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ON A FIXED POINT RESULT FOR φ-CONTRACTIONS IN b-METRIC SPACES

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Abstract In this paper, we provide a new proof to a fixed point result for φ-contractions on b-metric spaces published by S. Czerwik in his paper: [*Contraction mappings in b-metric spaces*, Acta Math. Inform. Univ. Ostrav. **1** (1993), 5-11]. We point out that our proof is completely different from the one given recently by S. Kajanto and A. Lukacs in [*A note on the paper "Contraction mappings in b-metric spaces" by Czerwik*, Acta Univ. Sapientiae, Mathematica, **10**, 1 (2018) , 85-89].

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1. Introduction

In fixed point theory, one can find many extensions of the notions of metric and metric space along with many generalizations of the Banach contraction principles in their setting. See for instance the books [16], [28], [34], or the survey papers [8], [7], [26] and the references therein.

We have gathered a list of references at the end of the paper where the reader could find other complements on the topics of b-metric spaces and other generalizations.

One of the ineteresting extensions of metric spaces is given by the notion of b-metric spaces (see for instance the nice and recent survey [8]).

A *b-metric* on a nonempty set X is a function $d : X \times X \to \mathbb{R}_+ := [0, \infty)$ satisfying the conditions

$$\begin{aligned} &\text{(i)} \quad d(x,y) = 0 \iff x = y; \\ &\text{(ii)} \quad d(x,y) = d(y,x); \\ &\text{(iii)} \quad d(x,y) \le s[d(x,z) + d(z,y)], \end{aligned} \tag{1.1}$$

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for all $x, y, z \in X$, and for some fixed number $s \geq 1$. The triple $(X, d; s)$ is called a *b-metric space* with parameter s.
Obviously, for $s = 1$ one obtains a metric on X.
The inequality (iii) is called the *s-relaxed triangle inequality*,

According to [8], the relaxed triangle inequality was introduced by Coifman and de Guzman [9] in their studies of some problems in harmonic analysis, where a b-metric was called a "distance" function. This work was continued in 1979 by Macías and Segovia [29, 30].

In 1989, Bakhtin [5] called them "quasi-metric spaces" and established a contraction principle for such spaces.

In 1993, Czerwik introduced them under the name "b-metric space", first for $s = 2$ in [10], and then for an arbitrary $s \geq 1$ in [11].

S. Czerwik [10] established the following fixed point result for b-metric spaces.

Theorem 1.1 ([10]). *Let $(X, d; s)$ be a complete b-metric space with parameter $s \geq 1$, and let T be a selfmap of X, satisfying*

$$d(T(x), T(y)) \leq \varphi(d(x, y)), \quad \forall x, y \in X, \tag{1.2}$$

where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a function satisfying:
(a) φ is non-decreasing, and
(b) $\lim_{n\to\infty} \varphi^n(t) = 0$, for all $t \in \mathbb{R}_+$.
Then T has a unique fixed point $z \in X$ such that the sequence of iterates $\left(T^n(x)\right)_{n\in\mathbb{N}_0}$ converges to z for all $x \in X$ as $n \to \infty$.

S. Cobzaş observed, in [8], that the proofs given in [10] and in [28] for Theorem 1.1 were not satisfactory. So, he established the following result.

Theorem 1.2. *[[8]] Let $(X, d; s)$ be a complete b-metric space with parameter $s \geq 1$ and let $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ be a function satisfying the conditions*

$$\begin{aligned} &\text{(a)} \quad \varphi \ \text{ is nondecreasing,} \\ &\text{(b)} \quad \lim_{n\to\infty} \varphi^n(t) = 0, \ \text{and} \\ &\text{(c)} \quad \varphi(t) < \frac{t}{s}, \end{aligned} \tag{1.3}$$

for all $t > 0$.
Then every mapping $T : X \to X$ satisfying the inequality

$$d(T(x), T(y)) \leq \varphi(d(x, y)), \tag{1.4}$$

for all $x, y \in X$, has a unique fixed point z and, for every $x \in X$, the sequence $\left(T^n(x)\right)_{n\in\mathbb{N}_0}$ converges to z as $n \to \infty$.

Theorem 1.2 was established in [8] by a proof adapting the arguments from [21]. In [8], the following question was addressed: What kind of conditions have to be added to the assumptions of Theorem 1.1, in order to render it valid ?

In this paper, we give an answer to this question. In fact, we shall give a proof for Theorem 1.1 without addition of any other condition. As a consequence, our proof will show that the condition (c) in (1.3) is superfluous and can be dropped.

In 2018, S. Kajántó and A. Lukács [24] proposed a new proof of this theorem. As, it will be seen later, the proof we give here is completely different from the proof of [24] for Theorem 1.1.

We point out that Theorem 1.1 is an extension, to b-metric spaces, of a well known result established in [21] for metric spaces. The analogous of this theorem for a class of semimetric spaces was established in [22].

As it was said above, the aim of this note is to give a new proof of Theorem 1.1.

We point out that the proof given here for Theorem 1.1 is completely different from the one given in [24].

In the second section, we recall some important facts concerning b-metric spaces. The proof of Theorem 1.1 is done in the third section. In the fourth section, we give some consequences and applications.

2. Recalls on b-metric spaces

In all this section, $(X, d; s)$ will be a b-metric space with parameter $s \geq 1$.

(1) By induction and using the inequality (iii), we obtain so called s-relaxed triangle inequality of order $n \geq 2$:

$$d(x_0, x_n) \leq sd(x_0, x_1) + s^2 d(x_1, x_2) + \cdots + s^n d(x_{n-1}, x_n)\,, \tag{2.1}$$

for all $n \in \mathbb{N}$ with $n \geq 2$ and all $x_0, x_1, \ldots, x_n \in X$.

By definition, the "open" ball $B(x, r)$ of center $x \in X$ and radius $r > 0$ is given by

$$B(x, r) = \{y \in X : d(x, y) < r\}\,.$$

(2) A subset Y of X is called open if for every $x \in Y$ there exists a number $r_x > 0$ such that $B(x, r_x) \subseteq Y$. Denoting by $\mathcal{T}_d$ (or $\mathcal{T}(d)$) the family of all open subsets of X it follows that τ_d satisfies the axioms of a topology.

(3) The b-metric d is called *continuous* if, for all sequences $(x_n), (y_n)$ in X and all $x, y \in X$, we have

$$d(x_n, x) \to 0 \text{ and } d(y_n, y) \to 0 \implies d(x_n, y_n) \to d(x, y)\,. \tag{2.2}$$

(4) The b-metric d is called *separately continuous* if the function $d(x, \cdot)$ is continuous on X for every $x \in X$, i.e.,

$$d(y_n, y) \to 0 \implies d(x, y_n) \to d(x, y)\,, \tag{2.3}$$

for all sequences $(x_n), (y_n)$ in X and all $x, y \in X$.

(5) The metrizability of $\mathcal{T}(d)$ was obtained in [1] and [33] by a slight modification of Frink's technique [20].

For $0 < p \leq 1$ define

$$\rho_p(x, y) := \inf \left\{ \sum_{k=0}^{n} d(x_{i-1}, x_i)^p \right\}, \tag{2.4}$$

where the infimum is taken over all $n \in \mathbb{N}$ and all chains $x = x_0, x_1, \ldots, x_n = y$ of elements in X.

The function ρ_p defined by (2.4) satisfies the conditions

(1) $\rho_p(x, y) = \rho_p(y, x)$,
(2) $\rho_p(x, y) \leq \rho_p(x, z) + \rho_p(z, y)$,
(3) $d^p(x, y) \geq \rho_p(x, y)$, for all $x, y \in X$.

The following is a part of general result established in [33].

Theorem 2.1 ([33]). *Let d be a b-metric on a nonempty set X satisfying the s-relaxed triangle inequality (1.1).(iii), for some $s \geq 1$. If the number $p \in (0,1]$ is given by the equation $(2s)^p = 2$, then the mapping $\rho_p : X \times X \to [0,\infty)$ defined by (2.4) is a metric on X satisfying the inequalities*

$$\rho_p(x,y) \leq d^p(x,y) \leq 2\rho_p(x,y)\,, \tag{2.5}$$

for all $x, y \in X$.

(6) Under the hypotheses of Theorem 2.1, by virtue of the inequalities (2.5), we have the following consequences:

(a) $\mathcal{T}_d = \mathcal{T}\rho_p$, that is, the topology of any b-metric space is metrizable, and the convergence of sequences with respect to $\mathcal{T}_d$ is characterized in the following way:

$$x_n \xrightarrow{\mathcal{T}_d} x \iff d(x,x_n) \longrightarrow 0\,,$$

for any sequence (x_n) in X and $x \in X$.

(b) $B(x,r) \in \mathcal{T}_d$ for every $r > 0 \iff d(x,\cdot)$ is upper semicontinuous on X.

Thus, the balls $B(x,r)$ need not be in $\mathcal{T}_d$ and the b-metric d could not be continuous on $X \times X$.

An example of a b-metric space where the balls are not necessarily open is given in [33]. For other examples, see [3].

3. Proof of Theorem 1.1

Let $(X, d; s)$ be a complete b-metric space with parameter $s \geq 1$, and let T be a selfmap of X, satisfying

$$d(T(x),T(y)) \leq \varphi(d(x,y))\,, \quad \forall x,y \in X, \tag{3.1}$$

where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a a function satisfying:

(a) φ is non-decreasing, and

(b) $\lim_{n\to\infty} \varphi^n(t) = 0$, for all $t \in \mathbb{R}_+$.

We start by observing that the function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies $\varphi(0) = 0$ and

$$\varphi(t) < t, \quad \forall t > 0. \tag{3.2}$$

Therefore, we have

$$\varphi(t) \leq t, \quad \forall t \in [0,+\infty). \tag{3.3}$$

Let $x \in X$ be fixed and define $x_n := T^n x$ for every $n \in \mathbb{N}$.

By the monotonicity of the function φ and by the inequality (1.2), we obtain that

$$d(T^n x, T^{n+m}x) \leq \varphi^n(d(x,T^m x)) \quad \forall n,m \in \mathbb{N}. \tag{3.4}$$

In particular, we have

$$d(T^n x, T^{n+1}x) \leq \varphi^n(d(x,Tx)) \quad \forall n \in \mathbb{N},$$

which implies, by virtue of the assumption (b) that $\lim_{n\to+\infty} d(T^n x, T^{n+1}x) = 0$.

Therefore, there exists an integer $q \in \mathbb{N}$ such that

$$d(T^q x, T^{q+1}x) \leq \frac{1}{2}. \tag{3.5}$$

We shall discuss two cases:

(i) Suppose that $\varphi(s) \leq \frac{1}{2}$. In this case, we prove by induction, with respect to n that, for all $n \in \mathbb{N}$ we have

$$d(T^q x, T^{q+n}) \leq s. \tag{3.6}$$

By definition of q, we know that the inequality (3.6) is true for $n = 1$.

Suppose that the inequality (3.6) holds for some integer n. Then, By (a) and (3.1) and (3.5), we have

$$d(T^{q+1}x, T^{q+n+1}) \leq \varphi(d(T^q x, T^{q+n}x)) \leq \varphi(s) \leq \frac{1}{2}. \tag{3.7}$$

By (3.7) and the s-relaxed triangle inequality satisfied by the b-metric d, we get

$$d(T^q x, T^{q+n+1}x) \leq s\left[d(T^q, T^{q+1}x) + d(T^{q+1}x, T^{q+n+1}x)\right] \leq s(\frac{1}{2} + \frac{1}{2}) = s.$$

This ends the induction.

By the inequalities (3.1) and (3.6), we get

$$d(T^{q+n}x, T^{q+n+m}x) \leq \varphi^n(s), \quad \forall n, m \in \mathbb{N}. \tag{3.8}$$

The inequality (3.8) implies that the sequence $(T^n(x))_{n\in\mathbb{N}}$ is a Cauchy sequence. By the completeness of the b-metric space $(X, d; s)$, there is point (say) z in X such that $(T^n(x))_{n\in\mathbb{N}}$ converges to z. That is $\lim_{n\to+\infty} d(T^n(x), z) = 0$.

By using the contraction property and the s-relaxed triangle inequality, we obtain

$$\begin{aligned} d(z, T(z)) &\leq s[d(z, x_{n+1}) + d(x_{n+1}, T(z))] \\ &< s[d(z, x_{n+1}) + \varphi(d(x_n, z))] \\ &\leq s[d(z, x_{n+1}) + d(x_n, z)], \end{aligned}$$

which, by taking the limits when $n \to +\infty$, gives $d(z, T(z)) = 0$. This is equivalent to say that $Tz = z$. Hence, z is a fixed point of T.

Let u another point of X, then we have, by the contraction property of T and the property (a) of φ, we have

$$d(T^n(u), z) = d(T^n(u), T^n(z)) \leq \varphi^n(d(u, x)), \quad \forall n \in \mathbb{N},$$

which shows that $\lim_{n\to+\infty} d(T^n(u), z) = 0$.

Let $w \in X$ another fixed point of T, then we have, by the contraction property of T and the property (a) of φ, we have

$$d(w, z) = d(T^n(w), T^n(z)) \leq \varphi^n(d(w, x)), \quad \forall n \in \mathbb{N},$$

which implies that $w = z$.

So we have proved that in the case (i) all the required conclusions are obtained.

(ii) General case: Since $\lim_{n\to+\infty} \varphi^n(s) = 0$, there exists an integer $p \in \mathbb{N}$ such that $\varphi^p(s) \leq \frac{1}{2}$.

We set $S := T^p$, $\psi(t) := \varphi^p(t)$ for all $t \geq 0$. Then the selfmap S of X satisfies the following contraction

$$d(S(x), S(y)) \leq \psi(d(x, y)), \quad \forall x, y \in X. \tag{3.9}$$

Obviously, the function ψ satisfies the properties (a) and (b) of Theorem 1.1. So, according to the analysis of the particular case (i), we deduce that S has a unique fixed point (say) z_* and for evey $u \in X$, the sequence $(S^n u = T^{pn}u)$ converges to z_*.

Let us show that z_* is a fixed point of T. To get a contradiction, suppose that $T(z_*) \neq z_*$, then we have

$$d(z_*, Tz_*) = d(Sz_*, S(Tz_*)) \leq \psi(d(z, Tz_*)) < d(z, Tz_*),$$

which is a contradiction. Hence $T(z_*) = z_*$.

Suppose that $x \in X$ is given. For every integer n, there is a unique couple $(m_n, r_n) \in \mathbb{N} \times \{0, 1, \ldots, p-1\}$, such that $n = pm_n + r_n$. Obviously, $n \to +\infty$ if, and only if, $m_n \to +\infty$. Then, for all integer n, we have

$$\begin{aligned} d(T^n x, z_*) &= d\big(S^{m_n}(T^{r_n}x), z_*\big) \\ &\leq \psi^{m_n} d\big(T^{r_n}x, T^{r_n}z_*\big) \\ &< \psi^{m_n}(\phi^{r_n}(d(x, z_*))) \\ &\leq \psi^{m_n}(d(x, z_*)), \end{aligned}$$

which shows that $\lim_{n\to+\infty} d(T^n x, z_*) = 0$. This ends the proof of Theorem 1.1.

4. Consequences

As a first consequence of Theorem 1.1, we obtain that the Banach's fixed point theorem actually holds for arbitrary contractions on complete b-metric spaces.

Theorem 4.1 ([2]). *Let $(X, d; s)$ be a complete b-metric space with parameter $s \geq 1$ and $0 < \alpha < 1$. If $T : X \to X$ satisfies the inequality*

$$d(f(x), f(y)) \leq \alpha d(x, y)\,, \tag{4.1}$$

for all $x, y \in X$, then T has a unique fixed point z and the sequence $(T^n(x))_{n\in\mathbb{N}}$ converges to z for every $x \in X$.

Theorem 4.1 was proved in [2] by using Theorem 2.1 which gives a metrization process for the b-metric space $(X, d; s)$ and by applying the classical Banach contraction principle for metric spaces. Also, we notice that Theorem 4.1 extends the result of I. A. Bakhtin published in [5]. By the proof of Theorem 1.1, we see that Theorem 4.1 can also be proved directly by using only the appropriate technics of b-metrics.

A second consequence of Theorem 1.1 is connected to a result of [32], where the following definition was introduced.

Definition 4.2. A function $\varphi : [0, \infty) \to [0, \infty)$ is called a x^γ-summable comparison function, where $\gamma > 0$, if:

i) φ is increasing;

ii) the series $\sum_{n=1}^{\infty} n^\gamma \varphi^n(r)$ is convergent for every $r \in [0, \infty)$.

We denote the family of x^γ-summable comparison functions by Γ^γ.

Every $\varphi \in \Gamma^\gamma$, where $\gamma > 0$, is a comparison function, so it satisfies conditions a) and b) from the hypotheses of Theorem 1.1. Therefore, we obtain the following corollary.

Corollary 4.3. *Let $(X, d; s)$ be a complete b-metric space with parameter $s \geq 1$, and let T be a selfmap of X, satisfying*

$$d(T(x), T(y)) \leq \varphi(d(x, y))\,, \quad \forall x, y \in X, \tag{4.2}$$

where $\varphi \in \Gamma^\gamma$ for some $\gamma > 0$.

Then T has a unique fixed point $z \in X$ such that the sequence of iterates $\big(T^n(x)\big)_{n\in\mathbb{N}_0}$ converges to z for all $x \in X$ as $n \to \infty$.

Conflict of Interests. The author declare that there is no conflict of interests regarding the publication of this paper.

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