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A STUDY ON HYPERSURFACES OF A COMPLEX SPACE FORM

Sunil Kumar Yadav[1,*]
[1] *Department of Mathematics*
Poornima College of Engineering,
ISI-6, RIICO Institution Area Sitapura,
Jaipur-302022, Rajasthan, India.
E-mails: prof_sky16@yahoo.com

**Corresponding author.*

Abstract The object of the present paper is to investigate conformal η-Ricci solitons and η-Yamabe solitons on hypersurface M^n of a complex space form $\hat{M}^{n+1}(4\kappa)$ restricted to the shape operator A with respect to $N{=}J\xi$ has only one eigenvalues.

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1. Introduction

In 1982, Hamilton [8] introduced the concept of Ricci flow and proved its existence. The Ricci flow is an evolution equation for the metrics on a Remannian manifold given by

$$\frac{\partial}{\partial t}g = -2Ric,$$

where g is the Riemannian metric and Ric denotes the Ricci tensor.
A self-similar solution to the Ricci flow (see, [8], [13]) is called Ricci soliton [9] if it move only by a one parameter family of diffeomorphism and scaling. The Ricci soliton equation is given by

$$\mathfrak{L}_V g + 2Ric + 2\tau_1 g = 0, \tag{1.1}$$

where $\mathfrak{L}$, g, Ric, V and τ_1 denote the Lie derivative, Riemannian metric, Ricci tensor, a complete vector field and a real scalar on a Riemannian manifold M respectively. A Ricci soliton (g, V, τ_1) is said to be shrinking, steady and expanding according as τ_1 is negative, zero and positive respectively. A Ricci soliton with $V = 0$ reduced to Einstein

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equation. It became more important when Grigory Perelman applied Ricci solitons to solve the long standing Poincaré conjecture posed in 1904.

Fischer [7] initiated the concept of conformal Ricci flow, which is a variation of the classical Ricci flow equation that modifies the unit volume constraint of that equation to a scalar curvature constraint. The conformal Ricci flow on a smooth closed connected oriented n-manifold M is defined by the equation

$$\frac{\partial g}{\partial t}+2\left(Ric+\frac{g}{n}\right)=-pg, \quad r(g)=-1, \tag{1.2}$$

where p is a non-dynamical field (time dependent scalar field), $r(g)$ is the scalar curvature of the n-dimensional manifold. The conformal Ricci flow equations are analogous to the Navier-Stokes equations of fluid mechanics and because of this analogy the time dependent scalar field p is called a conformal pressure and, as for the real physical pressure in fluid mechanics that serves to maintain the incompressibility of the fluid, the conformal pressure serves as a Lagrange multiplier to conformally deform the metric flow so as to maintain the scalar curvature constraint. The equilibrium points of the conformal Ricci flow equations are Einstein metrics with Einstein constant $-\frac{1}{n}$. Thus the conformal pressure p is zero at an equilibrium point and positive otherwise.

In 2015, Basu and Bhattacharyya [2] introduced the idea of conformal Ricci soliton defined by the equation as

$$\mathfrak{L}_V g+2Ric=[2\tau_1-(p+\frac{2}{n})]g, \tag{1.3}$$

where τ_1 is constant. This equation is the generalization of the Ricci soliton equation and it also satisfies the conformal Ricci flow equation.

In 2009, Jong Taek Cho and Makoto Kimura introduced the notion of η-Ricci soliton [14], given by the equation

$$\mathfrak{L}_V g+2Ric=2\tau_1 g+2\tau_2\eta\otimes\eta, \tag{1.4}$$

for constants τ_1 and τ_2.
In 2018, M. D. Siddiqi [12] introduced the notion of Conformal η-Ricci soliton as

$$\mathfrak{L}_V g+2Ric+[2\tau_1-(p+\frac{2}{n})]g+2\tau_2\eta\otimes\eta=0, \tag{1.5}$$

where $\mathfrak{L}_V$ is the Lie derivative along the vector field V, Ric is the Ricci tensor, τ_1 and τ_2 are constants, p is a scalar non-dynamical field (time dependent scalar field) and n is the dimension of manifold.
A solution to the Yamabe flow, $g(t)$, is called a Yamabe soliton if there exists a smooth function $\gamma(t)$ and a 1-parameter family of diffeomorphisms $\{\psi(t)\}$ of M such that

$$g(t)=\gamma(t)\psi_t^*(g_0),$$

with $\gamma(0)$=1 and ψ_0=idM. Here, the Yamabe flow, which is the parabolic analogue of the Yamabe equation, is defined as

$$\frac{d}{dt}g(t)=-\sigma(t)g(t), \quad g(0)=g_0. \tag{1.6}$$

We notice that Yamabe flow has been studied extensively by the several authors (see, [1],[3],[4],[10],[11],[6]). On substituting $g(t)=\gamma(t)\psi_t^*(g_0)$, into (1.6), and evaluating it at

t=0, we get

$$(\sigma - \tau_1)g = \frac{1}{2}\mathfrak{L}_V g, \tag{1.7}$$

where τ_1=$\acute{\gamma}(0)$, V is the vector field generated by the 1-parameter family $\{\psi(t)\}$ and $\mathfrak{L}_V g$ denotes the Lie-derivative of the metric g along the vector field V, σ is the scalar curvature. It is called shrinking, steady or expanding if $\tau_1 > 0$, steady if $\tau_1 = 0$ or $\tau_1 < 0$ respectively.
In this context we define the notion of η-Yamabe soliton as

$$\frac{1}{2}\mathfrak{L}_V g = (\sigma - \tau_1)g - \tau_2\eta \otimes \eta, \tag{1.8}$$

where τ_1 and τ_2 are contants and η is a 1-form. Moreover if τ_2=0, the above equation reduces to (1.7) and so the η-Yamabe soliton becomes Yamabe soliton.

2. Hypersurfaces of a complex space form

Definition 2.1. A Kähler manifold $\hat{M}^{n+1}$ is called a complex space form if it has constant holomorphic sectional curvature.

The Riemannian curvature tensor $\hat{R}$ of a complex space form as follows

$$\begin{aligned}\hat{R}(X,Y)Z = \kappa[&g(Y,Z)X - g(X,Z)Y\\ &+ g(JY,Z)JX - g(JX,Z)JY - 2g(JX,Y)JZ].\end{aligned} \tag{2.1}$$

Let M^n be a hypersurface of a complex space form $\hat{M}^{n+1}$ with constant holomorphic sectional curvature 4κ and N a unit normal vector field on M^n such that N=$J\xi$. We define a metric g on M^n as

$$g(X,Y) = \hat{g}(\iota X, \iota Y),$$

for any X, $Y \in TM$. The Riemannian metric g is said the induced metric from $\hat{g}$ on $\hat{M}^{n+1}(4\kappa)$ and the ι is called an isometric immersion and denotes by J the almost complex structure of the ambient manifold and by A the shape operator with respect to N=$J\xi$ of M^n. For any vector field $X \in \chi(M)$ the decomposition holds:

$$JX = \phi X + \eta(X)N. \tag{2.2}$$

The structure (ϕ, η, ξ, g) is an almost contact metric structure on M^n such that

$$\phi^2 = Id + \eta\xi, \;\; \eta(\xi) = 1, \;\; \phi\xi = 0, \;\; \eta\phi = 0. \tag{2.3}$$

and

$$\hat{g}(\phi X, \phi Y) = \hat{g}(X,Y) - \eta(X)\eta(Y), \;\; \eta(X) = \hat{g}(X, \xi). \tag{2.4}$$

A CR-submanifold is a submanifold M^n tangent to ξ that admits an invariant distribution D whose orthogonal complementary distribution $D^\perp$ is anti-invariant, that is, TM=$D \oplus D^\perp$ with condition $\phi(D_p) \subset D_p$ for all $p \in M$ and $\phi(D_p^\perp) \subset T_p^\perp M$ for all $p \in M$, where D= span$\{X_1, ..., X_m, \phi X_1, ..., \phi X_m\}$ and $D^\perp$=span(ξ) such that m=$\frac{n-1}{2}$.
The Gauss and Weingarten formula between $\hat{M}^{n+1}(4\kappa)$ and M^n are given as

$$\hat{\nabla}_X Y = \nabla_X Y + g(AX,Y)N, \;\; \hat{\nabla}_X N = -AX + D_X N, \tag{2.5}$$

for any tangent vector fields X, Y, where $\hat{\nabla}$ and ∇ denote the Levi-Civita connection of $(\hat{M}^{n+1}(4k), \hat{g})$ and (M, g), respectively. From (2.2) and $\hat{\nabla}J$=0, we obtain

$$(\nabla_X\phi)Y = \eta(Y)AX - g(AX, Y)\xi, \quad \nabla_X\xi = \phi AX. \tag{2.6}$$

Again from (2.2), we have the Gauss and Codazzi equations

$$\begin{aligned} g(R(X, Y, Z), U) = \kappa[&g(Y, Z)X - g(X, Z)Y \\ &+ g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z] \\ &+ g(AY, Z)AX - g(AX, Z)AY, \end{aligned} \tag{2.7}$$

$$(\nabla_X A)Y - (\nabla_Y X) = \kappa[\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi], \tag{2.8}$$

$$\begin{aligned} Ric(X, Y) = \kappa[&(2n + 1)g(X, Y) - 3\eta(X)\eta(Y)] \\ &+ (traceA)g(AX, Y) - g(AX, AY), \end{aligned} \tag{2.9}$$

for any tangent vector fields X,Y,Z on M, where R and Ric are the curvature and Ricci tensors of M, respectively.
As per the assumption, it follows that A=0, or AX=πX for all $X \in T(M)$, provided $\pi \neq 0$. Thus, equation (2.9), reduces for AX=πX, that

$$Ric(e_i, e_j) = (n - 1)\pi^2\delta_{i,j}, \quad (i, j = 1, \ldots, n - 1), \tag{2.10}$$

$$Ric(\xi, \xi) = (n - 1)\pi^2, \tag{2.11}$$

$$Ric(e_i, \xi) = 0, \quad (i = 1, \ldots, n - 1). \tag{2.12}$$

We recall the following theorem (see M. Djorić and M. Okumura [5]):

Theorem 2.2. *Let M^n be a hypersurface of a complex space form $\hat{M}^{n+1}(4\kappa)$. If the shape operator A for N has only one eigenvalue, then $\hat{M}^{n+1}$ is a complex Euclidean space.*

3. Main Results

Basically, we consider conformal η-Ricci solitons and η-Yamabe solitons on hypersurface M^n of a complex space form $\hat{M}^{n+1}(4\kappa)$ restricted to the shape operator A with respect to N=$J\xi$ has only one eigenvalues. The main purpose of this section is to prove the following:

Theorem 3.1. *Let M^n be a hypersurface of complex space form $\mathbb{C}^{n+1}$ with AX=πX.Then a conformal η-Ricci soliton (g, V, τ_1, τ_2) with potential field V=$\psi\xi$ is always shrinking Ricci soliton.*

Proof. Let the hypersurface M^n of a complex space form $\hat{M}^{n+1}$ admitting conformal η-Ricci soliton. Then from (1.5), we have

$$\mathfrak{L}_V g + 2Ric + [2\tau_1 - (p + \frac{2}{n})]g + 2\tau_2\eta \otimes \eta = 0, \tag{3.1}$$

We substitute

$$V = \psi\xi, \quad (\psi : M \to \Re, \psi \neq 0). \tag{3.2}$$

With the definition of Lie derivative and using (2.6), we obtain

$$(\mathfrak{L}_{\psi\xi}g)(X,Y) = d\psi(X)\eta(Y) + d\psi(Y)\eta(X). \tag{3.3}$$

From (3.3) we easily compute that

$$(\mathfrak{L}_{\psi\xi}g)(\xi,\xi) = 2d\psi(\xi), \tag{3.4}$$

$$(\mathfrak{L}_{\psi\xi}g)(\xi,e_i) = d\psi(e_i), \qquad (i = 1,2,\dots,n-1), \tag{3.5}$$

$$(\mathfrak{L}_{\psi\xi}g)(e_i,e_j) = 0, \qquad (i,j = 1,2,\dots,n-1), \tag{3.6}$$

With the help of (2.10), (2.11), (2.12), (3.4), (3.5) and (3.6), equation (3.1) equivalent to

$$d\psi(\xi) = -[\tau_1 + \tau_2 + (n-1)\pi^2 - \frac{1}{2}(p + \frac{2}{n})], \tag{3.7}$$

$$d\psi(e_i) = 0, \qquad (i = 1,2,\dots,n-1), \tag{3.8}$$

$$[(n-1)\pi^2 + \tau_1 + \frac{1}{2}(p + \frac{2}{n})]\delta_{i,j} = 0, \qquad (i,j = 1,2,\dots,n-1), \tag{3.9}$$

It is clear from (3.9), that $\tau_1 = -[(n-1)\pi^2 + \frac{1}{2}(p+\frac{2}{n})]$. So we have our result. ∎

Also, we have the corollary

Corollary 3.2. *Let M^n be a hypersurface of complex space form $\mathbb{C}^{n+1}$ with $AX=\pi X$. Then a conformal Ricci soliton (g,V,τ_1,τ_2) with potential field $V=\psi\xi$ is always shrinking Ricci soliton.*

Theorem 3.3. *Let M^n be a hypersurface of complex space form $\mathbb{C}^{n+1}$ with $A=0$. Then a conformal η-Ricci soliton (g,V,τ_1,τ_2) with potential field $V=\psi\xi$ is always expanding Ricci soliton.*

Proof. Let the hypersurface $M^n(n \geq 3)$ in a complex space form $\hat{M}^{n+1}(4\kappa)$ admitting conformal η-Ricci soliton with $A=0$. Then from (2.9), we get $Ric(X,Y)=0$, we calculate

$$d\psi(\xi) = -[\tau_1 + \tau_2 - \frac{1}{2}(p + \frac{2}{n})], \tag{3.10}$$

$$d\psi(e_i) = 0, \qquad (i = 1,2,\dots,n-1), \tag{3.11}$$

$$[2\tau_1 - (p + \frac{2}{n})]\delta_{i,j} = 0, \qquad (i,j = 1,2,\dots,n-1), \tag{3.12}$$

It is clear from (3.12), that $\tau_1 = \frac{1}{2}(p+\frac{2}{n})$. This finishes the proof of Theorem 3.2. ∎

We have the following corollary.

Corollary 3.4. *Let M^n be a hypersurface of complex space form $\mathbb{C}^{n+1}$ with $A=0$. Then a conformal Ricci soliton (g,V,τ_1,τ_2) with potential field $V=\psi\xi$ is always expanding Ricci soliton.*

Proof. It is well know that $\nabla g = 0$. Since $[2\tau_1 - (p + \frac{2}{n})]$, defined in equation (1.5), is a constant, therefore $\nabla[(2\tau_1 - (p + \frac{2}{n})]g = 0$. It means $\mathfrak{L}_V g + 2Ric$ is parallel. Therefore we conclude that $\mathfrak{L}_V g + 2Ric = \Phi$ (say) is a constant multiple of the metric tensor g, that is,

$$(\mathfrak{L}_V g + 2Ric)(X, Y) = \Phi(X, Y) = \Phi(X, Y)g(X, Y),$$

Taking

$$V = \psi\xi, \quad (\psi : M \to \Re, \psi \neq 0).$$

Using (2.10), (2.11), (2.12), (3.4), (3.5) and (3.6), we obtain $\Phi(X, Y)$ as follows

$$\Phi(\xi, \xi) = 2d\psi(\xi) + 2(n-1)\pi^2, \tag{3.13}$$

$$\Phi(\xi, e_i) = d\psi(e_i), \quad (i = 1, \dots, n-1), \tag{3.14}$$

$$\Phi(e_i, e_j) = 2(n-1)\pi^2\delta_{ij}, \;\; (i, j = 1, \dots, n-1). \tag{3.15}$$

By virtue of (3.14),(3.15) and (3.16), equation (3.1) turn up

$$d\psi(\xi) = -[(n-1)\pi^2 + \tau_1 + \tau_2 - \frac{1}{2}(p + \frac{2}{n})], \tag{3.16}$$

$$d\psi(e_i) = 0, \qquad (i = 1, 2, \dots, n-1), \tag{3.17}$$

$$[2(n-1)\pi^2 + 2\tau_1 - (p + \frac{2}{n})]\delta_{i,j} = 0, \qquad (i, j = 1, 2, \dots, n-1), \tag{3.18}$$

It is clear from (3.18), that $\tau_1 = [\frac{1}{2}(p + \frac{2}{n}) - (n-1)\pi^2]$. Thus we are in a condition to write the following. ∎

Theorem 3.5. *If the symmetric tensor $\mathfrak{L}_{\psi\xi} g + 2Ric=\Phi$ is parallel on the hypersurface of complex space form $\mathbb{C}^{n+1}$ with with $AX=\pi X$.Then a conformal η-Ricci soliton (g, V, τ_1, τ_2) with potential field $V=\psi\xi$ to be*
(i) expanding if $\frac{1}{2}\left(p + \frac{2}{n}\right) > (n-1)\pi^2$,
(ii) shrinking if $\frac{1}{2}\left(p + \frac{2}{n}\right) < (n-1)\pi^2$,
(iii) steady if $\frac{1}{2}\left(p + \frac{2}{n}\right) = (n-1)\pi^2$.

In view of Theorem 3.3. We have the following corollary

Corollary 3.6. *If the symmetric tensor $\mathfrak{L}_{\psi\xi} g + 2Ric=\Phi$ is parallel on the hypersurface of complex space form $\mathbb{C}^{n+1}$ with with $AX=\pi X$.Then a conformal Ricci soliton (M, g, V, τ_1) with potential field $V=\psi\xi$ to be*
(i) expanding if $\frac{1}{2}\left(p + \frac{2}{n}\right) > (n-1)\pi^2$,
(ii) shrinking if $\frac{1}{2}\left(p + \frac{2}{n}\right) < (n-1)\pi^2$,
(iii) steady if $\frac{1}{2}\left(p + \frac{2}{n}\right) = (n-1)\pi^2$.

Again, we consider A=0, then from (2.9), we have $Ric(X, Y)$=0. Let the hypersurface $M^n (n \geq 3)$ in a complex space form $\hat{M}^{n+1}(4\kappa)$ admitting conformal η-Ricci soliton, then we can find $\Phi(X, Y)$, i.e.,

$$\Phi(\xi, \xi) = 2d\psi(\xi), \tag{3.19}$$

$$\Phi(\xi, e_i) = d\psi(e_i), \quad (i = 1, \ldots, n-1), \tag{3.20}$$

$$\Phi(e_i, e_j) = 0, \qquad (i, j = 1, \ldots, n-1). \tag{3.21}$$

Using above these fact the equation (3.1), reduces to

$$d\psi(\xi) = -[\tau_1 + \tau_2 - \frac{1}{2}(p + \frac{2}{n})], \tag{3.22}$$

$$d\psi(e_i) = 0, \qquad (i = 1, 2, \ldots, n-1), \tag{3.23}$$

$$[2\tau_1 - (p + \frac{2}{n})]\delta_{i,j} = 0, \qquad (i, j = 1, 2, \ldots, n-1), \tag{3.24}$$

It is clear from (3.24), that τ_1=$[\frac{1}{2}(p + \frac{2}{n})]$. Thus we are in a condition to write the following

Theorem 3.7. *If the symmetric tensor $\mathfrak{L}_{\psi\xi}g + 2Ric$=$\Phi$ is parallel on the hypersurface of complex space form $\mathbb{C}^{n+1}$ with with A=0.Then a conformal η-Ricci soliton (g, V, τ_1, τ_2) with potential field $V = \psi\xi$ to be always expanding.*

Corollary 3.8. *If the symmetric tensor $\mathfrak{L}_{\psi\xi}g + 2Ric$=$\Phi$ is parallel on the hypersurface of complex space form $\mathbb{C}^{n+1}$ with with A=0.Then a conformal Ricci soliton (g, V, τ_1) with potential field $V = \psi\xi$ to be always expanding.*

At last, we mention the following result

Theorem 3.9. *If the hypersurface of complex space form $\mathbb{C}^{n+1}$ with AX=πX admitting η-Yamabe soliton (g, V, τ_1, τ_2) with potential field $V = \psi\xi$ then the scalar curvature is constant.*

Proof. With the help of (3.4),(3.5) and (3.6), the equation (1.7) transform that

$$d\psi(\xi) = \sigma - (\tau_1 + \tau_2), \tag{3.25}$$

$$d\psi(e_i) = 0, \qquad (i = 1, 2, \ldots, n-1), \tag{3.26}$$

$$(\sigma - \tau_1)\delta_{i,j} = 0, \qquad (i, j = 1, 2, \ldots, n-1), \tag{3.27}$$

It is clear from (3.27), that σ=τ_1. Now as τ_1 are constant, so σ is also constant. Therefore proof is completed. ∎

Also, we have the following corollary

Theorem 3.10. *If the hypersurface of complex space form $\mathbb{C}^{n+1}$ with AX=πX admitting Yamabe soliton (g, V, τ_1) with potential field V=$\psi\xi$ then V is a Killing vector field.*

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