)] \tag{2.23}$$

$$C(\xi,X)\xi = \left[1 - \frac{2(n-1)}{(n-2)} - \frac{r}{(n-1)(n-2)}\right][X - \eta(X)\xi] \tag{2.24}$$

$$C(\xi,\xi)X = 0. \tag{2.25}$$

3. Yamabe soliton in Weyl-Conformally flat para-Kenmotsu manifold

We use the following definition:

Definition 3.1. An n-dimensional $(n > 2)$ para-Kenmotsu manifold is called Weyl-conformally flat if $C(X,Y)Z = 0$ for any vector fields X, Y, Z.

we consider a para-Kenmotsu manifold which is Weyl-conformally flat. Then from (2.1) and (2.14) we have

$$\begin{aligned} R(X,Y)Z = &\frac{1}{(n-2)}[S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)QY] \\ &+ \frac{r}{(n-1)(n-2)}[g(Y,Z)X - g(X,Z)Y]. \end{aligned} \tag{3.1}$$

Taking inner product with U on both side in (**??**), we get

$$\begin{aligned} R(X,Y,Z,U) = &\frac{1}{(n-2)}[S(Y,Z)g(X,U) - S(X,Z)g(Y,U) \\ &+ g(Y,Z)S(X,U) - g(X,Z)S(Y,U)] \\ &+ \frac{r}{(n-1)(n-2)}[g(Y,Z)g(X,U) - g(X,Z)g(Y,U)], \end{aligned} \tag{3.2}$$

where $g(R(X,Y)Z,U) = R(X,Y,W,U)$ and $S(X,Y) = g(QX,Y)$.
Putting $X = U = e_i$ and summing over $i = 1, 2..., n$ in (3.2), we get the following equations

$$S(Y,Z) = \frac{n(n-1)}{(n-2)} g(Y,Z). \tag{3.3}$$

$$r = \frac{n^2(n-1)}{(n-2)}. \tag{3.4}$$

Therefore, we can state the following :

Theorem 3.2. *A Weyl-conformally flat para-Kenmotsu manifold scalar curvature is positive i.e.,* $\frac{n^2(n-1)}{(n-2)}$.

Let (M, g) be an n-dimenisonal $(n > 2)$ para-Kenmotsu manifold and Let (g, V, λ) be a Yamabe soliton on M. If V is conformal Killing vector filed, then by the definitions of conformal Killing vector filed we have

$$\mathcal{L}_V g(X,Y) = \rho g(X,Y) \tag{3.5}$$

where ρ is some scalar function and from equation (3.1), we have

$$r = \left(\lambda g - \frac{1}{2}\mathcal{L}_V g\right). \tag{3.6}$$

From equations (9.1) and (3.11), we get

$$r = -\left(\lambda - \frac{\rho}{2}\right) g(X,Y). \tag{3.7}$$

Taking inner product (2.14) with W and using equations (2.15) and (3.12), we obtain

$$C(X,Y,Z,W) = 0. \tag{3.8}$$

This shows that a para-Kenmotsu manifold M is Weyl-conformally flat.

Conversely, let M be an n-dimensional Weyl-conformally flat para-Kenmotsu manifold and (g, V, λ) be a Yamabe soliton on M $(n > 2)$, then from (7.1), we have

$$r = \frac{n^2(n-1)}{(n-2)}. \tag{3.9}$$

Substituting this in (3.1), we get

$$(\mathcal{L}_V g)(Y,Z) = \rho g(Y,Z) - \eta(Y)(Z). \tag{3.10}$$

Where $\rho = -2\left[-\lambda + \frac{n^2(n-1)}{(n-2)}\right]$ i.e V is conformal Killing.

If an n-dimensional $(n > 2)$Wely-conformally flat para-Kenmotsu manifold M admits a Yamabe soliton (g, ξ, λ), then from equation (3.1) we have

$$2r = -(\mathcal{L}_\xi g)(X,Y) + 2\lambda g(X,Y). \tag{3.11}$$

On an n-dimensional para-Kenmotsu manifold M, from equation (2.9) and (3.2), we obtain

$$r = (\lambda + 1)\, g(X,Y) - \eta(X)\eta(Y). \tag{3.12}$$

Then from equations (7.1) and (3.12), we get

$$\left[\frac{n^2(n-1) - (\lambda+1)(n-2)}{(n-2)}\right] g(X,Y) + \eta(X)\eta(Y) = 0. \tag{3.13}$$

Putting $X = \xi$ in equation (3.13), we get

$$\lambda = \frac{n^2(n-1)}{(n-2)}. \tag{3.14}$$

Since, λ is positive here. Therefore we can state the following theorem:

Theorem 3.3. *Let (g, V, λ) be a Yamabe soliton in an n-dimensional para-Kenmotsu manifold M $(n > 2)$. Then M is Weyl-conformally flat if and only if V is conformally killing. Further (g, ξ, λ) is expanding.*

4. Yamabe soliton in Weyl-semi-symmetric para-Kenmotsu manifold

Definition 4.1. An n-dimensional para-Kenmotsu manifold M is called Weyl-semi-symmetric if $R.C = 0$.

Let us assume Weyl-semi-symmetric para-Kenmotsu manifold. Then from the definition (4.1), we have

$$(R(X,Y).C)(U,V)W = 0. \tag{4.1}$$

Now, equation (4.1) can be express as

$$\begin{aligned} &R(X,Y)C(U,V)W - C(R(X,Y)U,V)W - C(U,R(X,Y)V)W \\ &- C(U,V)R(X,Y)W = 0. \end{aligned} \tag{4.2}$$

Putting $X = U = \xi$ in (4.2) and using (2.12), we get

$$\begin{aligned} &\eta(C(\xi,V)W)Y - g(Y,C(\xi,V)W)\xi - C(Y,V)W + \eta(Y)C(\xi,V)W \\ &- \eta(Y)C(\xi,Y)W + g(Y,V)C(\xi,\xi)W - \eta(W)C(\xi,V)Y + \varepsilon g(Y,W)C(\xi,V)\xi = 0. \end{aligned} \tag{4.3}$$

Using (2.22), (2.23), (2.24) and (2.25) in (4.3), we get

$$R(Y,V)W = [g(Y,W)V - g(V,W)Y] \tag{4.4}$$

Taking inner product with Z of (4.4), we have

$$g(R(Y,V)W,Z) = [g(Y,W)g(V,Z) - g(V,W)g(Y,Z)] \tag{4.5}$$

Taking $V = W = e_i$ and summing over $i = 1, 2, ..., n$ in (4.5), we get the following equations

$$S(Y,Z) = -(n-1)g(Y,Z) \tag{4.6}$$

$$r = -n(n-1). \tag{4.7}$$

Hence we state the following :

Theorem 4.2. *A Weyl semi-symmetric para-Kenmotsu manifold with a negative scalar curvature i.e., $-n(n-1)$.*

Let (M, g) be an n-dimensional para-Kenmotsu manifold and Let (g, V, λ) be a Yamabe soliton on M. If V is conformal Killing vector filed, then by the definitions of conformal Killing vector filed we have

$$\mathcal{L}_V g(X,Y) = \rho g(X,Y) \tag{4.8}$$

where ρ is some scalar function and from equation (3.2), we have

$$r = \frac{1}{2}\mathcal{L}_V g + \lambda g. \tag{4.9}$$

From equations (4.8) and (4.9), we get

$$r = \left(\lambda + \frac{\rho}{2}\right) g(X,Y). \tag{4.10}$$

Then equation (4.2), we have

$$\begin{aligned} R.C = R(X,Y)C(U,V)W - C(R(X,Y)U,V)W - C(U,R(X,Y)V)W \\ - C(U,V)R(X,Y)W = 0. \end{aligned} \tag{4.11}$$

Using (2.15) and (4.10), we get

$$R.C = 0. \tag{4.12}$$

This shows that n-dimensional para-Kenmotsu manifold M is Weyl semi-symmetric.

Conversely, Let M be a Weyl semi-symmetric para-Kenmotsu manifold and (g,V,λ) be a Yamabe soliton on M. then from (4.6), we have

$$r = -n(n-1). \tag{4.13}$$

Putting this in (3.1) we get

$$(\mathcal{L}_V g)(X,Y) = \rho g(X,Y), \tag{4.14}$$

where $\rho = (-2n(n-1) + \lambda)$ i.e V is conformal killing.
If an n-dimensional Wely-conformally flat para-Kenmotsu manifold M admits Yamabe soliton (g,ξ,λ), then from equation (3.1) we have

$$2r = (\mathcal{L}_\xi g)(X,Y) + (2\lambda)g(X,Y). \tag{4.15}$$

On an n-dimensional para-Kenmotsu manifold M, from equation (2.9) and (3.2), we obtain

$$r = (\lambda + 1)\, g(X,Y) - \eta(X)\eta(Y). \tag{4.16}$$

Then from equations (4.13) and (4.15), we get

$$[-n(n-1) - (\lambda+1)]g(X,Y) + \eta(X)\eta(Y) = 0. \tag{4.17}$$

Putting $X = \xi$ in equation (4.17), we get

$$\lambda = n(n-1). \tag{4.18}$$

Since, λ is positive here. Therefore we can state the following theorem:

Theorem 4.3. *Let (g,V,λ) be a Yamabe soliton in an n-dimensional para-Kenmotsu manifold M. Then M is Weyl semi-symmetric if and only if V is conformally killing. Further deduce that (g,ξ,λ) is expanding.*

5. Yamabe soliton in Einstein semi-symmetric para-Kenmotsu manifold

Now, we have the following definition:

Definition 5.1. An n-dimensional (ε)-Kenmotsu manifold M is called Einstein semi-symmetric if $R.E = 0$, where E is the Einstein tensor defined by

$$E(X,Y) = S(X,Y) - \frac{r}{n}g(X,Y), \tag{5.1}$$

where S is the Ricci tensor and r is the scalar curvature.

Let is assume that an n-dimensional Einstein semi-symmetric para-Kenmotsu manifold M. Then from the definition 5.1, we have

$$(R(X,Y).E)(Z,W) = 0. \tag{5.2}$$

Equation (5.2) can be written as

$$E(R(X,Y)Z,W) + E(Z,R(X,Y)W) = 0 \tag{5.3}$$

Now, using (5.1) in (5.3), we get

$$\begin{aligned} &S(R(X,Y)Z,W) + S(Z,R(X,Y)W) \\ &- \frac{r}{n}[g(R(X,Y)Z,W) + g(Z,R(X,Y)W)] = 0. \end{aligned} \tag{5.4}$$

Replacing $X = W = e_i$ where $\{e_i\}$ is an orthonormal basis of the tangent space at each point of the manifold and talking summation over $i = 1, 2, ...n$ we get the following equations

$$S(X,Z) = -\frac{r}{n}g(X,Z) \tag{5.5}$$

and

$$r = 0 \tag{5.6}$$

Hence, we state the following:

Theorem 5.2. *Scalar curvature of an n-dimensional Einstein semi-symmetric para-Kenmotsu manifold M is vanish.*

Now, let us consider an n-dimensional para-Kenmotsu manifold M and let data (g, V, λ) be a Yamabe soliton on M. If V is conformal killing. Then by the definition

$$(\mathcal{L}_V g)(X,Y) = \rho g(X,Y) \tag{5.7}$$

for some scalar function ρ and from (3.1), we have

$$r = \left[\lambda g + \frac{1}{2}\mathcal{L}_V g\right]. \tag{5.8}$$

From (5.7) and (5.9), we get

$$\lambda = \frac{1}{2}\rho. \tag{5.9}$$

Then from (5.3), we have

$$\begin{aligned} R.E = &S(R(X,Y)Z,W) + S(Z,(R(X,Y)W) \\ &- \frac{r}{n}[g(R(X,Y)Z,W) + g(Z,R(X,Y)W)]. \end{aligned} \tag{5.10}$$

Using (2.15) and (5.9) in (5.10), we get

$$R.E = 0. \tag{5.11}$$

This shows that an n-dimensional para-Kenmotsu manifold M is Einstein semi-symmetric.

Conversely, let an n-dimensional Einstein semi-symmetric para-Kenmotsu manifold M and (g, V, λ) be a Yamabe soliton on M, then using (5.5) in (3.1) we get

$$(\mathcal{L}_V g)(X, Y) = \rho g(X, Y) \tag{5.12}$$

where $\rho = 2\lambda$ i.e V is conformal killing.

Now, if an n-dimensional Einstein semi-symmetric para-Kenmotsu manifold M admits Yamabe soliton (g, ξ, λ), then from the equation (3.1), we have

$$r = (\lambda + 1)g(X, Y) - \eta(X)\eta(Y). \tag{5.13}$$

Then from (5.5) and (5.13), we get

$$[r - (\lambda + 1)]g(X, Y) + \eta(X)\eta(Y) = 0 \tag{5.14}$$

Putting $X = \xi$ in (5.13), we get

$$\lambda = 0 \tag{5.15}$$

Hence, λ is zero . Therefore we can state the following theorem:

Theorem 5.3. *Let (g, V, λ) be a Yamabe soliton in an n-dimensional para-Kenmotsu manifold M. Then M is Einstein semi-symmetric if and only if V is conformal killing. Further deduce that (g, ξ, λ) is steady.*

6. Yamabe soliton in projectively flat para-Kenmotsu manifold

Definition 6.1. The projective curvature tenosr P in an n-dimensional para-Kenmotsu manidold M is defined by

$$P(X, Y)Z = R(X, Y)Z - \frac{1}{(n-1)}[g(Y, Z)QX - g(X, Z)QY] \tag{6.1}$$

for any X, Y, Z on M, where Q is the Ricci operator defined by $S(X, Y) = g(QX, Y)$. The manifold is said to be projectively flat if P vanishes identically on M i e.

$$P(X, Y)Z = 0. \tag{6.2}$$

We consider an n-dimensional para-Kenmotsu manifold which is projectively flat that is $P(X, Y)Z = 0$. Then from the definition (6.1) and equation (7.1) we have

$$R(X, Y)Z = \frac{1}{(n-1)}[g(Y, Z)QX - g(X, Z)QY]. \tag{6.3}$$

Taking inner product with U of equation (7.5), we get

$$R(X, Y, Z, U) = \frac{1}{(n-1)}[g(Y, Z)S(X, U) - g(X, Z)S(Y, U)], \tag{6.4}$$

where $R(X, Y, Z, U) = g(R(X, Y)Z, U)$ and $S(X, Y) = g(QX, Y)$. Taking $X = U = e_i$ and summing over $i = 1, 2, ...n$ in (7.6), we obtain the following equations

$$S(Y, Z) = -ng(Y, Z). \tag{6.5}$$

$$r = -n^2 \tag{6.6}$$

Hence we can state the following:

Theorem 6.2. *The scalar curvature of a projectively flat n-dimensional para-Kenmotsu manifold M is negative i.e., $-n^2$.*

Let an n-dimensional para-Kenmotsu manifold M and Let (g, V, λ) be a Yamabe soliton on M. If V is conformal killing vector filed, then by the definition

$$(\mathcal{L}_V g)(X, Y) = \rho g(X, Y) \tag{6.7}$$

for some scalar function ρ, by (3.1), we have

$$r = (\lambda g + \frac{1}{2}\mathcal{L}_V g). \tag{6.8}$$

From (7.8) and (7.9), we get

$$r = (\lambda + \frac{1}{2}\rho)g(X, Y). \tag{6.9}$$

Taking inner product (7.1) with W and by the virtue of (2.15) and (7.10), we obtain

$$P(X, Y, Z, W) = 0. \tag{6.10}$$

This shows that an n-dimensional para-Kenmotsu manifold M is projectively flat.

Conversely, let M be a projectively flat n-dimensional para-Kenmotsu manifold and (g, V, λ) be a Yamabe soliton on M, then from (3.1), we have

$$r = -n^2. \tag{6.11}$$

Substituting this in (3.1), we get

$$(\mathcal{L}_V g)(X, Y) = \rho g(Y, Z). \tag{6.12}$$

Where $\rho = -2[n^2 + \lambda]$ that is V is conformal Killing.
Further, If a Projectively flat n-dimensional para-Kenmotsu manifold M admits Yamabe soliton (g, ξ, λ), then from virtue of equations (3.1) and (2.9), we get

$$r = (\lambda + 1))g(X, Y) - \eta(X)\eta(Y). \tag{6.13}$$

Then from equations (7.7) and (9.3), we get

$$[r - (\lambda + 1))g(X, Y) + \eta(X)\eta(Y) = 0. \tag{6.14}$$

Substituting $X = \xi$ in (9.4), we get

$$\lambda = -2n^2. \tag{6.15}$$

Hence λ is negative. Therefore we can state the following theorem:

Theorem 6.3. *Let (g, V, λ) be an Yamabe soliton in an n-dimensional para-Kenmotsu manifold M. Then M is projectively flat if and only if V is conformally killing. Further (g, ξ, λ) is shrinking.*

7. Yamabe soliton in Weyl pseudo-symmetric para-Kenmotsu manifold

Definition 7.1. An n-dimensional para-Kenmotsu manifold M is called Weyl pseudo-symmetric if the tensors $R.C$ and $Q(g, C)$ are linearly dependent. This is equivalent to

$$R.C = L_C Q(g, C) \tag{7.1}$$

holding on the set $U_C = \{x \in M : C \neq 0 \text{ at } x\}$, where L_C is some function on U_C and $Q(g, S)$, $(X \wedge Y)$ are respectively defined as

$$Q(g, S) = ((X \wedge_g Y).S)(U, V) \tag{7.2}$$

$$(X \wedge_g Y) = g(Y, Z)X - g(X, Z)Y. \tag{7.3}$$

for all X, Y, U and $V \in TM^n$.

Let us consider n-dimensional Weyl pseudo-symmetric para-Kenmotsu manifold M. Then from definition (7.1), we have

$$(R(X, Y)C)(U, V)W = L_C[Q(g, C)(U, V, W; X, Y)]. \tag{7.4}$$

Equation (7.4) can be written as

$$\begin{aligned} &R(X, Y)C(U, V)W - C(R(X, Y)U, V)W - C(U, R(X, Y)V)W \\ &- C(U, V)R(X, Y)W = L_C[(X \wedge Y)C(U, V)W - C((X \wedge Y)U, V))W \\ &- C(U, (X \wedge Y)V)W - C(U, V)(X \wedge Y)W]. \end{aligned} \tag{7.5}$$

Putting $X = U = \xi$ in (7.5) and using (2.13), (2.14), (7.2) and (7.3), we get

$$\begin{aligned} &[L_C + 1][g(Y, C(\xi, V)W)\xi - \eta(C(\xi, V)W)Y + C(Y, V)W \\ &- \eta(Y)C(\xi, V)W + \eta(V)C(\xi, Y)W - g(Y, V)C(\xi, \xi)W \\ &+ \eta(W)C(\xi, V)Y - g(Y, W)C(\xi, V)\xi] = 0. \end{aligned} \tag{7.6}$$

Using (2.22), (2.23), (2.24) and (2.25) in (7.6), we get

$$[L_C + 1][R(Y, V)W + g(V, W)Y - g(Y, W)V] = 0. \tag{7.7}$$

Therefore, either

$$L_C = -1 \qquad \text{or} \quad R(Y, V) = g(Y, W)V - g(V, W)Y. \tag{7.8}$$

Taking inner product with Z of (7.8), we get

$$R(Y, V, W, Z) = g(Y, W)g(V, Z) - g(V, W)g(Y, Z) \tag{7.9}$$

Taking $V = W = e_i$ and summing over $i = 1, 2, ...n$ in (7.9), we obtain the following equations

$$S(Y, Z) = -(n - 1)g(Y, Z) \tag{7.10}$$

and

$$r = -n(n - 1) \tag{7.11}$$

Hence, we state the following

Theorem 7.2. *The scalar curvature of a Weyl pseudo-symmetric para-Kenmotsu manifold in negative i.e., $-n(n - 1)$ with $L_C \neq -1$.*

Let a n-dimensional Weyl pseudo-symmetric para-Kenmotsu manifold M admits Yamabe soliton on (g, ξ, λ). Then from equations (3.13) and (7.11), we have

$$[-n(n - 1) - (\lambda + 1)]g(X, Y) + \eta(X)\eta(Y) = 0. \tag{7.12}$$

Substitution of $X = \xi$ in (9.12), we get the relation:

$$\lambda = -n(n - 1). \tag{7.13}$$

Therefore, λ is negative. Hence we state the following theorem:

Theorem 7.3. *A Yamabe soliton in (g, ξ, λ) n-dimensional Weyl-semi-symmetric para-Kenmotsu manifold M is shrinking provided $L_C \neq -1$.*

8. Yamabe soliton in partially Ricci-pseudo-symmetric para-Kenmotsu manifold

Definition 8.1. An n-dimensional para-Kenmotsu manifold M is called partially Ricci-pseudo-symmetric if and only if the relation

$$R.S = f(p)Q(g, S) \tag{8.1}$$

holds on the set $A = \{x \in M : Q(g, S) \neq 0 \text{ at } x\}$, where $f \in C^\infty(M)$ for $p \in A$.
R.S is defined as

$$(R(X, Y).S)(U, V) = -S(R(X, Y)U, V) - S(U, R(X, Y)V \tag{8.2}$$

for all X, Y, U and $V \in TM^n$.

Let us consider n-dimensional partially Ricci-pseudo symmetric para-Kenmotsu manifold M. Then from the definition (9.1), we have

$$(R(X, Y).S)(Z, U) = f(p)[(X \wedge_g Y).S)(Z, U)] \tag{8.3}$$

From (7.2), (7.3) and (9.2), it follows that

$$\begin{aligned} &S(R(X, Y)Z, U) + S(Z, R(X, Y)U) \\ &= f(p)[S((X \wedge_g Y)Z, U) + S(Z, (X \wedge_g Y)U)]. \end{aligned} \tag{8.4}$$

Taking $Y = U = \xi$ in (9.4), we have

$$\begin{aligned} &S(R(X, \xi)Z, \xi) + S(Z, R(X, \xi)\xi) \\ &= f(p)[S((X \wedge_g \xi)Z, \xi) + S(Z, (X \wedge_g \xi)\xi)]. \end{aligned} \tag{8.5}$$

Applying (2.16) and (9.2) in (9.5), we obtain

$$\begin{aligned} &-(n-1)\eta(R(X, \xi)Z, \xi) + \eta(X)S(Z, \xi) - S(X, Z) \\ &= f(p)[\eta(Z)S(X, \xi) - g(X, Z)S(\xi, \xi) + S(X, Z) - \eta(X)S(Z, \xi)]. \end{aligned} \tag{8.6}$$

Using (2.10) and (2.16) in (9.6), we get

$$\begin{aligned} &(n-1)\{\eta(Z)\eta(X) - g(X, Z)\} - S(X, Z) - (n-1)\eta(X)\eta(Z) \\ &= f(p)[-(n-1)\eta(X)\eta(Z) + (n-1)g(X, Z) + S(X, Z) + (n-1)\eta(X)\eta(Z)]. \end{aligned} \tag{8.7}$$

This can be written as

$$-[S(X, Z) + (n-1)g(X, Z)] = f(p)[S(X, Z) + (n-1)g(X, Z)]. \tag{8.8}$$

Thus, we have

$$[f(p) + 1][S(X, Z) + (n-1)g(X, Z)] = 0. \tag{8.9}$$

This can be hold only if either

$$f(p) = -1 \qquad or \quad S(X, Z) = -(n-1)g(X, Z) \tag{8.10}$$

Also ,

$$f(p) = -1 \qquad or \quad r = -n(n-1). \tag{8.11}$$

Hence, we state the following

Theorem 8.2. *The scalar curvature r of a partially Ricci pseudo-symmetric para-Kenmotsu manifold in negative i.e., $-n(n-1)$ with $f(p) \neq -1$.*

Let a n-dimensional partially Ricci pseudo-symmetric para-Kenmotsu manifold M admits Yamabe soliton on (g, ξ, λ). Then from equations (3.13) and (9.11), we have

$$[-n(n-1) - (\lambda + 1)]g(X, Y) + \eta(X)\eta(Y) = 0. \tag{8.12}$$

Substitution of $X = \xi$ in (9.12), we get the relation:

$$\lambda = -n(n-1). \tag{8.13}$$

Therefore, λ is negative. Hence we state the following theorem:

Theorem 8.3. *A Yamabe soliton in (g, ξ, λ) n-dimensional partially Ricci pseudo-symmetric para-Kenmotsu manifold M is shrinking provided $f(p) \neq -1$.*

9. Yamabe soliton in partially Ricci-pseudo-symmetric para-Kenmotsu manifold

Definition 9.1. An n-dimensional para-Kenmotsu manifold M is called partially Ricci-pseudo-symmetric if and only if the relation

$$R.S = f(p)Q(g, S) \tag{9.1}$$

holds on the set $A = \{x \in M : Q(g, S) \neq 0 \text{ at } x\}$, where $f \in C^\infty(M)$ for $p \in A$.
R.S is defined as

$$(R(X, Y).S)(U, V) = -S(R(X, Y)U, V) - S(U, R(X, Y)V \tag{9.2}$$

for all X, Y, U and $V \in TM^n$.

Let us consider n-dimensional partially Ricci-pseudo symmetric para-Kenmotsu manifold M. Then from the definition (9.1), we have

$$(R(X, Y).S)(Z, U) = f(p)[(X \wedge_g Y).S)(Z, U)] \tag{9.3}$$

From (7.2), (7.3) and (9.2), it follows that

$$\begin{aligned} &S(R(X, Y)Z, U) + S(Z, R(X, Y)U) \\ &= f(p)[S((X \wedge_g Y)Z, U) + S(Z, (X \wedge_g Y)U)]. \end{aligned} \tag{9.4}$$

Taking $Y = U = \xi$ in (9.4), we have

$$\begin{aligned} &S(R(X, \xi)Z, \xi) + S(Z, R(X, \xi)\xi) \\ &= f(p)[S((X \wedge_g \xi)Z, \xi) + S(Z, (X \wedge_g \xi)\xi)]. \end{aligned} \tag{9.5}$$

Applying (2.16) and (9.2) in (9.5), we obtain

$$\begin{aligned} &-(n-1)\eta(R(X, \xi)Z, \xi) + \eta(X)S(Z, \xi) - S(X, Z) \\ &= f(p)[\eta(Z)S(X, \xi) - g(X, Z)S(\xi, \xi) + S(X, Z) - \eta(X)S(Z, \xi)]. \end{aligned} \tag{9.6}$$

Using (2.10) and (2.16) in (9.6), we get

$$\begin{aligned} &(n-1)\{\eta(Z)\eta(X) - g(X, Z)\} - S(X, Z) - (n-1)\eta(X)\eta(Z) \\ &= f(p)[-(n-1)\eta(X)\eta(Z) + (n-1)g(X, Z) + S(X, Z) + (n-1)\eta(X)\eta(Z)]. \end{aligned} \tag{9.7}$$

This can be written as

$$-[S(X,Z)+(n-1)g(X,Z)] = f(p)[S(X,Z)+(n-1)g(X,Z)]. \tag{9.8}$$

Thus, we have

$$[f(p)+1][S(X,Z)+(n-1)g(X,Z)] = 0. \tag{9.9}$$

This can be hold only if either

$$f(p) = -1 \qquad or \quad S(X,Z) = -(n-1)g(X,Z) \tag{9.10}$$

Also ,

$$f(p) = -1 \qquad or \quad r = -n(n-1). \tag{9.11}$$

Hence, we state the following

Theorem 9.2. *The scalar curvature r of a partially Ricci pseudo-symmetric para-Kenmotsu manifold in negative i.e., $-n(n-1)$ with $f(p) \neq -1$.*

Let a n-dimensional partially Ricci pseudo-symmetric para-Kenmotsu manifold M admits Yamabe soliton on (g,ξ,λ). Then from equations (3.13) and (9.11), we have

$$[-n(n-1)-(\lambda+1)]g(X,Y)+\eta(X)\eta(Y) = 0. \tag{9.12}$$

Substitution of $X=\xi$ in (9.12), we get the relation:

$$\lambda = -n(n-1). \tag{9.13}$$

Therefore, λ is negative. Hence we state the following theorem:

Theorem 9.3. *A Yamabe soliton in (g,ξ,λ) n-dimensional partially Ricci pseudo-symmetric para-Kenmotsu manifold M is shrinking provided $f(p) \neq -1$.*

10. Yamabe solitons in Weyl Ricci-pseudo symmetric para-Kenmotsu manifold

Definition 10.1. An n-dimensional para-Kenmotsu manifold M is called Weyl Ricci-pseudo-symmetric if the tensors $C.S$ and $Q(g,S)$ are linearly dependent. This is equivalent to

$$C.S = L_S Q(g,S) \tag{10.1}$$

holding the set $U_S = \{x \in M : \quad C \neq 0 \quad at \quad x\}$, where L_S is some function on U_S.

Let us consider an n-dimensional Weyl Ricci-pseudo-symmetric para-Kenmotsu manifold M. Then from definition (10.1), we have

$$(C(X,Y).S)(U,V) = L_S Q(g,S). \tag{10.2}$$

Equation (10.2) can be express as

$$\begin{aligned} &S(C(X,Y)U,V)+S(U,C(X,Y)V) \\ &= L_S[S((X\wedge Y)U,V)+S(U,(X\wedge Y)V)]. \end{aligned} \tag{10.3}$$

Putting $X=V=\xi$ in (10.3), we get

$$S(C(\xi,Y)U,\xi)+S(U,C(\xi,Y)\xi) = L_S[S((\xi\wedge Y)U,\xi+S(U,(\xi\wedge Y)\xi)] \tag{10.4}$$

Using (2.16), (2.17), (2.24), (2.25) and (7.3) in (10.4), we obtain

$$\left[L_S + 1 - \frac{2(n-1)}{(n-2)} - \frac{r}{(n-1)(n-2)}\right] S(Y,U) + (n-1)g(Y,U) = 0 \tag{10.5}$$

This can be hold only if either

$$L_S = \left[\frac{2(n-1)}{(n-2)} + \frac{r}{(n-1)(n-2)} - 1\right] \quad or \quad S(Y,U) = -(n-1)g(Y,U). \tag{10.6}$$

Also,

$$r = -n(n-1) \tag{10.7}$$

Hence, we state the following:

Theorem 10.2. *The scalar curvature r of a Weyl Ricci pseudo-symmetric para-Kenmotsu manifold in negative i.e., $-n(n-1)$ with $L_S \neq \left[\frac{2(n-1)}{(n-2)} + \frac{r}{(n-1)(n-2)} - 1\right]$.*

Let a n-dimensional Weyl Ricci pseudo-symmetric para-Kenmotsu manifold M admits Yamabe soliton on (g, ξ, λ). Then from equations (3.13), (10.6) and (10.7), we have

$$[-n(n-1) - (\lambda+1)]g(X,Y) + \eta(X)\eta(Y) = 0. \tag{10.8}$$

Substitution of $X = \xi$ in (10.8), we get the relation:

$$\lambda = -n(n-1). \tag{10.9}$$

Therefore, λ is negative. Hence we state the following theorem:

Theorem 10.3. *A Yamabe soliton in (g, ξ, λ) n-dimensional Weyl Ricci pseudo-symmetric para-Kenmotsu manifold M is shrinking provided $L_S \neq \left[\frac{2(n-1)}{(n-2)} + \frac{r}{(n-1)(n-2)} - 1\right]$.*

11. Example of para-Kenmotsu manifolds admitting Yamabe soliton

Example 11.1. Let the three dimensional manifold $M = [(x,y,z) \in \mathbb{R}^3 \mid z \neq 0]$, where (x,y,z) are the cartesian coordinates in $\mathbb{R}^3$. Choosing the vector fields

$$e_1 = \frac{\partial}{\partial x}, \quad e_2 = \frac{\partial}{\partial y}, \quad e_3 = -\frac{\partial}{\partial z},$$

which are linearly independent at each point of M. Set

$$\phi = \frac{\partial}{\partial} \otimes dx + \frac{\partial}{\partial x} \otimes dy, \quad \xi = -\frac{\partial}{\partial Z}, \quad \eta = -dz.$$

Let g be the Riemannian metric define by

$$g = (dx \otimes dx - dy \otimes dy) + dz \otimes dz$$

$$g(e_1, e_3) = g(e_2, e_3) = g(e_2, e_2) = 0, \quad g(e_1, e_1) = g(e_2, e_2) = g(e_3, e_3) = 1,$$

where $\epsilon = \pm 1$. That is the form of the metric becomes Let η be the 1-form defined by $\eta(Z) = g(Z, e_3)$ for any vector field Z on TM and ϕ be the $(1,1)$ tensor field defined by $\phi(e_1) = e_2, \quad \phi(e_2) = e_1, \quad \phi(e_3) = 0$. Then by the linearity property of ϕ and g, we have

$$\phi^2 Z = Z - \eta(Z)e_3, \qquad \eta(e_3) = 1 \qquad g(\phi Z, \phi W) = g(Z,W) - \eta(Z)\eta(W)$$

for any vector fields Z, W on M.

Let ∇ be the Levi-Civita connection with respect to the metric g. Then we have

$$[e_1, e_2] = [e_1, e_3] = [e_2, e_3] = 0.$$

The Riemannian connection ∇ with respect to the metric g is given by

$$2g(\nabla_X Y, Z) = Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) + g([X, Y], Z) - g([Y, Z], X) + g([Z, X], Y).$$

From above equation which is known as Koszul's formula, we have

$$\nabla_{e_1} e_3 = e_1, \qquad \nabla_{e_2} e_3 = e_2, \qquad \nabla_{e_3} e_3 = 0,$$

$$\nabla_{e_1} e_2 = 0, \qquad \nabla_{e_2} e_2 = e_3, \qquad \nabla_{e_3} e_2 = e_2,$$

$$\nabla_{e_1} e_1 = -e_3, \qquad \nabla_{e_2} e_1 = 0, \qquad \nabla_{e_3} e_1 = e_1.$$

Using the above relations, for any vector field X on M, we have

$$\nabla_X \xi = (X - \eta(X)\xi)$$

for $\xi \in e_3$. Hence the manifold M under consideration is a para-Kenmotsu manifold of dimension three.

Thus it can be easily seen that (M^3, ϕ, ξ, η) is a para-Kenmotsu manifold. Hence one can easily obtain by simple calculation that the curvature tensor, Ricci tensor components and scalar curvature are as follows

$$R(e_1, e_2)e_2 = e_1, \qquad R(e_1, e_3)e_3 = -e_1, \qquad R(e_2, e_1)e_1 = -e_2,$$

$$R(e_2, e_3)e_3 = -e_2, \qquad R(e_3, e_1)e_1 = e_3, \qquad R(e_3, e_2)e_2 = -e_3,$$

and

$$S(e_1, e_1) = S(e_2, e_2) = 0, \quad S(e_3, e_3,) = -2.$$

Thus the scalar curvature r is constant. Therefore we have

$$r = S(e_1, e_1) + S(e_2, e_2) + S(e_3, e_3,) = -2.$$

From the expression (1.2) we have

$$2[g(e_i, e_i) - \eta(e_i)\eta(e_i)] - 2(r - \lambda)g(e_i, e_i) = 0$$

for all $i \in \{1, 2, 3\}$, and we have

$$2(1 - \delta_{i3}) - 2(r - \lambda)g(e_i, e_i) = 0$$

for all $i \in \{1, 2, 3\}$,
we get $\lambda = 1$ (*i.e.* $\lambda > 0$). Thus the data (g, ξ, λ) is a Yamabe soliton on $(M^3, \phi, \xi, \eta, g)$, i.e, expanding.

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