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EQUIVALENCE OF HENSTOCK AND CERTAIN SEQUENTIAL HENSTOCK INTEGRALS

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Abstract In this paper, we define and show the equivalence between certain class of Henstock integrals and Sequential Henstock integrals for real valued functions. In order to get some applications of our results, we also provide a specific example.

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1. Introduction

The symbols used in this paper are as follows: $\mathbb{R}$ and $\mathbb{N}$ for a set of real and natural numbers respectively, X as a topological space, which is a subset of the real line $\mathbb{R}$, $\{\delta_n(x)\}_{n=1}^{\infty}$ as set of gauge functions of $x \in [a,b]$, P_n as set of partitions of subintervals of a compact interval $[a,b]$ for $n = 1, 2, 3, ...$ and $\ll$ as much more smaller.

Firstly, we recall a few basic definitions and results concerning Sequential Henstock integral. see[2],[3] and [7].

Definition 1.1. [2](Darboux Henstock integral).
A function $f : [a,b] \to \mathbb{R}$ is said to be Darboux Henstock integrable on $[a,b]$ to $\alpha \in \mathbb{R}$, if for any $\varepsilon > 0$ there exists a function $\delta(x) > 0$ such that for every $\delta(x) - fine$ tagged partitions $P = \{(u_{i-1}, u_i), t_i\}$ where $[u_{i-1}, u_i] \in [a,b]$ we have $|U(f,P) - L(f,P)| < \varepsilon$. In this case $U(f,P) = L(f,P) = \alpha$, where the upper sum $U(f,P) = sup \sum_{i=1}^{n} f(t_i)(u_i -$

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$u_{i-1})$, the lower sum $L(f,P) = inf \sum_{i=1}^{n} f(t_i)(u_i - u_{i-1})$ and $[u_{i-1}, u_i] \in [a,b]$.
Hence, $(DH)\int_a^b f = \alpha$.

Remark 1.1. Darboux Henstock integral is a special kind of integral in the family of Henstock that applies the concept of upper and lower integrals to solve most integral problems of a function defined on a compact interval. The condition for the existence of this integral depends on the equivalence of the lower and upper integrals.(see [2] and [3])

These following concepts are well known to the case of functions defined in a Topological space.

Let X be a locally compact Hausdorff space with subspace $\Omega \subset X$. We denote the closure of Ω as $\bar{\Omega}$ and the interior as $Int\Omega$. Let $\triangle$ be a family of subsets of X such that

(i) If $\Omega \in \triangle$, then $\bar{\Omega}$ is compact.
(ii) For each $x \in X$, the collection $\triangle(x) = \{\Omega \in \triangle | x \in Int\Omega\}$ is a neighbourhood base at x.
(iii) If $\Omega, \omega \in \triangle$, then $\Omega \cap \omega \in \triangle$ and there exist disjoint sets $C_1, ..., C_k \in \triangle$ such that $\Omega - \omega = \bigcup_{i=1}^{k} C_i$.

A gauge (topological) on $\Omega \in \triangle$ is a map U assigning to each $x \in \bar{\Omega}$ a neighbourhood $U(x)$ of x contained in X.

A division (topological) on $\Omega \in \triangle$ is a disjoint collection $\{\Omega_1, ..., \Omega_k\} \subset \triangle$ such that $\bigcup_{i=1}^{k} \Omega_i = \Omega$.

A partition (topological) on Ω is a set $P = \{(\Omega_1, t_1), ..., (\Omega_k, t_k)\}$ such that $\{\Omega_1, ..., \Omega_k\}$ is a division of Ω and $\{x_1, ..., x_k\} \subset \bar{\Omega}$. If U is a gauge on Ω, we say the partition P is U-fine if $\Omega_i \subset U(x_i)$, for $i = 1, 2, ..., k$.

A volume is a non-negative function such that $V(U) = \sum_{i=1}^{k} v(U_i)$ for every $\Omega \in \triangle$ and each division $\{\Omega_1, ..., \Omega_k\}$of Ω.
Note: Volume here can intuitively be defined to represent the "length" of the "interval".

Definition 1.2. [3] (Topological Henstock integral).
Let X be a locally compact Hausdorff space and let $\Omega \in \triangle$ with $f : \bar{\Omega} \to \mathbb{R}$, then f is Topological Henstock integrable to $\alpha \in \mathbb{R}$ if for any $\varepsilon > 0$ there exists a neighbourhood $U(x) > 0$ such that $|\sum_{i=1}^{n} f(t_i)v(U_i) - \int_\Omega f| = |\sigma(f,P) - \int_\Omega f| < \varepsilon$, for every $U(x)-$ fine partition P of Ω, where $\int_\Omega f = \alpha$

This Henstock integral uses the concept of neighbourhood system of a Topological space to define the integral value of the Topological space valued functions.

Definition 1.3. [7] (Henstock integral).
A real valued function $f : [a,b] \to \mathbb{R}$ is Henstock integrable to $\alpha \in \mathbb{R}$ on $[a,b]$ if for any $\varepsilon > 0$ there exists a function $\delta(t) > 0$ such that for every $\delta(x) - fine$ division $P = \{(u_{i-1}, u_i), t_i\}$ we have $|U(f,P) - \alpha| < \varepsilon$.
where $U(f,P) = \sum_{i=1}^{n} f(t_i)(u_i - u_{i-1})$, $(H)\int_a^b f(t)d(t) = \alpha$ and $[u_{i-1}, u_i] \in [a,b]$ for $u_{i-1} \le t_i \le u_i$.

Definition 1.4. [7] (Sequential Henstock Integral).
A function $f : [a,b] \to \mathbb{R}$ is Sequential Henstock integrable on $[a,b]$ to $\alpha \in \mathbb{R}$ if for any $\varepsilon > 0$ there exists a sequence of gauge functions $\delta_\mu \in \{\delta_n(x)\}_{n=1}^{\infty}$ such that for every

$\delta_n(x) - fine$ tagged partitions $P_n = \{(u_{(i-1)_n}, u_{i_n}), t_{i_n}\}$, we have $|U(f, P_n) - \alpha| < \varepsilon$, where $U(f, P_n) = \sum_{i=1}^{n\in\mathbb{N}} f(t_{i_n})(u_{i_n} - u_{(i-1)_n}) \to \alpha$ as $n \to \infty$ i.e $\alpha = \int_{[a,b]} f$, $[u_{(i-1)_n}, u_{i_n}] \in [a, b]$ for $u_{(i-1)_n} \leq t_{i_n} \leq u_{i_n}$ and $n = 1, 2, 3, ...$

The Sequential Henstock integral is a theory developed by Laramie [7]. It is a sequential approach of defining and proving theorems on Henstock integral, in which both had been shown to be equivalent. Moreso, it has the potential of expanding the overall theory of Henstock integration into more abstract mathematical settings which in turn may lead to further applications of the Henstock integral.

In the last one decade, several papers have been published on the equivalence of certain classes of integrals involving Henstock and Henstock-type, see [1]-[10] for a recent survey. These papers were motivated by certain conditions which are generalisation of the Riemann integrals. However, there exists a large class of integrals, as for example, that of the Sequential Henstock integral introduced in Laramie[7] for which its equivalence to the family of Henstock integrals is yet to be shown. In such situation, it is of theoretical and practical importance to study if possible, if these classes of integrals are equivalent.

The following results were proved by Laramie [7]. Ying [10] and Chartfield [2].

(R_1). The Henstock integral is equivalent to the Sequential Henstock integral. i.e. $H_f[a, b] = SH_f[a, b]$

(R_2). The Topological Henstock integral is equivalent to the Sequential Henstock integral on $[a, b] \subset \mathbb{R}$. i.e. $TH_f[a, b] = SH_f[a, b]$

(R_3). The Darboux Henstock integral is equivalent to the Sequential Henstock integral on $[a, b] \subset \mathbb{R}$. i.e. $DH_f[a, b] = SH_f[a, b]$.

Remark 1.2 Clearly, $H_f[a, b] = TH_f[s, b] = DH_f[a, b] = SH_f[a, b]$.

As far as we know, there are only a few articles devoted to these very important theoretical problems, see[4], [5]. It is the main purpose of this paper to establish the equivalence of the class of Henstock integral to a new class of Sequential Henstock integrals. This new class is of significant importance to the family of Sequential Henstock integral introduced in Laramie [7].

2. Main Results

Firstly, we consider the following new concepts which are analogous of definitions 1.1 and 1.2.

Definition 2.1. (Sequential Darboux Henstock Integral).
A function $f : [a, b] \to \mathbb{R}$ is Sequential Darboux Henstock ($SDH_f[a, b]$) integrable on $[a, b]$ to $\alpha \in \mathbb{R}$ if for any $\varepsilon > 0$, there exists a sequence of gauge functions $\delta_\mu \in \{\delta_n(x)\}_{n=1}^{\infty}$ for $\mu \leq n \in \mathbb{N}$ such that for every $\delta_n(x)$-fine tagged partitions $P_n = \{(u_{(i-1)_n}, u_{i_n}), t_{i_n}\}$, we have $|U(f, P_n) - L(f, P_n)| < \varepsilon$,
where $U(f, P_n) = sup \sum_{i=1}^{n\in\mathbb{N}} f(t_{i_n})(u_{i_n} - u_{(i-1)_n})$, $L(f, P_n) = inf \sum_{i=1}^{n\in\mathbb{N}} f(t_{i_n})(u_{i_n} - u_{(i-1)_n})$. i.e $\overline{\int}_{[a,b]} f = \alpha = \underline{\int}_{[a,b]} f$, $[u_{(i-1)_n}, u_{i_n}] \in [a, b]$ for $u_{(i-1)_n} \leq t_{i_n} \leq u_{i_n}$ and $n = 1, 2, 3, ...$

Definition 2.2. (Sequential Topological Henstock integral).
Let X be a locally compact Hausdorff space and let $\Omega \in \triangle$ with $f : \bar{\Omega} \to \mathbb{R}$, then f

is Sequential Topological Henstock integrable to $\alpha \in \mathbb{R}$ if for any $\varepsilon > 0$, there exists a sequence of neighbourhood $U_\mu \in \{U_n(x)\}_{n=1}^{\infty}$ for $n \le \mu$ such that

$$|\sum_{i=1}^{n} f(t_{i_n})v(U_n) - \int_\Omega f| = |\sigma(f, P_n) - \int_\Omega f| < \varepsilon.$$

For every $U_n(x) - fine$ partition P_n of X.

Theorem 2.3. *If $f : [a, b] \to \mathbb{R}$ is Sequential Darboux Henstock integrable, then it is Sequential Henstock integrable there. In fact, $(SDH)\int_a^b f = (SH)\int_a^b f$.*

Proof. Suppose $f \in SDH_f[a, b]$, we want to show that $f \in SH_f[a, b]$.
Let $\varepsilon > 0$, there exists a $\delta_\mu \in \{\delta_n(x)\}_{n=1}^{\infty}$ for $\mu \le n$ such that for all $\delta_n(x) - fine$ partitions P_n of $[a, b]$, we have

$$|\overline{\int}_{u_{(i-1)_n}}^{u_{i_n}} f - \underline{\int}_{u_{(i-1)_n}}^{u_{i_n}} f| < \varepsilon,$$

so that $\overline{\int}_{u_{(i-1)_n}}^{u_{i_n}} f = \underline{\int}_{u_{(i-1)_n}}^{u_{i_n}} f = \int_{u_{(i-1)_n}}^{u_{i_n}} f$. Since $\underline{\int}_{u_{(i-1)_n}}^{u_{i_n}} f$ is the greatest lower bound of the upper sums $U(f, P_n)$, there exists $\delta_{\mu_1} \in \{\delta_{n_1}(x)\}_{n=1}^{\infty}$ such that for all $\delta_{\mu_1}(x) - fine$ partitions P_{n_1} of $[a, b]$, we have the upper Riemann integral

$$U(f, P_n) = \underline{\int}_{u_{(i-1)_n}}^{u_{i_n}} f + \frac{\varepsilon}{2}.$$

Similarly, Since, $\overline{\int}_{u_{(i-1)_n}}^{u_{i_n}} f$ is the least upper bound of the lower sums $L(f, P_n)$, there exists $\delta_{\mu_2} \in \{\delta_{n_2}(x)\}_{n=1}^{\infty}$ such that for all $\delta_{\mu_2}(x) - fine$ partitions P_{n_2} of $[a, b]$, we have the lower Riemann integral

$$L(f, P_n) = \overline{\int}_{u_{(i-1)_n}}^{u_{i_n}} f - \frac{\varepsilon}{2}.$$

We choose $\delta_\mu(x) = min\{\delta_{\mu_1}(x), \delta_{\mu_2}(x)\}$ such that, for $\varepsilon > 0$, there exists a $\delta_\mu(x)$ such that $P_n \ll \delta_\mu(x)$, we have

$$|U(f, P_n) - L(f, P_n)| < \frac{\varepsilon}{2}. \tag{2.1}$$

That is,

$$\begin{aligned}
|U(f, P_n) - L(f, P_n)| &= |U(f, P_n) - \int_{u_{(i-1)_n}}^{u_{i_n}} f + \int_{u_{(i-1)_n}}^{u_{i_n}} f - L(f, P_n)| \\
&\le |U(f, P_n) - \int_{u_{(i-1)_n}}^{u_{i_n}} f| + |L(f, P_n) - \int_{u_{(i-1)_n}}^{u_{i_n}} f| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

For the same, $P \ll \delta_\mu(x)$, we have

$$L(f, P_n) = S(f, P_n) = U(f, P_n), \tag{2.2}$$

where $S(f, P_n)$ is the Riemann sum over P_n.
By hypothesis, we have

$$L(f, P_n) \leq \sup L(f, P_n) \leq \underline{\int}_{u_{(i-1)_n}}^{u_{i_n}} f = \int_{u_{(i-1)_n}}^{u_{i_n}} f = \overline{\int}_{u_{(i-1)_n}}^{u_{i_n}} f = \inf U(f, P_n) \leq U(f, P_n),$$

which implies

$$L(f, P_n) \leq \int_{u_{(i-1)_n}}^{u_{i_n}} f \leq U(f, P_n) \Rightarrow -U(f, P_n) \leq -\int_{u_{(i-1)_n}}^{u_{i_n}} f \leq -L(f, P_n). \quad (2.3)$$

Combining equations (2.2) and (2.3) and taking

$$U(f, P_n) \geq L(f, P_n) + \frac{\varepsilon}{2}$$

we have

$$|S(f, P_n) - \int_{u_{(i-1)_n}}^{u_{i_n}} f| \leq |U(f, P_n) - L(f, P_n)| \quad (2.4)$$

Hence, for $\varepsilon > 0$, there exists a $\delta_\mu \in \{\delta_n(x)\}_{n=1}^{\infty}$ for $\mu \leq n$ such that for all $\delta_n(x) - fine$ partitions P_n of $[a, b]$, we have

$$|\overline{\int}_{u_{(i-1)_n}}^{u_{i_n}} f - \underline{\int}_{u_{(i-1)_n}}^{u_{i_n}} f| < \varepsilon,$$

by equations (2.1) and (2.3). Hence, $SDH_f[a, b] = SH_f[a, b]$.

Theorem 2.4. *If $f : [a, b] \to \mathbb{R}$ is Sequential Henstock integrable, then it is Sequential Darboux Henstock integrable there. In fact, $(SH) \int_a^b f = (SDH) \int_a^b f$.*

Proof Suppose $f \in SH_f[a, b]$, we want to show that $f \in SDH_f[a, b]$.
Let $\varepsilon > 0$, there exists a $\delta_\mu \in \{\delta_n(x)\}_{n=1}^{\infty}$ for $\mu \leq n$ and $P_n \ll \delta_n(x)$, then

$$|U(f, P_n) - \underline{\int}_{u_{(i-1)_n}}^{u_{i_n}} f| < \varepsilon.$$

We say that the sequence $U(f, P_n) \to \int_{u_{(i-1)_n}}^{u_{i_n}} f$ as $n \to \infty$. Therefore, the upper and lower bounds of the sequence must also converge to $\int_a^b f$ and we have
$L(f, P_n) \to \overline{\int}_{u_{(i-1)_n}}^{u_{i_n}} f$ and $U(f, P_n) \to \underline{\int}_{u_{(i-1)_n}}^{u_{i_n}} f$ as $n \to \infty$. By order of relationship among the lower and upper bounds and lower and upper integrals of f as $n \to \infty$, we have

$$\begin{aligned}
\int_a^b f = L(f, P_n) \leq supL(f, P_n) &= \underline{\int}_{u_{(i-1)_n}}^{u_{i_n}} f \leq \overline{\int}_{u_{(i-1)_n}}^{u_{i_n}} f \\
&= infU(f, P_n) \leq U(f, P_n) \\
&= \int_{u_{(i-1)_n}}^{u_{i_n}} f.
\end{aligned}$$

Then, we have $\underline{\int}_{u_{(i-1)_n}}^{u_{i_n}} f \to \int_{u_{(i-1)_n}}^{u_{i_n}} f$ and $\overline{\int}_{u_{(i-1)_n}}^{u_{i_n}} f \to \int_{u_{(i-1)_n}}^{u_{i_n}} f$ as $n \to \infty$ over the set of all $\delta_\mu(x)'s$ and partitions P_n on $[a, b]$. Now, Let $\varepsilon > 0$, there exists a $\delta_\mu \in \{\delta_n(x)\}_{n=1}^{\infty}$

for $\mu \leq n$ such that for all $\delta_n(x) - fine$ partitions P_n of $[a, b]$, we have

$$|\underline{\int}_{u_{(i-1)_n}}^{u_{i_n}} f - \int_{u_{(i-1)_n}}^{u_{i_n}} f| < \frac{\varepsilon}{2}$$

and

$$|\overline{\int}_{u_{(i-1)_n}}^{u_{i_n}} f - \int_{u_{(i-1)_n}}^{u_{i_n}} f| < \frac{\varepsilon}{2}.$$

This yields

$$\begin{aligned} |\overline{\int}_{u_{(i-1)_n}}^{u_{i_n}} f - \underline{\int}_{u_{(i-1)_n}}^{u_{i_n}} f| &= |\overline{\int}_{u_{(i-1)_n}}^{u_{i_n}} f - \int_{u_{(i-1)_n}}^{u_{i_n}} f + \int_{u_{(i-1)_n}}^{u_{i_n}} f - \underline{\int}_{u_{(i-1)_n}}^{u_{i_n}} f| \\ &\leq |\overline{\int}_{u_{(i-1)_n}}^{u_{i_n}} f - \int_{u_{(i-1)_n}}^{u_{i_n}} f| + |\int_{u_{(i-1)_n}}^{u_{i_n}} f - \underline{\int}_{u_{(i-1)_n}}^{u_{i_n}} f| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

i.e. $SH_f[a, b] = SDH_f[a, b]$.

Corollary 2.5. *A function $f : [a, b] \to \mathbb{R}$ is Sequential Darboux Henstock integrable in $[a, b]$, if and only if it is Sequential Henstock integrable there.*

Proof. It follows easily from Theorems 2.1 and 2.2. This completes the proof.

Interestingly, the techniques applied in obtaining these results have been used in the analysis of applied dynamics fields, see [11] and [12].

Theorem 2.6. *Let X be a locally compact Hausdorff space. The Topological Sequential Henstock integral is equivalent to Sequential Henstock integral on $I = [a, b] \subset \mathbb{R}$. In fact, $(SH)\int_a^b f = (STH)\int_\Omega f$.*

Since X is a locally compact Hausdorff space, then by Heine - Borel's theorem, each $[u_{i-1}, u_i] \subset [u_{(i-1)_n}, u_{i_n}] \subset [a, b] \subset \mathbb{R}$ is compact. Hence, any point $x \in \mathbb{R}$ is contained in the open interval (a,b), which in turn is contained in the compact subspace $[u_{(i-1)_n}, u_{i_n}] \subset X$, so that $\mathbb{R}$ is made into a locally compact Hausdorff space. Therefore $\triangle = \{[u_{(i-1)_n}, u_{i_n}] \subset [a, b] : a, b \in \mathbb{R}, a < b\}$ is proved. This shows that under this condition, the Sequential Topological Henstock integral reduces to Sequential Henstock integral. Thus, $(SH)\int_a^b f = (STH)\int_\Omega f$.

Proof. For $\Omega \in \triangle$, let $v(\Omega_{i_n}) = u_{i_n} - u_{(i-1)_n}$ and define a sequence of positive gauges $U_{\delta_n}(t_{i_n})$ on Ω such that $U_{\delta_n}(t_{i_n}) = (t_{i_n} - \delta_n(t_{i_n}), t_{i_n} + \delta_n(t_{i_n})), (i = 1, 2, ...k)$ is a U_n- fine partitions P_n on Ω with $\delta_\mu \in \{\delta_n(x)\}_{n=1}^\infty$ for $\mu \leq n$ for each $x \in [a, b]$. Thus, in this case, we can say $P_n = \{(u_{(i-1)_n}, u_{i_n}), t_{i_n}\}$ where $u_{(i-1)_n} \leq t_{i_n} \leq u_{i_n}$ is $\delta_n(x) - fine$. Since, each subinterval $[u_{(i-1)_n}, u_{i_n}] = (t_{i_n} - \delta_n(t_{i_n}), t_{i_n} + \delta_n(t_{i_n})), i = 1, 2, ...k$. Therefore, for closed intervals $[a, b] \subset \mathbb{R}$ and by Definition 2.2, for any $\varepsilon > 0$, there exists a $\delta_\mu \in \{\delta_n(x)\}_{n=1}^\infty$ for $\mu \leq n$ for each $x \in [a, b]$ such that for all $\delta_n(x) - fine$ tagged partitions $P_n = \{(u_{(i-1)_n}, u_{i_n}), t_{i_n}\}$, we have

$$|\sum_{i=1}^{k} f(t_{i_n})(u_{i_n} - u_{(i-1)_n}) - \int_{u_{(i-1)_n}}^{u_{i_n}} f| = |U(f, P_n) - \int_{u_{(i-1)_n}}^{u_{i_n}} f| < \varepsilon.$$

Hence, the Topological Sequential Henstock integral is equivalent to the Sequential Henstock integral on $X \subset \mathbb{R}$ i.e. $(STH)\int_{\Omega} f = (SH)\int_a^b f$, when the sets in $\triangle$ are $\Omega = [a,b] \subset \mathbb{R}$. This completes the proof.

Example 2.7
Consider the Dirichlet's function defined on $I = [0,1]$ by

$$f(x) = \begin{cases} 1, & \text{if } x \in [0,1], x \in Q \\ 0, & \text{if } x \in [0,1], x \notin Q \end{cases}$$

Suppose we define our gauge

$$\delta_n(x) = \begin{cases} \frac{1}{n2^n}, & \text{if } x \in [0,1] \\ 1, & \text{if } x \notin [0,1], \end{cases}$$

So, we have our

$$\begin{aligned} U(f,P_n) &= \sum_{i\in\Pi\cup\Pi'} f(t_{i_n})(u_{i_n} - u_{(i-1)_n}) \\ &= \sum_{i\in\Pi} 1.\frac{1}{i2^i} + \sum_{i\in\Pi'} 0.1 \\ &= \sum_{i=1}^{n} \frac{1}{i2^i}, \end{aligned}$$

if we evaluate numerically with Python programming, we get the following numerical values.

n	$U(f,P_n)$
1	0.5
2	0.125
3	0.41667
8	0.0004882
20	0.00000004768
30	0.00000000003130
75	0.0000000000000000003529
100	0.00000000000000000000078886
.	.
.	.
.	.
∞	0

Table 1

Clearly, f is Sequential Henstock integral on $[0,1]$ and that $\int_0^1 f(x)dx = 0$ as shown by table 2.8. So by Theorems 2.2 and 2.4. it follows that $f(x)$ is also Sequential Darboux Henstock integrable and Sequential Topological Henstock integrable on $[0,1]$.

3. Conclusion and Suggestion for Further Study

In this paper, we studied the equivalence of Henstock integral to certain family of Sequential Henstock integral by introducing new concepts like Sequential Darboux Henstock and Sequential Topological Henstock integrals and proving the theorems on their equivalence to the Sequential Henstock integral. The results obtained show that there is equivalence between these family of Henstock integrals. To this end, an example to show the applicability of the result is also given.

Up until this research work, however, the theory of equivalence of family of Henstock integral did not include definitions and theorems based on sequence and it is in our viewpoint that the Sequential Henstock integral can be used to renew the interest of integration theorists and researchers on equivalence of Henstock integral. In line with this, the results of this research can now be extended to studies in more abstract spaces and applications arising from this as well as to the conclusion of Sequential Henstock integral in introductory Calculus can be assessed for possible pedagogical benefits. In conclusion, can equivalence of Henstock and Certain Sequential Henstock Integrals hold for classes of functions, such as step functions, measuration functions, absolutely integrable functions? It is of the view of the authors that these problems could be considered for further research.

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