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SOME COMMON FIXED POINT THEOREMS FOR F-CONTRACTION MAPPINGS WITH APPLICATIONS TO FUNCTIONAL EQUATIONS IN THE DYNAMIC PROGRAMMING

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Abstract The purpose of this work is to prove the existence and uniqueness of a common fixed point of an F-contraction mapping in complete metric spaces. Moreover, we also extend our common fixed point results to an α-coupled common fixed point result. Finally, we discuss the existence and uniqueness for a common solution of coupled systems of functional equations in terms of F-contraction mappings.

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1. Introduction

Many years ago, fixed point theory played an important role in various problems occurring in different areas of mathematics. It is beneficial for solving various equations for a self-mapping f of the type $fx = x$ in a metric space. One of the problems in mathematics is to discuss the existence of a unique solution of integral equations, which are solved by the well-known fixed point result, namely the Banach contraction principle [1]. Later, many mathematicians generalized the Banach contraction principle in many directions (see [2–13] and references therein). In 1970, Ćirić [14, 15] established fixed point results for Ćirić contraction mappings and quasi-contraction mappings, which are the generalization of the Banach contraction principle. In 2003, Ran and Reurings [16] established fixed point results in partially ordered metric spaces and showed some applications to matrix equations. In 2012, Wardowski [17] introduced the concept of an F-contraction mapping and proved the fixed point result for F-contraction mappings in complete metric spaces as follows.

Definition 1.1 ([17])**.** A mapping T from a metric space (X, d) into itself is said to be an *F-contraction mapping* if there exist $\tau > 0$ and $F \in \mathcal{F}$ such that

$$\forall x, y \in X \quad [\ d(Tx, Ty) > 0 \quad \Longrightarrow \quad \tau + F(d(Tx, Ty)) \leq F(d(x, y))\ ], \tag{1.1}$$

where $\mathcal{F}$ is the family of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ satisfying the following properties:

(F_1): F is strictly increasing;

(F_2): for each sequence $\{\alpha_n\}$ of positive numbers, we obtain

$$\lim_{n\to\infty} \alpha_n = 0 \text{ if and only if } \lim_{n\to\infty} F(\alpha_n) = -\infty;$$

(F_3): there exists $k \in (0, 1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

The following ones are examples of functions belonging to $\mathcal{F}$.

- $F_1(\alpha) = \ln \alpha$ for all $\alpha > 0$;
- $F_2(\alpha) = \alpha + \ln \alpha$ for all $\alpha > 0$;
- $F_3(\alpha) = -\frac{1}{\sqrt{\alpha}}$ for all $\alpha > 0$;
- $F_4(\alpha) = \ln(\alpha^2 + \alpha)$ for all $\alpha > 0$.

Theorem 1.2 ([17])**.** *Let (X, d) be a complete metric space and let $T : X \to X$ be an F-contraction mapping. Then T has a unique fixed point in X. Moreover, for each $x_0 \in X$, the Picard sequence $\{T^n x_0\}$ is convergent to the fixed point of T.*

On the other hand, Guo and Lakshmikantham [18] first introduced the notion of a coupled fixed point and proved the existence and uniqueness results of a coupled fixed point in complete metric spaces. Later, Harjani *et al.* [19] improved and generalized the notion of a coupled fixed point by defining the following new notion.

Definition 1.3 ([19])**.** Let S be a nonempty set and $\alpha : S \to S$ be a given mapping. An element $(u, v) \in B(S)$, where $B(S)$ stands for the set of all bounded real-valued functions on S, is called an *α-coupled fixed point* of a mapping $G : B(S) \times B(S) \to B(S)$ if $G(u, v) = u$ and $G(u(\alpha), v(\alpha)) = v$.

Moreover, they proved the existence and uniqueness of an α-coupled fixed point in complete metric spaces and applied it to a dynamic programming.

Subsequently, Işik and Sintunavarat [20] improved the notion of α-coupled fixed point and a new results as follows:

Definition 1.4 ([20]). Let S be a nonempty set and $\alpha : S \to S$ be a given mapping. An element $(u, v) \in B(S)$, where $B(S)$ stands for the set of all bounded real-valued functions on S, is called an *α-coupled common fixed point* of mappings $K, G : B(S) \times B(S) \to B(S)$ if $K(u, v) = G(u, v) = u$ and $K(u(\alpha), v(\alpha)) = G(u(\alpha), v(\alpha)) = v$.

In this paper, we establish a new common fixed point theorem for F-contraction mappings and show that our common fixed point theorem can be extended to an α-coupled common fixed point theorem F-contraction mappings. Furthermore, we apply our results to a dynamic programming and show some new applications for solving the existence and uniqueness of coupled systems of functional equations.

2. Auxiliary notions

Throughout this paper, X, $\mathbb{R}^+$, $\mathbb{N}$ and $\mathbb{N}_0$ denote a nonempty set, the set of positive real numbers, the set of positive integers and the set of nonnegative integers, respectively. Also, we use the notation $B(S)$ stands for the set of all bounded real-valued functions on a fixed nonempty set S. Supremum metric on $B(S)$ is a mapping $d : B(S) \times B(S) \to [0, \infty)$ defined by

$$d(f, g) = \sup_{x \in S} |fx - gx|$$

for all $f, g \in B(S)$.

Remark 2.1. $B(S)$ endowed with the supremum metric d is a complete metric space.

Definition 2.2. Any pair of self-mappings f and g on a nonempty set X have a common fixed point if $fx = gx = x$.

3. Main results

In this section, we use a function $F \in \mathcal{F}$ to establish the new common fixed point results in complete metric spaces. We start our consideration by giving the following useful lemma.

Lemma 3.1. *Let (X, d) be a metric space and let f and g be self-mappings on X. Suppose that there are $\tau > 0$ and $F \in \mathcal{F}$ such that*

$$\forall x, y \in X \quad [\ d(fx, gy) > 0 \quad \Longrightarrow \quad \tau + F(d(fx, gy)) \leq F(d(x, y))\ ]. \tag{3.1}$$

If z is a fixed point of f or z is a fixed point of g, then z is a common fixed point of f and g.

Proof. First, we prove that if z is a fixed point of g, then z is a fixed point of f. Prove by contradiction that z is not a fixed point of f. Then $d(fz, gz) = d(fz, z) > 0$. Now a contradiction is appeared from the contractive condition (3.1). Hence, z is also a fixed point of f and then z is a common fixed point of f and g. Similarly, it is easy to prove that if z is a fixed point of f, then z is a common fixed point of f and g. ∎

In our main result, we use the utility of functions in the class $\mathcal{F}$ to consider in terms of a common fixed point, which is a generalization of Wardowski's fixed point result [17].

Theorem 3.2. *Let (X,d) be a complete metric space and let f and g be self-mappings on X. If there are $\tau > 0$ and $F \in \mathcal{F}$ such that*

$$\forall x,y \in X \ \ [\ d(fx,gy) > 0 \quad \Longrightarrow \quad \tau + F(d(fx,gy)) \leq F(d(x,y))\ ]. \tag{3.2}$$

Then f and g have a unique common fixed point.

Proof. Let $x_0 \in X$. Define the sequence $\{x_n\}$ in X by

$$x_{2n+1} = fx_{2n}$$

and

$$x_{2n+2} = gx_{2n+1}$$

for all $n \in \mathbb{N}_0$. If $x_{2n} = x_{2n+1}$ for some $n \in \mathbb{N}_0$, then $x_{2n} = fx_{2n}$ and so x_{2n} is a fixed point of f. It follows from Lemma 3.1 that x_{2n} is a common fixed point of f and g, that is, $x_{2n} = fx_{2n} = gx_{2n}$. Similarly, if $x_{2n+1} = x_{2n+2}$ for some $n \in \mathbb{N}_0$, then $x_{2n+1} = fx_{2n+1} = gx_{2n+1}$. So we may assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}_0$.

Let $n = 2m+1$, where $m \in \mathbb{N}_0$. By (3.2), we have

$$\begin{aligned} F(d(x_n,x_{n+1})) &= F(d(x_{2m+1},x_{2m+2})) \\ &= F(d(fx_{2m},gx_{2m+1})) \\ &\leq F(d(x_{2m},x_{2m+1})) - \tau \\ &= F(d(fx_{2m-1},gx_{2m})) - \tau \\ &\leq F(d(x_{2m-1},x_{2m})) - 2\tau \\ &\ \ \vdots \\ &\leq F(d(x_0,x_1)) - (2m+1)\tau \\ &= F(d(x_0,x_1)) - n\tau. \end{aligned}$$

By a similar method, for $n = 2m$, where $m \in \mathbb{N}_0$, we get

$$\begin{aligned} F(d(x_n,x_{n+1})) &\leq F(d(x_0,x_1)) - (2m)\tau \\ &= F(d(x_0,x_1)) - n\tau. \end{aligned}$$

Thus, for each $n \in \mathbb{N}$, we have

$$F(d(x_n,x_{n+1})) \leq F(d(x_0,x_1)) - n\tau. \tag{3.3}$$

From (3.3), we obtain

$$\lim_{n\to\infty} F(d(x_n,x_{n+1})) = -\infty, \tag{3.4}$$

and it follows from (F_2) that

$$\lim_{n\to\infty} d(x_n,x_{n+1}) = 0. \tag{3.5}$$

From (F_3), there exists $k \in (0,1)$ such that

$$\lim_{n\to\infty} [d(x_n,x_{n+1})]^k F(d(x_n,x_{n+1})) = 0. \tag{3.6}$$

From the inequality (3.3), we obtain

$$[d(x_n,x_{n+1})]^k F(d(x_n,x_{n+1})) - [d(x_n,x_{n+1})]^k F(d(x_0,x_1)) \leq -[d(x_n,x_{n+1})]^k n\tau \leq 0. \tag{3.7}$$

for all $n \in \mathbb{N}$. Letting $n \to \infty$ in (3.7) and using (3.5) and (3.6), we have

$$\lim_{n\to\infty} n[d(x_n, x_{n+1})]^k = 0. \tag{3.8}$$

From the equality (3.8), there is $n_1 \in \mathbb{N}$ such that $n[d(x_n, x_{n+1})]^k \leq 1$ for all $n \geq n_1$. Then

$$d(x_n, x_{n+1}) \leq \frac{1}{n^{1/k}}. \tag{3.9}$$

for all $n \geq n_1$. Now, we will show that $\{x_n\}$ is a Cauchy sequence in X. Let $m > n \geq n_1$. Using the triangle inequality and (3.9), we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{m-1}, x_m) \\ &\leq \sum_{i=n}^{m-1} \frac{1}{i^{1/k}}. \end{aligned}$$

Since $\sum_{i=n}^{\infty} \frac{1}{i^{1/k}} < \infty$, it follows that $\{x_n\}$ is a Cauchy sequence in X. By the completeness of X, there exists an element x^* in X such that $x_n \to x^*$ as $n \to \infty$.

Next, we claim that x^* is a common fixed point of f and g. If $\Omega := \{n \in \mathbb{N} : d(fx_{2n}, gx^*) = 0\}$ is an infinite set, then a subsequence $\{x_{2n+1} = fx_{2n}\}_{n\in\Omega}$ of $\{x_{2n+1}\}_{n\in\mathbb{N}}$ converges to gx^* as $n \to \infty$ and so $x^* = gx^*$. In the other hand, if $\mathbb{N} \setminus \Omega$ is an infinite set, for each $n \in \mathbb{N} \setminus \Omega$, we have $d(fx_{2n}, gx^*) > 0$. This implies that

$$\begin{aligned} F(d(x_{2n+1}, gx^*)) &= F(d(fx_{2n}, gx^*)) \\ &\leq F(d(x_{2n}, x^*)) - \tau \\ &< F(d(x_{2n}, x^*)). \end{aligned}$$

Again, by (F_1) we obtain that

$$d(x_{2n+1}, gx^*) < d(x_{2n}, x^*)$$

for all $n \in \mathbb{N}$ with $d(fx_{2n}, gx^*) > 0$. It yields that

$$d(x_{2n+1}, gx^*) \leq d(x_{2n}, x^*)$$

for all $n \in \mathbb{N}$. Letting limit $n \to \infty$ in the above inequality, we get $d(x^*, gx^*) = 0$ and then $x^* = gx^*$. Therefore, x^* is a fixed point of g. From bothe cases, by Lemma 3.1, we obtain x^* is a common fixed point of f and g.

Finally, we will show that x^* is a unique common fixed point of f and g. Suppose that y^* is a common fixed point of f and g with $x^* \neq y^*$. Then $d(fx^*, gy^*) > 0$ and so

$$F(d(x^*, y^*)) = F(d(fx^*, gy^*)) \leq F(d(x^*, y^*)) - \tau < F(d(x^*, y^*)),$$

which is a contradiction. Thus, $x^* = y^*$ and so f and g have a unique fixed point. This completes the proof. ∎

Now, we use our common fixed point theorem to solve the existence and uniqueness of an α-coupled common fixed point as follows:

Corollary 3.3. *Let S be a nonempty set, $\alpha : S \to S$ and $K, G : B(S) \times B(S) \to B(S)$ be given mappings. Supppose that there are $\tau > 0$ and a function F in $\mathcal{F}$ such that*

$$\tau + F(d(K(x, y), G(u, v))) \leq F(\max\{d(x, u), d(y, v)\}) \tag{3.10}$$

for all $x, y, u, v \in B(S)$ with $d(K(x, y), G(u, v)) > 0$. Then K and G have a unique α-coupled common fixed point.

Proof. Let $\rho : B(S) \times B(S) \to [0, \infty)$ be defined by

$$\rho((x, y), (u, v)) = \max\{d(x, u), d(y, v)\}$$

for all $x, y, u, v \in B(S)$. It is easy to see that $(B(S) \times B(S), \rho)$ is a complete metric space.

Define the mappings $T_K, T_G : B(S) \times B(S) \to B(S)$ by

$$T_K(A) = (F(x, y), F(x(\alpha), y(\alpha)))$$

and

$$T_G(A) = (G(x, y), G(x(\alpha), y(\alpha)))$$

for all $A = (x, y) \in B(S) \times B(S)$. Let $U, V \in B(S) \times B(S)$, where $U = (x, y)$ and $V = (u, v)$. Then

$$\begin{aligned}
& F(\rho(T_K(U), T_G(V))) \\
= \;& F(\rho((K(x, y), K(x(\alpha), y(\alpha))), (G(u, v), G(u(\alpha), v(\alpha))))) \\
= \;& F(\max\{d(K(x, y), G(u, v)), d(K(x(\alpha), y(\alpha)), G(u(\alpha), v(\alpha)))\}) \\
\leq \;& \max\{F(d(K(x, y), G(u, v))), F(d(K(x(\alpha), y(\alpha)), G(u(\alpha), v(\alpha))))\} \\
\leq \;& \max\{F(\max\{d(x, u), d(y, v)\}) - \tau, F(\max\{d(x(\alpha), u(\alpha)), d(y(\alpha), v(\alpha)) - \tau\})\} \\
= \;& \max\{F(\max\{d(x, u), d(y, v)\}), F(\max\{d(x(\alpha), u(\alpha)), d(y(\alpha), v(\alpha))\})\} - \tau \\
\leq \;& \max\{F(\max\{d(x, u), d(y, v)\}), F(\max\{\sup_{s \in S} |x((\alpha)(s)) - u((\alpha)(s))|, \sup_{s \in S} |y((\alpha)(s)) - v((\alpha)(s))|\})\} \\
\leq \;& \max\{F(\max\{d(x, u), d(y, v)\}), F(\max\{\sup_{s \in S} |x(s) - u(s)|, \sup_{s \in S} |y(s) - v(s)|\})\} - \tau \\
= \;& \max\{F(\max\{d(x, u), d(y, v)\}), F(\max\{d(x, u), d(y, v)\})\} - \tau \\
= \;& F(\max\{d(x, u), d(y, v)\}) - \tau \\
= \;& F(\rho(d(x, u), d(y, v))) - \tau \\
= \;& F(\rho(U, V)) - \tau.
\end{aligned}$$

Then T_K and T_G satisfy the inequality (3.2). By Theorem 3.2, T_K and T_G have a unique common fixed point $Z^* = (x^*, y^*) \in B(S) \times B(S)$, that is, $T_F(Z^*) = T_G(Z^*) = Z^*$. Therefore, $K(x^*, y^*) = G(x^*, y^*) = x^*$ and $K(x^*(\alpha), y^*(\alpha)) = G(x^*(\alpha), y^*(\alpha)) = y^*$. This completes the proof. ∎

4. Application to a Dynamic Programming

In this section, we use the following notations.

- S is a state space;
- D is a decision space;
- $\gamma : S \times D \to S$ is a given mapping;
- $p : S \times D \to \mathbb{R}$ is a given mapping;
- $P, Q : S \times D \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given mappings;
- $\alpha : S \to S$ is a given mapping.

Now, we employ our main results to obtain a unique common solution of the following coupled systems of functional equations arising in the dynamic programming:

$$\begin{aligned}
u_1(x) &= \sup_{y \in D}\{p(x, y) + P(x, y, u_1(\gamma(x, y)), v_1(\gamma(x, y))\}, \\
v_1(x) &= \sup_{y \in D}\{p(x, y) + P(x, y, u_1(\alpha(\gamma(x, y))), v_1(\alpha(\gamma(x, y)))\},
\end{aligned} \tag{4.1}$$

$$\begin{aligned} u_2(x) &= \sup_{y\in D}\{p(x,y)+Q(x,y,u_2(\gamma(x,y)),v_2(\gamma(x,y))\}, \\ v_2(x) &= \sup_{y\in D}\{p(x,y)+Q(x,y,u_2(\alpha(\gamma(x,y))),v_2(\alpha(\gamma(x,y)))\} \end{aligned} \tag{4.2}$$

for all $x \in S$.

Theorem 4.1. *Consider the systems of functional equations (4.1) and (4.2). Assume that the following conditions are satisfied:*

(i) p, P, and Q are bounded;
(ii) for each points $x \in S$, $y \in D$ and $h_1, k_1, h_2, k_2 \in \mathbb{R}$,

$$\begin{aligned} & |P(x,y,h_1,k_1)-Q(x,y,h_2,k_2)|e^{|P(x,y,h_1,k_1)-Q(x,y,h_2,k_2)|} \\ \leq\; & \max\{|h_1-h_2|,|k_1-k_2|\}e^{-\tau+\max\{|h_1-h_2|,|k_1-k_2|\}} \end{aligned}$$

where $\tau > 0$.

Then the equations (4.1) and (4.2) have a unique common solution in $B(S)\times B(S)$.

Proof. In the total of this proof, we define the mappings $K, G : H(n) \to H(n)$ for each $(u, v) \in B(S) \times B(S)$ by

$$\begin{aligned} (K(u,v))(x) &= \sup_{y\in D}\{p(x,y)+P(x,y,u(\gamma(x,y)),v(\gamma(x,y))\}, \\ (G(u,v))(x) &= \sup_{y\in D}\{p(x,y)+Q(x,y,u(\gamma(x,y)),v(\gamma(x,y))\} \end{aligned} \tag{4.3}$$

for all $x \in S$. Then K and G are well defined since functions p, P and Q are bounded.

Next, we will show that the condition (3.10) in Corollary 3.3 holds with K, G, and the supremum metric d. Let $(u_1, v_1), (u_2, v_2) \in B(S)\times B(S)$ with $d(K(u_1, v_1), G(u_2, v_2)) > 0$. By (ii), we get

$$\begin{aligned} & d(K(u_1,v_1),G(u_2,v_2))e^{d(K(u_1,v_1),G(u_2,v_2))} \\ =\; & \sup_{x\in S}|K(u_1,v_1)(x)-G(u_2,v_2)(x)|e^{\left(\sup_{x\in S}|K(u_1,v_1)(x)-G(u_2,v_2)(x)|\right)} \\ \leq\; & \sup_{x\in S}\left\{\sup_{y\in D}|A-B|\right\}e^{\sup_{x\in S}\left\{\sup_{y\in D}|A-B|\right\}} \\ \leq\; & \sup_{x\in S}\left\{\sup_{y\in D}\left\{\max\{C,D\}e^{-\tau+\max\{C,D\}}\right\}\right\}, \end{aligned}$$

where

$$A = P(x,y,u_1(\gamma(x,y)),v_1(\gamma(x,y)) \text{ and } B = Q(x,y,u_2(\gamma(x,y)),v_2(\gamma(x,y)),$$

$$C = |u_1(\gamma(x,y)) - u_2(\gamma(x,y))| \text{ and } D = |v_1(\gamma(x,y)) - v_2(\gamma(x,y))|.$$

This yields that

$$\begin{aligned} & d(K(u_1,v_1),G(u_2,v_2))e^{d(K(u_1,v_1),G(u_2,v_2))} \\ \leq\; & \max\{d(u_1,u_2),d(v_1,v_2)\}e^{-\tau+\max\{d(u_1,u_2),d(v_1,v_2)\}}. \end{aligned}$$

From the above inequality, we have

$$\begin{aligned} & \ln\left(d(K(u_1,v_1),G(u_2,v_2))\right) + d(K(u_1,v_1),G(u_2,v_2)) \\ \leq\ & \ln\left(\max\left\{d(u_1,u_2),d(v_1,v_2)\right\}\right) - \tau + \max\left\{d(u_1,u_2),d(v_1,v_2)\right\}. \end{aligned}$$

By setting the mapping $F \in \mathcal{F}$ by $F(t) = \ln(t) + t$ for all $t > 0$, we have

$$\tau + F(d(K(u_1,v_1),G(u_2,v_2))) \leq F(\max\{d(u_1,u_2),d(v_1,v_2)\}).$$

for all $(u_1,v_1),(u_2,v_2) \in B(S) \times B(S)$ with $d(K(u_1,v_1),G(u_2,v_2)) > 0$. This means that the contractive condition (3.10) in Corollary 3.3 is satisfied. Hence, K and G have a unique α-coupled common fixed point. That is, the equation (4.1) and (4.2) have a unique common solution in $B(S) \times B(S)$. ∎

Theorem 4.2. *Consider the systems of functional equations (4.1) and (4.2). Assume that the following conditions are satisfied:*

(i) *$p, P,$ and Q are bounded;*
(ii) *for each points $x \in S$, $y \in D$ and $h_1,k_1,h_2,k_2 \in \mathbb{R}$,*

$$|P(x,y,h_1,k_1) - Q(x,y,h_2,k_2)| \leq \sqrt[n]{e^{-\tau}(\sup\{|h_1-h_2|,|k_1-k_2|\})^n},$$

where $\tau > 0$ and $n \in \mathbb{N}$.

Then the equations (4.1) and (4.2) have a unique common solution in $B(S) \times B(S)$.

Proof. In the total of this proof, we define the mappings $K, G : H(n) \to H(n)$ for each $(u,v) \in B(S) \times B(S)$ by

$$\begin{aligned} (K(u,v))(x) &= \sup_{y\in D}\{p(x,y) + P(x,y,u(\gamma(x,y)),v(\gamma(x,y))\}, \\ (G(u,v))(x) &= \sup_{y\in D}\{p(x,y) + Q(x,y,u(\gamma(x,y)),v(\gamma(x,y))\} \end{aligned} \tag{4.4}$$

for all $x \in S$. Then K and G are well defined since functions p, P and Q are bounded.

Next, we will show that the condition (3.10) in Corollary 3.3 holds with K, G, and the supremum metric d. Let $(u_1,v_1),(u_2,v_2) \in B(S) \times B(S)$ with $d(K(u_1,v_1),G(u_2,v_2)) > 0$. By (ii), we can show that

$$d(K(u_1,v_1),G(u_2,v_2)) \leq \sqrt[n]{e^{-\tau}(\max\{d(u_1,u_2),d(v_1,v_2)\})^n}$$

and so

$$[d(K(u_1,v_1),G(u_2,v_2))]^n \leq e^{-\tau}(\max\{d(u_1,u_2),d(v_1,v_2)\})^n.$$

This implies that

$$\frac{[d(K(u_1,v_1),G(u_2,v_2))]^n}{(\max\{d(u_1,u_2),d(v_1,v_2)\})^n} \leq e^{-\tau}.$$

and then

$$\ln\frac{[d(K(u_1,v_1),G(u_2,v_2))]^n}{(\max\{d(u_1,u_2),d(v_1,v_2)\})^n} \leq -\tau.$$

This yields that

$$\ln[d(K(u_1,v_1),G(u_2,v_2))]^n - \ln(\max\{d(u_1,u_2),d(v_1,v_2)\})^n \leq -\tau,$$

that is,

$$\tau + \ln[d(K(u_1,v_1),G(u_2,v_2))]^n \leq \ln(\max\{d(u_1,u_2),d(v_1,v_2)\})^n.$$

Therefore,

$$\tau + F(d(K(u_1, v_1), G(u_2, v_2))) \leq F(\max\{d(u_1, u_2), d(v_1, v_2)\}).$$

for all $(u_1, v_1), (u_2, v_2) \in B(S) \times B(S)$ with $d(K(u_1, v_1), G(u_2, v_2)) > 0$. This means that the contractive condition (3.10) in Corollary 3.3 is satisfied with the mapping $F \in \mathcal{F}$ defined by $F(t) = n \ln t$ for all $t > 0$. Hence, K and G have a unique α-coupled common fixed point, that is, the equations (4.1) and (4.2) have a unique common solution in $B(S) \times B(S)$. ∎

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

All authors contributed equally and significantly in writing this paper. All authors read and approved the final manuscript.

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