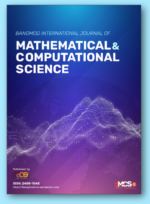
BANGMOD INTERNATIONAL JOURNAL OF MATHEMATICAL & COMPUTATIONAL SCIENCE
Volume. 7 Number. 1 & 2 (2021)
Pages 102–125

BANGMOD INTERNATIONAL JOURNAL OF
MATHEMATICAL & COMPUTATIONAL SCIENCE ISSN: 2408-154X
https://bangmodmcs.wordpress.com/



EXISTENCE AND STABILITY ANALYSIS OF A FRACTIONAL-ORDER COVID-19 MODEL

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Received: 24 November 2021 / Accepted: 25 December 2021

Abstract After the first confirmed case of coronavirus disease COVID-19 in Pakistan on February 26, 2020, the number of cases increases rapidly and as of April 20, 2020, 11:00 AM, the number of confirmed cases reached 271,887 out of which 5,787 died. Considering the situation, we develop a modified SEIR model of novel coronavirus (nCoV-19) keeping in view the transmission of pandemics in Pakistan. We generalize the proposed model to fractional-order derivatives in the Atangana-Baleanu sense. Moreover, we show the existence and uniqueness of solutions of the proposed fractional model using Schaefer's and Banach's fixed point theory, and utilizing the Sumudu transform and Picard's successive approximation method, we explore the iterative solutions and their stability. In addition, using the least square curve fitting method together with the *fminsearch* function in the **MATLAB** optimization toolbox, we obtain the best values for some of the unknown biological parameters involved in the proposed model. Furthermore, we solved the fractional model numerically using the Atangana-Toufik numerical scheme and presented the different forms of graphical results that can be useful in minimizing the infection.

MSC: 34A08, 92B10, 33E30
Keywords: Atangana-Baleanu derivative; Coronavirus (nCoV-2019); Sumudu transform; Real data; Numerical results

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1. Introduction

In December 2019, pandemic of novel coronavirus in the city of Wuhan, China was outbreak. After that it spread almost in the whole world during the month of March 2020. Since the virus was transmitting from human to human so the whole world has taken preventive measures. But the pandemic spread globally at a large scale affecting more than 15.947 million confirmed cases as of July 25, 2020, 11:00 GMT, out of which more than 0.642 million has passed away, 9.743 million were recovered and 66,263 are in serious critical situation [2].

In order to tackle and understand the pandemic behavior of such infectious diseases, mathematical modeling play very important role. More precisely, the developing of SEIR model of a certain infectious disease has significance importance. Several studies involving SEIR models of nCoV-19 are available in the literature to analyze the nCoV-19 transmission dynamics (see for example [10, 12, 26, 27, 30–33]. In all of the cited studies, models presented therein based on classical derivatives which have some limitations according to the order of differential equations involved. Keeping in view such limitations Khan and Atangana [18] used fractional calculus and analyze the outbreak as in fractional calculus the differential operators used are non-integer or fractional order which possesses memory impacts and are valuable to demonstrate many natural phenomena, nature-related truths and facts having non-local dynamics and anomalous behavior. In recent decades, many authors have developed and suggested more efficient techniques to obtained real and approximate solutions of the differential equation involving fractional operators [11, 17, 21, 24, 25, 28]. More precisely, to study the complex biological systems and diseases, fractional calculus played an important role as it provides better results than the integer order models (see e.g. [4, 8, 13–16, 20]).

We start with some basic notions. First, we recall the definition of Caputo fractional derivative which can be found in many books (see, e.g., [19]).

Definition 1.1. For a differentiable function h, the Caputo derivative of order $\alpha \in (0,1)$ is defined by

$$^{C}\mathfrak{D}^{\alpha}h(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t h^{'}(s)\frac{1}{(t-s)^{\alpha}}ds. \tag{1.1}$$

Definition 1.2. [6] Let $h \in F^1(0,1)$ and $\alpha \in [0,1]$ then the Atangana-Baleanu-Caputo (ABC) fractional derivative is defined by

$$^{ABC}\mathfrak{D}^{\alpha}h(t) = \frac{M(\alpha)}{(1-\alpha)}\int_0^t h^{'}(\omega)E_{\alpha}[-\frac{\alpha}{1-\alpha}(t-\omega)^{\alpha}]d\omega. \tag{1.2}$$

Published online: 31 December 2021
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Published by Center of Excellence in Theoretical and Computational Science (TaCS-CoE)

Please cite this article as: A. Hussain et al., Existence and Stability Analysis of a Fractional-Order COVID-19 Model, Bangmod Int. J. Math. & Comp. Sci., Vol. 7 No. 1 & 2 (2021) 102–125.

Definition 1.3. [6] The integral operator associated with ABC-fractional derivative is defined by

$$^{ABC}\mathfrak{J}^{\alpha}h(t) = \frac{(1-\alpha)}{M(\alpha)}h(t) + \frac{\alpha}{M(\alpha)\Gamma(\alpha)}\int_0^t h(\omega)(t-\omega)^{\alpha-1}d\omega, \tag{1.3}$$

where $M(\alpha)$ is the normalization function.

Definition 1.4. [22] For a function $\xi(t)$ in

$$A = \{\xi(t) : \text{there exist } \chi, t_1, t_2 > 0, |\xi(t)| < \chi exp\left(\frac{|t|}{t_i}\right), \text{ if } t \in (-1)^j \times [0,\infty)\},$$

the Sumudu transform (S_T) of $\xi(u) \in A$ is given by

$$S_T(\xi(t)) = \int_0^{\infty} exp\,(-t)\,\xi(st)ds\ t \in (-t_1, t_2). \tag{1.4}$$

Lemma 1.5. *[9] Assume $h \in H^1(a,b)$, $b > a$, $\alpha \in (0,1)$ and $h(t) \in A$, the Sumudu transform (S_T) of Atangana-Baleanu fractional derivative in Caputo sense is*

$$S_T(^{ABC}\mathfrak{D}^{\alpha}h(t)) = \frac{N(\alpha)}{1-\alpha+\alpha(t)^{\alpha}}(S_T(h(t)) - h(0)). \tag{1.5}$$

2. Formulation of nCoV-19 model

The very first case of the COVID-19 was reported on February 26, 2020 in Pakistan. In the beginning of the outbreak, the reported cases have travel history from different countries. During the month of March, the local transmission which have connection with the people who has travel history, were reported as well. Taking into account the situation of epidemic, Pakistan government announced lock down initially for 14 days from March 16, 2020 to March 30, 2020 but then extended for next fifteen days till April 14, 2020. As of July 25, 2020, 11:00 AM (GMT+5), there are 271,887 confirmed cases were reported out of which 5,787 were died, 236,596 were recovered and 29,504 are active cases. Figure 1 is the graphical view of COVID-19 cases in Pakistan [1].

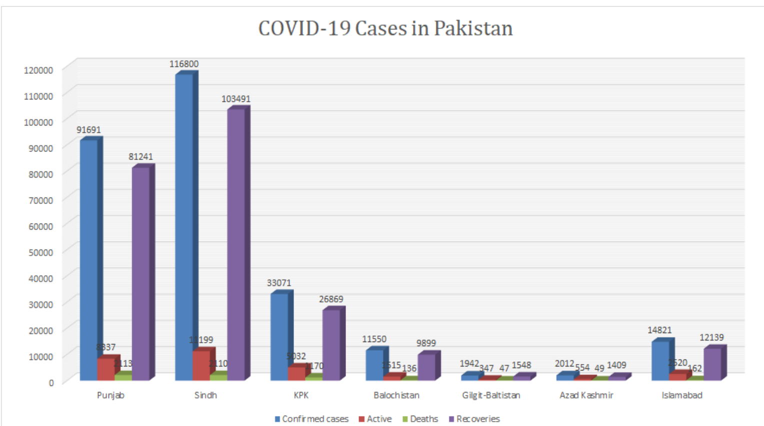


FIGURE 1. Province wise detail of COVID-19 in Pakistan

To describe an outbreak of a certain epidemic, compartmental models are strong enough. We develop a modified SEIR model for the epidemic dynamic of nCoV-19 according to the situation of outbreak in Pakistan. We use six state variables and to avoid over-fitting minimum number of parameters. The model is modified SEIR because in the case of coronavirus pandemic, number of people remains undetected due to having no or mild symptoms. The dynamics of nCoV-19 is describe graphically in Figure 2 and the description of the different compartments and parameters is as describe in Table 1 and 2 respectively. Using the above detail, we formulate the model in the form of set of

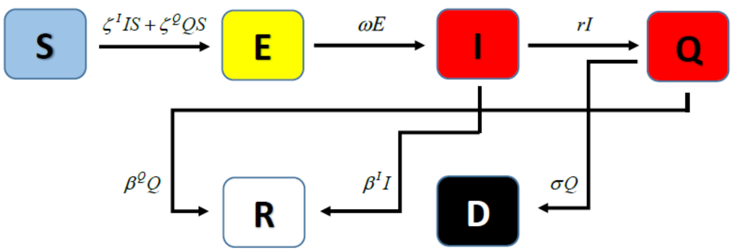


FIGURE 2. Transmission of COVID-19

TABLE 1. Transmission of COVID-19.

Compartment	Description
S_p	Susceptible people
E_p	Exposed people
I_p	Infected but not reported people
Q_p	Infected, detected and reported people
R_p	Recovered people
D_p	Dead people

TABLE 2. Parameters with their description.

Parameters	Description
ζ_I	Infection rate among compartment I_p
ζ_Q	Infection rate among compartment Q_p
ω	Rate of exposed people which are not yet infected
r	Rate of infected people moved to compartment Q_p
β_I	Recovery rate among compartment I_p
β_Q	Recovery rate among compartment Q_p
σ	Death rate from compartment Q_p

nonlinear differential equations as follows:

$$\begin{aligned}\frac{dS_p}{dt} &= -\zeta_I S_p I_p - \zeta_Q S_p Q_p,\\ \frac{dE_p}{dt} &= \zeta_I S_p + \zeta_Q S_p Q_p - \omega E_p,\\ \frac{dI_p}{dt} &= \omega E_p - \beta_I I_p - r I_p,\\ \frac{dQ_p}{dt} &= r I_p - \beta_Q Q_p - \sigma Q_p,\\ \frac{dR_p}{dt} &= \beta_I I_p + \beta_Q Q_p,\\ \frac{dD_p}{dt} &= \sigma Q_p,\end{aligned} \tag{2.1}$$

with the initial conditions

$$S_p(0) \geq 0, E_p(0) \geq 0, I_p(0) \geq 0, Q_p(0) \geq 0, R_p(0) \geq 0, D_p(0) \geq 0.$$

We now generalize the model (2.1) to a fractional order model using Atangana-Baleanu derivative in Caputo sense as follows:

$$\begin{aligned}{}^{ABC}\mathfrak{D}^\alpha S_p &= -\zeta_I S_p I_p - \zeta_Q S_p Q_p,\\ {}^{ABC}\mathfrak{D}^\alpha E_p &= \zeta_I S_p + \zeta_Q S_p Q_p - \omega E_p,\\ {}^{ABC}\mathfrak{D}^\alpha I_p &= \omega E_p - \beta_I I_p - r I_p,\\ {}^{ABC}\mathfrak{D}^\alpha Q_p &= r I_p - \beta_Q Q_p - \sigma Q_p,\\ {}^{ABC}\mathfrak{D}^\alpha R_p &= \beta_I I_p + \beta_Q Q_p\\ {}^{ABC}\mathfrak{D}^\alpha D_p &= \sigma Q_p,\end{aligned} \tag{2.2}$$

where α denotes the the fractional order parameter and the model variables in (2.2) are nonnegative and the initial conditions are given by

$$S_p(0) \geq 0, E_p(0) \geq 0, I_p(0) \geq 0, Q_p(0) \geq 0, R_p(0) \geq 0, D_p(0) \geq 0.$$

Using the initial conditions and fractional integral operator, we convert model (2.2) into integral equations

$$\begin{aligned}S_p(u) - S_p(0) &= {}^{ABC}\mathfrak{I}^\alpha\left[-\zeta_I S_p I_p - \zeta_Q S_p Q_p\right],\\ E_p(u) - E_p(0) &= {}^{ABC}\mathfrak{I}^\alpha[\zeta_I S_p + \zeta_Q S_p Q_p - \omega E_p],\\ I_p(u) - I_p(0) &= {}^{ABC}\mathfrak{I}^\alpha[\omega E_p - \beta_I I_p - r I_p],\\ Q_p(u) - Q_p(0) &= {}^{ABC}\mathfrak{I}^\alpha[r I_p - \beta_Q Q_p - \sigma Q_p],\\ R_p(u) - R_p(0) &= {}^{ABC}\mathfrak{I}^\alpha[\beta_I I_p + \beta_Q Q_p].\\ D_p(u) - D_p(0) &= {}^{ABC}\mathfrak{I}^\alpha[\sigma Q_p].\end{aligned} \tag{2.3}$$

For simplicity, we write the kernels

$$\begin{aligned} F_1(u,S_p(u)) &= -\zeta_I S_p I_p - \zeta_Q S_p Q_p, \\ F_2(u,E_p(u)) &= \zeta_I S_p + \zeta_Q S_p Q_p - \omega E_p, \\ F_3(u,I_p(u)) &= \omega E_p - \beta_I I_p - r I_p, \\ F_4(u,Q_p(u)) &= r I_p - \beta_Q Q_p - \sigma Q_p, \\ F_5(u,R_p(u)) &= \beta_I I_p + \beta_Q Q_p, \\ F_6(u,D_p(u)) &= \sigma Q_p. \end{aligned} \tag{2.4}$$

and the functions

$$\Psi(\alpha) = \frac{1-\alpha}{N(\alpha)}, \ \Phi(\alpha) = \frac{\alpha}{\Gamma(\alpha)N(\alpha)}. \tag{2.5}$$

Using (1.3), (2.4) and (2.5) in (2.3) and writing state variables in terms of kernels, we obtain

$$\begin{aligned} S_p(u) &= S_p(0) + \Psi(\alpha)F_1(u,S_p(u)) + \Phi(\alpha)\int_0^u F_1(x,S_p(x))(u-x)^{\alpha-1}dx, \\ E_p(u) &= E_p(0) + \Psi(\alpha)F_2(u,E_p(u)) + \Phi(\alpha)\int_0^u F_2(x,E_p(x))(u-x)^{\alpha-1}dx, \\ I_p(u) &= I_p(0) + \Psi(\alpha)F_3(u,I_p(u)) + \Phi(\alpha)\int_0^u F_3(x,I_p(x))(u-x)^{\alpha-1}dx, \\ Q_p(u) &= Q_p(0) + \Psi(\alpha)F_4(u,Q_p(u)) + \Phi(\alpha)\int_0^u F_4(x,Q_p(x))(u-x)^{\alpha-1}dx, \\ R_p(u) &= R_p(0) + \Psi(\alpha)F_5(u,R_p(u)) + \Phi(\alpha)\int_0^u F_5(x,R_p(x))(u-x)^{\alpha-1}dx, \\ D_p(u) &= D_p(0) + \Psi(\alpha)F_6(u,D_p(u)) + \Phi(\alpha)\int_0^u F_6(x,D_p(x))(u-x)^{\alpha-1}dx. \end{aligned} \tag{2.6}$$

The Picard iterations are given by

$$\begin{aligned} S_p^{m+1}(u) &= \Psi(\alpha)F_1(u,S_p^m(u)) + \Phi(\alpha)\int_0^u F_1(x,S_p^m(x))(u-x)^{\alpha-1}dx, \\ E_p^{m+1}(u) &= \Psi(\alpha)F_2(u,E_p^m(u)) + \Phi(\alpha)\int_0^u F_2(x,E_p^m(x))(u-x)^{\alpha-1}dx, \\ I_p^{m+1}(u) &= \Psi(\alpha)F_3(u,I_p^m(u)) + \Phi(\alpha)\int_0^u F_3(x,I_p^m(x))(u-x)^{\alpha-1}dx, \\ Q_p^{m+1}(u) &= \Psi(\alpha)F_4(u,Q_p^m(u)) + \Phi(\alpha)\int_0^u F_4(x,Q_p^m(x))(u-x)^{\alpha-1}dx, \\ R_p^{m+1}(u) &= \Psi(\alpha)F_5(u,R_p^m(u)) + \Phi(\alpha)\int_0^u F_5(x,R_p^m(x))(u-x)^{\alpha-1}dx, \\ D_p^{m+1}(u) &= \Psi(\alpha)F_6(u,D_p^m(u)) + \Phi(\alpha)\int_0^u F_6(x,D_p^m(x))(u-x)^{\alpha-1}dx. \end{aligned} \tag{2.7}$$

In order to show the existence and uniqueness of solution of the model (2.2), we make use of fixed point theory. First, we re-write the model (2.2) in the following way:

$$\begin{cases} {}^{ABC}\mathfrak{D}^{\alpha}\varphi(u) = F(u, \varphi(u)), \\ \varphi(0) = \varphi_0, \ 0 < u < T < \infty. \end{cases} \tag{2.8}$$

The vector $\varphi(u) = (S_p, E_p, I_p, Q_p, R_p, D_p)$ and F in (2.8) represent the state variables and a continuous vector function respectively defined as follows:

$$F = (F_1, F_2, F_3, F_4, F_5, F_6), \tag{2.9}$$

with initial conditions $\varphi_0(u) = (S_p(0), E_p(0), I_p(0), Q_p(0), R_p(0), D_p(0))$. Corresponding to (2.8), the integral equation is give by

$$\varphi(u) = \varphi_0 + \Psi(\alpha)F(u, \varphi(u)) + \Phi(\alpha)\int_0^u F(x, \varphi(x))(u-x)^{\alpha-1}dx. \tag{2.10}$$

3. Existence and Uniqueness Results

Consider $A = [0, T]$, $\mathcal{W} = \mathcal{C}(B, \mathbb{R}^6)$ and the Picard operator $\mathcal{T} : \mathcal{W} \to \mathcal{W}$ be given by

$$\mathcal{T}[\varphi(u)] = \varphi_0 + \Psi(\alpha)F(u, \varphi(u)) + \Phi(\alpha)\int_0^u F(x, \varphi(x))(u-x)^{\alpha-1}dx. \tag{3.1}$$

Together with the supremum norm $\|\cdot\|_{\mathcal{C}}$, on φ is defined by

$$\|\varphi(u)\|_{\mathcal{C}} = \sup_{t\in A}\|\varphi(u)\|, \ \varphi(u) \in \mathcal{W}, \tag{3.2}$$

$\mathcal{W}$ defines a Banach space. Assume the following

[$\mathcal{B}_1$**:**] Let $F : A \times \mathbb{R}^6 \to \mathbb{R}^6$ is continuous.
[$\mathcal{B}_2$**:**] There exists $C_F > 0$ such that

$$|F(u, \varphi) - F(u, \varphi')| \le C_F|\varphi - \varphi'|$$

for all $\varphi, \varphi' \in \mathbb{R}^6, \in A$.
[$\mathcal{B}_3$**:**] There exist a constant $L > 0$ such that $|F(x, \varphi)| \le L(1 + |\varphi|)$ for each $x \in A$ and all $\varphi \in \mathbb{R}^6$.

We prove the existence of solution of (2.8) by Schaefer's fixed point theorem.

Theorem 3.1. *Assuming [$\mathcal{B}_1$]-[$\mathcal{B}_3$] together with $1 - \Psi(\alpha)L > 0$, then the problem (2.8) which is equivalent with the proposed model (2.2) has at least one solution.*

Proof. We first show that the operator $\mathcal{T}$ given in (3.1) is continuous. Consider a sequence (φ_j) such that $\varphi_j \to \varphi$ in $\mathcal{W}$. Now

$$\begin{aligned}
|\mathcal{T}\varphi_j(u) - \mathcal{T}\varphi(u)| &= \Bigg| \Psi(\alpha)F(u,\varphi_j(u)) + \Phi(\alpha)\int_0^u F(x,\varphi_j(x))(u-x)^{\alpha-1}dx \\
&- \Psi(\alpha)F(u,\varphi(u)) - \Phi(\alpha)\int_0^u F(x,\varphi(x))(u-x)^{\alpha-1}dx \Bigg| \\
&\leq \Psi(\alpha)\Big|(F(u,\varphi_j(u)) - F(u,\varphi(u)))\Big| \\
&+ \Phi(\alpha)\int_0^u |F(x,\varphi_j(x)) - F(x,\varphi(x))|(u-x)^{\alpha-1}dx \\
&\leq \Psi(\alpha)C_F\|F(x,\varphi_j(x)) - F(x,\varphi(x))\|_{\mathcal{C}} \\
&+ \Phi(\alpha)C_F\|F(x,\varphi_j(x)) - F(x,\varphi(x))\|_{\mathcal{C}}\frac{t^\alpha}{\alpha} \\
&\leq \left(\Psi(\alpha) + \frac{\Phi(\alpha)T^\alpha}{\alpha}\right)C_F\|F(x,\varphi_j(x)) - F(x,\varphi(x))\|_{\mathcal{C}}.
\end{aligned}$$

Continuity of F implies the continuity of $\mathcal{T}$.
Now suppose that $W = \{\varphi \in \mathcal{W} : \|\varphi\| \leq c > 0\}$. We now show that $\mathcal{T}[W]$ is bounded, i.e. there exists $d > 0$ such that for every $\varphi \in W$, $\|\mathcal{T}\varphi\| \leq d$. For any $t \in A$, we have

$$\begin{aligned}
|\mathcal{T}\varphi(u)| &= \left|\varphi_0 + \Psi(\alpha)F(u,\varphi_j(u)) + \Phi(\alpha)\int_0^u F(x,\varphi_j(x))(u-x)^{\alpha-1}dx\right| \\
&\leq |\varphi_0| + \Psi(\alpha)|F(u,\varphi(u))| + \Phi(\alpha)\left|\int_0^u F(x,\varphi(x))(u-x)^{\alpha-1}dx\right| \\
&= |\varphi_0| + \left(\Psi(\alpha) + \Phi(\alpha)\frac{T^\alpha}{\alpha}\right)L(1+c) = d,
\end{aligned}$$

which implies

$$|\mathcal{T}\varphi(u)| \leq d.$$

For the equicontinuity of $\mathcal{T}$, let $t_1, t_2 \in A$ with $0 \leq t_1, t_2 \leq T$ and $\varphi \in W$. Utilizing $[\mathcal{B}_3]$, we have

$$\begin{aligned}
|\mathcal{T}\varphi(u_1) - \mathcal{T}\varphi(u_2)| &= \Big|\Psi(\alpha)F(u_1,\varphi(u_1)) + \Phi(\alpha)\int_0^{t_1} F(x,\varphi(x))(u_1-x)^{\alpha-1}dx \\
&\quad - \Psi(\alpha)F(u_2,\varphi(u_2)) - \Phi(\alpha)\int_0^{t_2} F(x,\varphi(x))(u_2-x)^{\alpha-1}dx\Big| \\
&\leq \Psi(\alpha)\Big|(F(u_1,\varphi(u_1)) - F(u_2,\varphi(u_2)))\Big| + \Phi(\alpha)\Big|\int_0^{t_1} F(x,\varphi(x))(u_1-x)^{\alpha-1}dx \\
&\quad - \int_0^{t_2} F(x,\varphi(x))(u_2-x)^{\alpha-1}dx\Big| \\
&\leq \Psi(\alpha)\Big|(F(u_1,\varphi(u_1)) - F(u_2,\varphi(u_2)))\Big| + \Phi(\alpha)L(1+|\varphi|)\int_{t_1}^{t_2} |(u_2-x)^{\alpha-1}|dx \\
&\quad + \Phi(\alpha)\Big|\int_0^{t_1} F(x,\varphi(x))[(u_1-x)^{\alpha-1} - (u_2-x)^{\alpha-1}]dx\Big| \\
&\leq \Psi(\alpha)\Big|(F(u_1,\varphi(u_1)) - F(u_2,\varphi(u_2)))\Big| + \Phi(\alpha)L(1+d)\frac{(u_2-t_1)^{\alpha}}{\alpha} \\
&\quad + \Phi(\alpha)\Big|\int_0^{t_1} F(x,\varphi(x))[(u_1-x)^{\alpha-1} - (u_2-x)^{\alpha-1}]dx\Big|.
\end{aligned}$$

As t_1 tends to t_2, continuity of F tends R.H.S of above inequality to zero. Hence $\mathcal{T}$ is equicontinuous. Therefore, we conclude by Arzela-Ascoli Theorem that $\mathcal{T}$ is completely continuous.
Finally, we show that the set $Q(\mathcal{T}) = \{\varphi \in \mathcal{W} : \varphi = \vartheta\mathcal{T}\varphi \text{ for some } \vartheta \in (0,1)\}$ is bounded. For each $t \in A$, we have

$$\begin{aligned}
|\mathcal{T}\varphi(u)| &= \Big|\varphi_0 + \Psi(\alpha)F(u,\varphi_j(u)) + \Phi(\alpha)\int_0^u F(x,\varphi_j(x))(u-x)^{\alpha-1}dx\Big| \\
&\leq |\varphi_0| + \Psi(\alpha)|F(u,\varphi(u))| + \Phi(\alpha)\Big|\int_0^u F(x,\varphi(x))(u-x)^{\alpha-1}dx\Big| \\
&\leq |\varphi_0| + \Psi(\alpha)L(1+|\varphi(u)|) + \Phi(\alpha)L\int_0^u (1+|\varphi(x)|)(u-x)^{\alpha-1}dx \\
&= |\varphi_0| + \Psi(\alpha)L + \Psi(\alpha)L|\varphi(u)| + \Phi(\alpha)L\frac{T^{\alpha}}{\alpha} + \Phi(\alpha)L\int_0^u |\varphi(x)|(u-x)^{\alpha-1}dx.
\end{aligned}$$

Writing $S = |\varphi_0| + \Psi(\alpha)L + \Phi(\alpha)L\frac{T^{\alpha}}{\alpha}$ and since $1 - \Psi(\alpha)L > 0$, we can have

$$|\mathcal{T}\varphi(u)| \leq \frac{S}{1-\Psi(\alpha)L} + \frac{\Phi(\alpha)L}{1-\Psi(\alpha)L}\int_0^u |\varphi(x)|(u-x)^{\alpha-1}dx,$$

utilizing Gronwall's inequality, we obtain

$$|\mathcal{T}\varphi(u)| \leq \frac{S}{1-\Psi(\alpha)L} exp\left(\frac{\Phi(\alpha)LT^{\alpha}}{(1-\Psi(\alpha)L)\alpha}\right).$$

Therefore $Q(\mathcal{T})$ is bounded. Consequently, by Schaefer's theorem $\mathcal{T}$ has a fixed point which is in fact a solution of (2.8). ∎

We now show by using Banach contraction principle that solution of (2.8) is unique.

Theorem 3.2. *Assuming [$\mathcal{B}_1$]-[$\mathcal{B}_2$] together with* $\left(\Psi(\alpha)+\frac{\Phi(\alpha)T^{\alpha}}{\alpha}\right)C_F<1$, *there exists a unique solution of (2.8).*

Proof. Considering (1.3) together with (2.8), we have

$$\varphi(u)=\mathcal{T}[\varphi(u)]. \tag{3.3}$$

The operator $\mathcal{T}$ given in (3.1), is well defined by [$\mathcal{B}_1$]. Now for all $\varphi,\varphi'\in\mathcal{W}$, we have

$$\begin{aligned}|\mathcal{T}[\varphi(u)]-\mathcal{T}[\varphi'(u)]| &\leq \Psi(\alpha)|F(u,\varphi(u))-F(u,\varphi'(u))|\\ &\quad+\Phi(\alpha)\int_0^u|F(x,\varphi(x))-F(x,\varphi(x))|(u-x)^{\alpha-1}dx\\ &\leq \Psi(\alpha)C_F\|\varphi-\varphi'\|_{\mathcal{C}}+\Phi(\alpha)C_F\|\varphi-\varphi'\|_{\mathcal{C}}\int_0^u(u-x)^{\alpha-1}dx\\ &\leq \left(\Psi(\alpha)+\frac{\Phi(\alpha)T^{\alpha}}{\alpha}\right)C_F\|\varphi-\varphi'\|_{\mathcal{C}}\\ &= \mathcal{A}\|\varphi-\varphi'\|_{\mathcal{C}},\end{aligned}$$

where

$$\mathcal{A}=\left(\Psi(\alpha)+\frac{\Phi(\alpha)T^{\alpha}}{\alpha}\right)C_F.$$

This implies

$$\|\mathcal{T}[\varphi(u)]-\mathcal{T}[\varphi'(u)]\|_{\mathcal{C}}\leq\mathcal{A}\|\varphi-\varphi'\|_{\mathcal{C}}, \tag{3.4}$$

Thus the defined operator $\mathcal{T}$ is a contraction, and hence Banach contraction principle guarantees that $\mathcal{T}$ has a unique fixed point which is the solution model (2.8). ∎

4. Special Solutions by Iterative Approach

We obtain iterative solution of the model (2.2). Apply Sumudu transforms (S_T) on both sides of (2.2), we get

$$\begin{aligned}S_T[{}^{ABC}\mathfrak{D}^{\alpha}S_p] &= S_T[-\zeta_I S_p I_p-\zeta_Q S_p Q_p],\\ S_T[{}^{ABC}\mathfrak{D}^{\alpha}E_p] &= S_T[\zeta_I S_p+\zeta_Q S_p Q_p-\omega E_p],\\ S_T[{}^{ABC}\mathfrak{D}^{\alpha}I_p] &= S_T[\omega E_p-\beta_I I_p-rI_p],\\ S_T[{}^{ABC}\mathfrak{D}^{\alpha}Q_p] &= S_T[rI_p-\beta_Q Q_p-\sigma Q_p],\\ S_T[{}^{ABC}\mathfrak{D}^{\alpha}R_p] &= S_T[\beta_I I_p+\beta_Q Q_p],\\ S_T[{}^{ABC}\mathfrak{D}^{\alpha}D_p] &= S_T[\sigma Q_p],\end{aligned} \tag{4.1}$$

Using definition of Shehu transforms of ABC-derivative, we get

$$\begin{aligned}
\frac{N(\alpha)}{1-\alpha+\alpha u^{\alpha}}\left[S_T(S_p(u))-S_p(0)\right] &= S_T[-\zeta_I S_p I_p-\zeta_Q S_p Q_p],\\
\frac{N(\alpha)}{1-\alpha+\alpha\left(\frac{u}{s}\right)^{\alpha}}\left[S_T(E_p(u))-E_p(0)\right] &= S_T[\zeta_I S_p+\zeta_Q S_p Q_p-\omega E_p],\\
\frac{N(\alpha)}{1-\alpha+\alpha u^{\alpha}}\left[S_T(I_p(u))-I_p(0)\right] &= S_T[\omega E_p-\beta_I I_p-r I_p],\\
\frac{N(\alpha)}{1-\alpha+\alpha\left(\frac{u}{s}\right)^{\alpha}}\left[S_T(Q_p(u))-Q_p(0)\right] &= S_T[r I_p-\beta_Q Q_p-\sigma Q_p],\\
\frac{N(\alpha)}{1-\alpha+\alpha\left(\frac{u}{s}\right)^{\alpha}}\left[S_T(R_p(u))-R_p(0)\right] &= S_T[\beta_I I_p+\beta_Q Q_p],\\
\frac{N(\alpha)}{1-\alpha+\alpha\left(\frac{u}{s}\right)^{\alpha}}\left[S_T(D_p(u))-D_p(0)\right] &= S_T[\sigma Q_p].
\end{aligned}$$

On rearranging

$$\begin{aligned}
S_T(S_p(u)) &= S_p(0)+\frac{1-\alpha+\alpha u^{\alpha}}{N(\alpha)}S_T[-\zeta_I S_p I_p-\zeta_Q S_p Q_p],\\
S_T(E_p(u)) &= E_p(0)+\frac{1-\alpha+\alpha u^{\alpha}}{N(\alpha)}S_T[\zeta_I S_p+\zeta_Q S_p Q_p-\omega E_p],\\
S_T(I_p(u)) &= I_p(0)+\frac{1-\alpha+\alpha u^{\alpha}}{N(\alpha)}S_T[\omega E_p-\beta_I I_p-r I_p],\\
S_T(Q_p(u)) &= Q_p(0)+\frac{1-\alpha+\alpha u^{\alpha}}{N(\alpha)}S_T[r I_p-\beta_Q Q_p-\sigma Q_p],\\
S_T(R_p(u)) &= R_p(0)+\frac{1-\alpha+\alpha u^{\alpha}}{N(\alpha)}S_T[\beta_I I_p+\beta_Q Q_p],\\
S_T(D_p(u)) &= D_p(0)+\frac{1-\alpha+\alpha u^{\alpha}}{N(\alpha)}S_T[\sigma Q_p].
\end{aligned} \tag{4.2}$$

Operating S_T^{-1} on both sides of (4.2), we get

$$\begin{aligned}
S_p(u) &= S_p(0)+S_T^{-1}\left\{\frac{1-\alpha+\alpha u^{\alpha}}{N(\alpha)}S_T[-\zeta_I S_p I_p-\zeta_Q S_p Q_p]\right\},\\
E_p(u) &= E_p(0)+S_T^{-1}\left\{\frac{1-\alpha+\alpha u^{\alpha}}{N(\alpha)}S_T[\zeta_I S_p+\zeta_Q S_p Q_p-\omega E_p]\right\},\\
I_p(u) &= I_p(0)+S_T^{-1}\left\{\frac{1-\alpha+\alpha u^{\alpha}}{N(\alpha)}S_T[\omega E_p-\beta_I I_p-r I_p]\right\},\\
Q_p(u) &= Q_p(0)+S_T^{-1}\left\{\frac{1-\alpha+\alpha u^{\alpha}}{N(\alpha)}S_T[r I_p-\beta_Q Q_p-\sigma Q_p]\right\},\\
R_p(u) &= R_p(0)+S_T^{-1}\left\{\frac{1-\alpha+\alpha u^{\alpha}}{N(\alpha)}S_T[\beta_I I_p+\beta_Q Q_p]\right\},\\
D_p(u) &= D_p(0)+S_T^{-1}\left\{\frac{1-\alpha+\alpha u^{\alpha}}{N(\alpha)}S_T[\sigma Q_p]\right\}.
\end{aligned} \tag{4.3}$$

The recursive formula is given by

$$
\begin{aligned}
S_p^{n+1}(u) &= S_p^n(0) + S_T^{-1}\left\{\frac{1-\alpha+\alpha u^\alpha}{N(\alpha)} S_T[-\zeta_I S_p^n I_p^n - \zeta_Q S_p^n Q_p^n]\right\},\\
E_p^{n+1}(u) &= E_p^n(0) + S_T^{-1}\left\{\frac{1-\alpha+\alpha u^\alpha}{N(\alpha)} S_T[\zeta_I S_p^n + \zeta_Q S_p^n Q_p^n - \omega E_p^n]\right\},\\
I_p^{n+1}(u) &= I_p^n(0) + S_T^{-1}\left\{\frac{1-\alpha+\alpha u^\alpha}{N(\alpha)} S_T[\omega E_p^n - \beta_I I_p^n - r I_p^n]\right\},\\
Q_p^{n+1}(u) &= Q_p^n(0) + S_T^{-1}\left\{\frac{1-\alpha+\alpha u^\alpha}{N(\alpha)} S_T[r I_p^n - \beta_Q Q_p^n - \sigma Q_p^n]\right\},\\
R_p(u) &= R_p(0) + S_T^{-1}\left\{\frac{1-\alpha+\alpha u^\alpha}{N(\alpha)} S_T[\beta_I I_p^n + \beta_Q Q_p^n]\right\},\\
D_p^{n+1}(u) &= D_p^n(0) + S_T^{-1}\left\{\frac{1-\alpha+\alpha u^\alpha}{N(\alpha)} S_T[\sigma Q_p^n]\right\}.
\end{aligned} \tag{4.4}
$$

The approximate solution of (4.4) is given by

$$
\begin{aligned}
S_p &= \lim_{n\to\infty} S_p^n, \quad E_p = \lim_{n\to\infty} E_p^n, \quad I_p = \lim_{n\to\infty} I_p^n,\\
Q_p &= \lim_{n\to\infty} Q_p^n, \quad R_p = \lim_{n\to\infty} R_p^n, \quad D_p = \lim_{n\to\infty} D_p^n.
\end{aligned}
$$

5. Stability Analysis and Iterative Solution

Consider a Banach space $\mathcal{X}$ together with norm $\|x\| = \max_{t\in[a,b]} |x(u)|$, $x \in \mathcal{X}$ and $\mathcal{P}$ a self map on $\mathcal{X}$. The recursive procedure is

$$R_{n+1} = h(\mathcal{P}, R_n). \tag{5.1}$$

The set of fixed points $Fix(\mathcal{P})$ of $\mathcal{P}$ is nonempty and R_n converges to a point of $Fix(\mathcal{P})$. Choose a sequence (f_n) in $\mathcal{X}$ and $e_n = \|f_{n+1} - h(\mathcal{P}, R_n)\|$. The recursive procedure (5.1) is $\mathcal{P}$-stable if $\lim_{n\to\infty} e_n = 0$. We suppose that the sequence (f_n) is bounded above, else it will diverge. Under these conditions, $R_{n+1} = \mathcal{P}R_n$ is Picard's iteration as described in [29], implies it is $\mathcal{P}$-stable.

Theorem 5.1. *Let $(\mathcal{X}, \|\cdot\|)$ be a Banach space and $\mathcal{P}$ be a self map on $\mathcal{X}$ satisfying*

$$\|\mathcal{P}_x - \mathcal{P}_y\| \leq Z\|x - \mathcal{P}_x\| + z\|x - y\| \tag{5.2}$$

for all $x, y \in \mathcal{X}$, where $Z \geq 0$ and $0 \leq z < 1$. Then $\mathcal{P}$ is Picard $\mathcal{P}$-stable.

Theorem 5.2. *A self map $\mathcal{P}$ given by*

$$\begin{aligned}
\mathcal{P}(S_p^n(u)) &= S_p^{n+1}(u)\\
&= S_p^n(u) + S_T^{-1}\left\{\frac{1-\alpha+\alpha u^\alpha}{N(\alpha)} S_T\left[-\zeta_I S_p^n I_p^n - \zeta_Q S_p^n Q_p^n\right]\right\}\\
\mathcal{P}(E_p^n(u)) &= E_p^{n+1}(u)\\
&= E_p^n(u) + S_T^{-1}\left\{\frac{1-\alpha+\alpha u^\alpha}{N(\alpha)} S_T\left[\zeta_I S_p^n + \zeta_Q S_p^n Q_p^n - \omega E_p^n\right]\right\}\\
\mathcal{P}(I_p^n(u)) &= I_p^{n+1}(u)\\
&= I_p^n(u) + S_T^{-1}\left\{\frac{1-\alpha+\alpha u^\alpha}{N(\alpha)} S_T[\omega E_p^n - \beta_I I_p^n - r I_p^n]\right\}\\
\mathcal{P}(Q_p^n(u)) &= Q_p^{n+1}(u)\\
&= Q_p^n(u) + S_T^{-1}\left\{\frac{1-\alpha+\alpha u^\alpha}{N(\alpha)} S_T[r I_p^n - \beta_Q Q_p^n - \sigma Q_p^n]\right\}\\
\mathcal{P}(R_p(u)) &= R_p(u)\\
&= R_p(u) + S_T^{-1}\left\{\frac{1-\alpha+\alpha u^\alpha}{N(\alpha)} S_T[\beta_I I_p^n + \beta_Q Q_p^n]\right\}\\
\mathcal{P}(D^n(u)) &= D^{n+1}(u)\\
&= D_p^n(u) + S_T^{-1}\left\{\frac{1-\alpha+\alpha u^\alpha}{N(\alpha)} S_T[\sigma Q_p^n]\right\}
\end{aligned}$$

is $\mathcal{P}$-stable in $H^1(a,b)$ if the following conditions holds:

$$\begin{cases}
1-(\zeta_I+\zeta_Q)l_1 g_1(\pi) - \zeta_I l_3' g_2(\pi) - \zeta_Q l_{4'} g_3(\pi) < 1\\
1+\zeta_I g_4(\pi) + \zeta_Q l_1 g_5(\pi) + \zeta_Q l_4' g_6(\pi) - \omega g_7(\pi) < 1\\
1+\omega g_8(\pi) - \beta_I g_9(\pi) - r g_{10}(\pi) < 1\\
1+r g_{11}(\pi) - \beta_Q g_{12}(\pi) - \sigma g_{13}(\pi) < 1\\
1+\beta_I g_{14}(\pi) + \beta_Q g_{15}(\pi) < 1\\
1+\sigma g_{16}(\pi) < 1.
\end{cases}$$

Proof. We first show that $\mathcal{P}$ has a fixed point. For $m, n \in \mathbb{N}$, we have

$$\begin{aligned}
\mathcal{P}(S_p^n(u)) - \mathcal{P}(S_p^m(u)) &= S_p^n(u) - S_p^m(u) + S_T^{-1}\left\{\frac{1-\alpha+\alpha u^\alpha}{N(\alpha)} S_T[-\zeta_I S_p^n I_p^n - \zeta_Q S_p^n Q_p^n]\right\} \\
&\quad - S_T^{-1}\left\{\frac{1-\alpha+\alpha u^\alpha}{N(\alpha)} S_T[-\zeta_I S_p^m I_p^m - \zeta_Q S_p^m Q_p^m]\right\} \\
\mathcal{P}(E_p^n(u)) - \mathcal{P}(E_p^m(u)) &= E_p^n(u) - E_p^m(u) + S_T^{-1}\left\{\frac{1-\alpha+\alpha u^\alpha}{N(\alpha)} S_T[\zeta_I S_p^n + \zeta_Q S_p^n Q_p^n - \omega E_p^n]\right\} \\
&\quad - S_T^{-1}\left\{\frac{1-\alpha+\alpha u^\alpha}{N(\alpha)} S_T[\zeta_I S_p^m + \zeta_Q S_p^m Q_p^m - \omega E_p^m]\right\} \\
\mathcal{P}(I_p^n(u)) - \mathcal{P}(I_p^m(u)) &= I_p^n(u) - I_p^m(u) + S_T^{-1}\left\{\frac{1-\alpha+\alpha u^\alpha}{N(\alpha)} S_T[\omega E_p^n - \beta_I I_p^n - r I_p^n]\right\} \\
&\quad - S_T^{-1}\left\{\frac{1-\alpha+\alpha u^\alpha}{N(\alpha)} S_T[\omega E_p^m - \beta_I I_p^m - r I_p^m]\right\} \\
\mathcal{P}(Q_p^n(u)) - \mathcal{P}(Q_p^m(u)) &= Q_p^n(u) - Q_p^m(u) + S_T^{-1}\left\{\frac{1-\alpha+\alpha u^\alpha}{N(\alpha)} S_T[r I_p^n - \beta_Q Q_p^n - \sigma Q_p^n]\right\} \\
&\quad - S_T^{-1}\left\{\frac{1-\alpha+\alpha u^\alpha}{N(\alpha)} S_T[r I_p^m - \beta_Q Q_p^m - \sigma Q_p^m]\right\} \\
\mathcal{P}(R_p^n(u)) - \mathcal{P}(R_p^m(u)) &= R_p^n(u) - R_p^m(u) + S_T^{-1}\left\{\frac{1-\alpha+\alpha u^\alpha}{N(\alpha)} S_T[\beta_I I_p^n + \beta_Q Q_p^n]\right\} \\
&\quad - S_T^{-1}\left\{\frac{1-\alpha+\alpha u^\alpha}{N(\alpha)} S_T[\beta_I I_p^m + \beta_Q Q_p^m]\right\} \\
\mathcal{P}(D^n(u)) - \mathcal{P}(D^m(u)) &= D_p^n(u) - D_p^m(u) + S_T^{-1}\left\{\frac{1-\alpha+\alpha u^\alpha}{N(\alpha)} S_T[\sigma Q_p^n]\right\} \\
&\quad - S_T^{-1}\left\{\frac{1-\alpha+\alpha u^\alpha}{N(\alpha)} S_T[\sigma Q_p^m]\right\}
\end{aligned}$$

Taking norm, we have

$$\begin{aligned}
\|\mathcal{P}(S_p^n(u)) - \mathcal{P}(S_p^m(u))\| &\le \|S_p^n(u) - S_p^m(u)\| + \left\| S_T^{-1}\left\{\frac{1-\alpha+\alpha u^\alpha}{N(\alpha)} S_T[-\zeta_I S_p^n I_p^n - \zeta_Q S_p^n Q_p^n]\right\}\right. \\
&\quad \left. - S_T^{-1}\left\{\frac{1-\alpha+\alpha u^\alpha}{N(\alpha)} S_T[-\zeta_I S_p^m I_p^m - \zeta_Q S_p^m Q_p^m]\right\}\right\| \\
&\le \|S_p^n(u) - S_p^m(u)\| + S_T^{-1}\left\{\frac{1-\alpha+\alpha u^\alpha}{N(\alpha)} S_T\Big[-\zeta_I \|S_p^n(I_p^n - I_p^m)\|\right. \\
&\quad \left. - \zeta_I \|I_p^m(S_p^n - S_p^m)\| - \zeta_Q \|S_p^n(Q_p^n - Q_p^m)\| - \zeta_Q \|Q_p^m(S_p^n - S_p^m)\|\Big]\right\}
\end{aligned}$$

Due to similar functioning of both solutions, we have

$$\begin{aligned}
\|S_p^n(u) - S_p^m(u)\| &\cong \|E_p^n(u) - E_p^m(u)\| \\
\|S_p^n(u) - S_p^m(u)\| &\cong \|I_p^n(u) - I_p^m(u)\| \\
\|S_p^n(u) - S_p^m(u)\| &\cong \|Q_p^n(u) - Q_p^m(u)\| \\
\|S_p^n(u) - S_p^m(u)\| &\cong \|R_p^n(u) - R_p^m(u)\| \\
\|S_p^n(u) - S_p^m(u)\| &\cong \|D_p^n(u) - D^m(u)\|.
\end{aligned} \tag{5.3}$$

Replacing (5.3) in (5.3), we get

$$\begin{aligned}
\|\mathcal{P}(S_p^n(u)) - \mathcal{P}(S_p^m(u))\| &\le \|S_p^n(u) - S_p^m(u)\| \\
&+ S_T^{-1}\left\{\frac{1-\alpha+\alpha u^\alpha}{N(\alpha)} S_T\Big[-\zeta_I \|S_p^n(S_p^n - S_p^m)\|\right. \\
&- \zeta_I \|I_p^m(S_p^n - S_p^m)\| - \zeta_Q \|S_p^n(S_p^n - S_p^m)\| \\
&- \left. \zeta_Q \|Q_p^m(S_p^n - S_p^m)\|\Big]\right\}.
\end{aligned} \tag{5.4}$$

The sequences S_p^n, I_p^m, Q_p^m are bounded being convergent, so there exist l_1, l_3', L_4' for all t such that

$$\|S_p^n\| < l_1, \|I_p^m\| < l_3', \|Q_p^m\| < l_4'.$$

Together with this, (5.4) become

$$\|\mathcal{P}(S_p^n(u)) - \mathcal{P}(S_p^m(u))\| \leq (1 - (\zeta_I + \zeta_Q)l_1 g_1(\pi) - \zeta_I l_3' g_2(\pi) - \zeta_Q l_{4'} g_3(\pi))\|S_p^n - S_p^m\|, \tag{5.5}$$

where g_i are the functions obtained by $S_T^{-1}\left\{\frac{1-\alpha+\alpha u^{\alpha}}{N(\alpha)} S_T[\cdot]\right\}$. In a similar fashion, we can have

$$\begin{aligned}
\|\mathcal{P}(E_p^n(u)) - \mathcal{P}(E_p^m(u))\| &\leq (1 + \zeta_I g_4(\pi) + \zeta_Q l_1 g_5(\pi) + \zeta_Q l_4' g_6(\pi) - \omega g_7(\pi))\|E_p^n - E_p^m\|,\\
\|\mathcal{P}(I_p^n(u)) - \mathcal{P}(I_p^m(u))\| &\leq (1 + \omega g_8()\pi - \beta_I g_9(\pi) - r g_{10}(\pi))\|I_p^n - I_p^m\|,\\
\|\mathcal{P}(Q_p^n(u)) - \mathcal{P}(Q_p^m(u))\| &\leq (1 + r g_{11}(\pi) - \beta_Q g_{12}(\pi) - \sigma g_{13}(\pi))\|Q_p^n - Q_p^m\|,\\
\|\mathcal{P}(R_p^n(u)) - \mathcal{P}(R_p^m(u))\| &\leq (1 + \beta_I g_{14}(\pi) + \beta_Q g_{15}(\pi))\|R_p^n - R_p^m\|,\\
\|\mathcal{P}(D_p^n(u)) - \mathcal{P}(D^m(u))\| &\leq (1 + \sigma g_{16}(\pi))\|D^n - D^m\|,
\end{aligned} \tag{5.6}$$

where

$$\begin{cases}
1 - (\zeta_I + \zeta_Q)l_1 g_1(\pi) - \zeta_I l_3' g_2(\pi) - \zeta_Q l_{4'} g_3(\pi) < 1\\
1 + \zeta_I g_4(\pi) + \zeta_Q l_1 g_5(\pi) + \zeta_Q l_4' g_6(\pi) - \omega g_7(\pi) < 1\\
1 + \omega g_8(\pi) - \beta_I g_9(\pi) - r g_{10}(\pi) < 1\\
1 + r g_{11}(\pi) - \beta_Q g_{12}(\pi) - \sigma g_{13}(\pi) < 1\\
1 + \beta_I g_{14}(\pi) + \beta_Q g_{15}(\pi) < 1\\
1 + \sigma g_{16}(\pi) < 1.
\end{cases}$$

Hence, $\mathcal{P}$ possesses a fixed point. Thus to prove that the assumptions of Theorem 5.1 are satisfied by $\mathcal{P}$, we assume inequalities (5.5)-(5.6) holds, denote $r = (0,0,0,0,0,0)$ and

$$R = \begin{cases}
1 - (\zeta_I + \zeta_Q)l_1 g_1(\pi) - \zeta_I l_3' g_2(\pi) - \zeta_Q l_{4'} g_3(\pi) < 1\\
1 + \zeta_I g_4(\pi) + \zeta_Q l_1 g_5(\pi) + \zeta_Q l_4' g_6(\pi) - \omega g_7(\pi) < 1\\
1 + \omega g_8(\pi) - \beta_I g_9(\pi) - r g_{10}(\pi) < 1\\
1 + r g_{11}(\pi) - \beta_Q g_{12}(\pi) - \sigma g_{13}(\pi) < 1\\
1 + \beta_I g_{14}(\pi) + \beta_Q g_{15}(\pi) < 1\\
1 + \sigma g_{16}(\pi) < 1.
\end{cases}$$

Hence all the conditions of Theorem 5.1 are satisfied, therefor $\mathcal{P}$ is Picard $\mathcal{P}$-stable. ∎

6. Model fitting and parameter estimation

The validation of a newly developed epidemiological model is one of the essential mechanisms for analyzing the transmission dynamics of a disease. The availability of real data for the underlying disease significantly contributes to completing this task. And the real data gives us an insight into how to determine the best values of certain unknown biological parameters involved in the model. To this end, we employ nonlinear least-squares curve fitting method with the help of "*fminsearch*" function from the `MATLAB` Optimization Toolbox. This approach states that, if a theoretical model $t \mapsto \Xi(t, q_1, q_2, \ldots, q_n)$ is attained and depend on a few unknown parameters $q_1, q_2, \ldots, q_n$ and a sequence of actual data points $(t_0, y_0), \ldots, (t_j, y_j)$ is also at hand then the aim is to obtain values of the parameters so that the error calculated can,

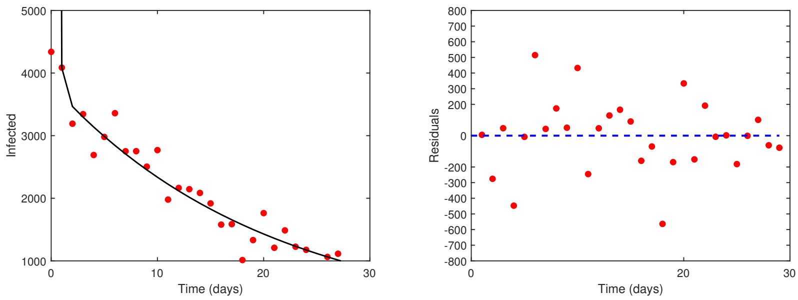
$$E := \sqrt{\sum_{i=0}^{j}\Big(\Xi(t, q_1, q_2, \ldots, q_n) - y_i\Big)^2}, \tag{3.5}$$

attain a minimum.

7 biological parameters are associated with the introduced model. Some of these parameters have been assumed while some have been best fitted. As can be seen in the Table 3, the parameters ζ_I, ζ_Q, ω and β_I have been best fitted using the above approach while the parameters β_Q, σ and r have been assumed. The initial conditions for the state variables are $S_p(0) = 150000000$, $E_p(0) = 100000$, $I_p(0) = 50000$, $Q_p(0) = 230$, $R_p(0) = 20$ and $D_p(0) = 10$.

TABLE 3. Baseline values of the parameters used in the model (2.1).

Fitted parameter	Value (Range)	Units/remarks	Sources
ζ_I	2.26043	day^{-1}	Estimated
ζ_Q	0.82928	day^{-1}	Estimated
ω	0.04916	day^{-1}	Estimated
β_I	2.09280	day^{-1}	Estimated
β_Q	0.91000	day^{-1}	Assumed
σ	0.57000	day^{-1}	Assumed
r	2.55000	day^{-1}	Assumed



(A) (B)

FIGURE 3. The daily COVID-19 cumulative cases time series in Pakistan from 1 July to July 31, 2020 with the best fitted curve from simulations of the proposed model and (B) the residuals for the best fitted curve.

7. NUMERICAL SIMULATIONS

We give a numerical procedure for the solution of the proposed fractional model (2.2) by adopting the techniques shown in [28]. The application of this scheme can be seen in many real word problems, see for example [3, 5, 7, 23] and the references therein. The numerical scheme used in the present analysis is as follows:

$$\begin{aligned} S_p(u_{k+1}) = S_p(u_0) + \frac{1-\alpha}{N(\alpha)} F_1(u_k, S_p(u_k)) \\ + \frac{\alpha}{N(\alpha)} \sum_{m=0}^{k} \Big[\frac{h^\alpha F_1(u_m, S_p(u_m))}{\Gamma(\alpha+2)} [(k+1-m)^\alpha (k-m+2+\alpha) \\ - (k-m)^\alpha (k-m+2+2\alpha)] \\ - \frac{h^\alpha F_1(u_{m-1}, S_p(u_{m-1}))}{\Gamma(\alpha+2)} [(k+1-m)^{\alpha+1} - (k-m)^\alpha (k-m+1+\alpha)] \Big]. \end{aligned} \tag{7.1}$$

$$\begin{aligned}E_p(u_{k+1}) &= S_p(u_0) + \frac{1-\alpha}{N(\alpha)}F_2(u_k, E_p(u_k)) \\ &+ \frac{\alpha}{N(\alpha)}\sum_{m=0}^{k}\Big[\frac{h^\alpha F_2(u_m, E_p(u_m))}{\Gamma(\alpha+2)}[(k+1-m)^\alpha(k-m+2+\alpha) \\ &- (k-m)^\alpha(k-m+2+2\alpha)] \\ &- \frac{h^\alpha F_2(u_{m-1}, E_p(u_{m-1}))}{\Gamma(\alpha+2)}[(k+1-m)^{\alpha+1} - (k-m)^\alpha(k-m+1+\alpha)]\Big].\end{aligned} \tag{7.2}$$

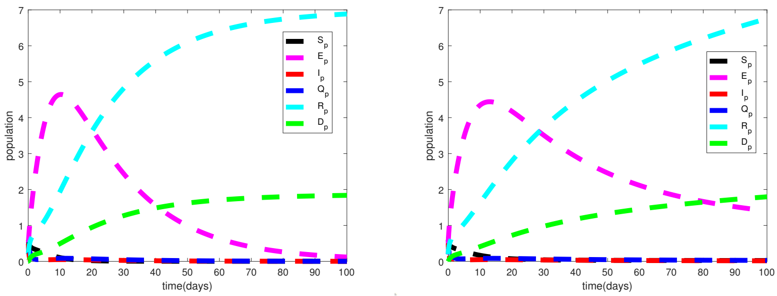
$$\begin{aligned}I_p(u_{k+1}) &= I_p(u_0) + \frac{1-\alpha}{N(\alpha)}F_3(u_k, I_p(u_k)) \\ &+ \frac{\alpha}{N(\alpha)}\sum_{m=0}^{k}\Big[\frac{h^\alpha F_3(u_m, I_p(u_m))}{\Gamma(\alpha+2)}[(k+1-m)^\alpha(k-m+2+\alpha) \\ &- (k-m)^\alpha(k-m+2+2\alpha)] \\ &- \frac{h^\alpha F_3(u_{m-1}, I_p(u_{m-1}))}{\Gamma(\alpha+2)}[(k+1-m)^{\alpha+1} - (k-m)^\alpha(k-m+1+\alpha)]\Big].\end{aligned} \tag{7.3}$$

$$\begin{aligned}Q_p(u_{k+1}) &= Q_p(u_0) + \frac{1-\alpha}{N(\alpha)}F_4(u_k, Q_p(u_k)) \\ &+ \frac{\alpha}{N(\alpha)}\sum_{m=0}^{k}\Big[\frac{h^\alpha F_4(u_m, Q_p(u_m))}{\Gamma(\alpha+2)}[(k+1-m)^\alpha(k-m+2+\alpha) \\ &- (k-m)^\alpha(k-m+2+2\alpha)] \\ &- \frac{h^\alpha F_4(u_{m-1}, Q_p(u_{m-1}))}{\Gamma(\alpha+2)}[(k+1-m)^{\alpha+1} - (k-m)^\alpha(k-m+1+\alpha)]\Big].\end{aligned} \tag{7.4}$$

$$\begin{aligned}R_p(u_{k+1}) &= R_p(u_0) + \frac{1-\alpha}{N(\alpha)}F_5(u_k, R_p(u_k)) \\ &+ \frac{\alpha}{N(\alpha)}\sum_{m=0}^{k}\Big[\frac{h^\alpha F_5(u_m, R_p(u_m))}{\Gamma(\alpha+2)}[(k+1-m)^\alpha(k-m+2+\alpha) \\ &- (k-m)^\alpha(k-m+2+2\alpha)] \\ &- \frac{h^\alpha F_5(u_{m-1}, R_p(u_{m-1}))}{\Gamma(\alpha+2)}[(k+1-m)^{\alpha+1} - (k-m)^\alpha(k-m+1+\alpha)]\Big].\end{aligned} \tag{7.5}$$

$$\begin{aligned}
D_p(u_{k+1}) = D_p(u_0) &+ \frac{1-\alpha}{N(\alpha)} F_6(u_k, D_p(u_k)) \\
&+ \frac{\alpha}{N(\alpha)} \sum_{m=0}^{k} \Big[\frac{h^\alpha F_6(u_m, D_p(u_m))}{\Gamma(\alpha+2)} [(k+1-m)^\alpha (k-m+2+\alpha) \\
&- (k-m)^\alpha (k-m+2+2\alpha)] \\
&- \frac{h^\alpha F_6(u_{m-1}, D_p(u_{m-1}))}{\Gamma(\alpha+2)} [(k+1-m)^{\alpha+1} - (k-m)^\alpha (k-m+1+\alpha)] \Big].
\end{aligned} \tag{7.6}$$

Using the baseline values of the parameters as displayed in Table 3, we simulate the proposed COVID-19 model for both classical and fractional order derivatives which shows the dynamic trajectories of each of the compartment.



(A) (B)

Figure 4. The dynamics of the state variables: (A) for the classical version (B) for the ABC version of the model.

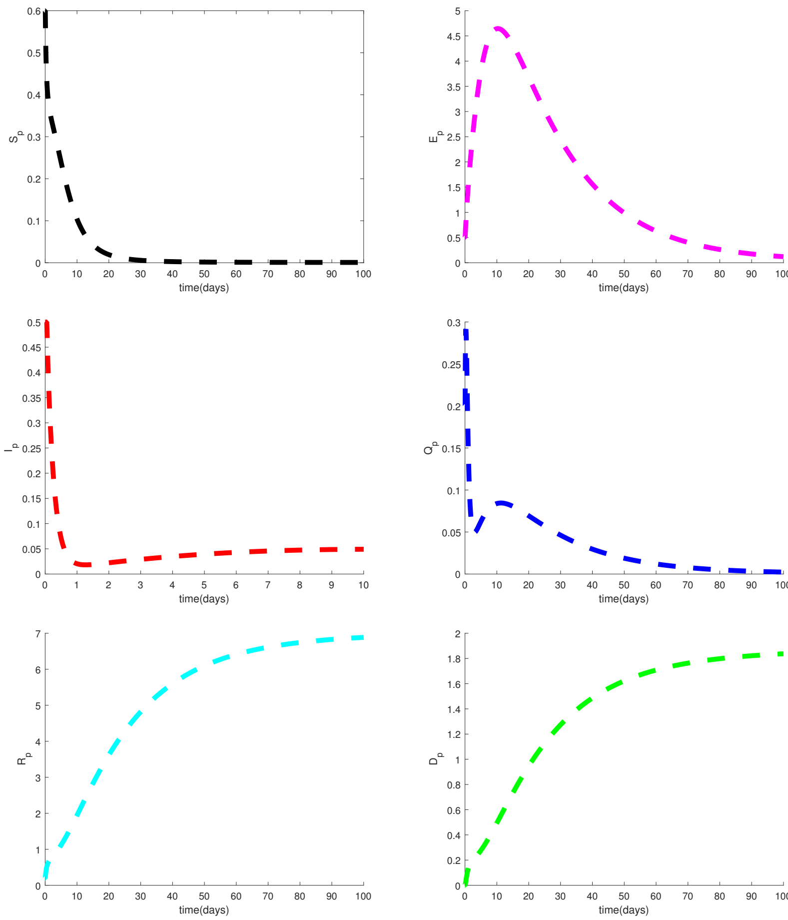


FIGURE 5. Profiles for behavior of each state variable for the classical version of the model.

In Figure 4, we presented the dynamics trajectories of the state variables for classical and ABC version respectively, which shows strong correlations between the integer and non-integer case. Also, Figures 5-6 deficit the epidemic trajectories for the proposed classical and fractional order COVID-19 model. To push the epidemic investigation one step further, we vary the fractional-order for different value of $\alpha = 1, 0.95, 0.90, 0.88$, which shows clearly the effect of the fractional-order as shown in Figure 7. The impacts of α are even more pronounced for example; in Figure 7(A), a decrease of the fractional-order α leads to the decrease of the number of the susceptible individual in the populations. Similarly, from 0-20 days, the number of exposed individuals increase and then start decreasing and becomes stable as displayed in Figure 7(B). An interesting scenario occurs in the infected and Quarantined compartment which shows the decrease of the fractional-order leads to the increase of each of the compartment as shown in Figures 7(C)-7(D). This situation has been observed in Pakistan, Malaysia, Turkey, Brazil, and Mexico as reported by John Hopkins University and Medicine on July 1, 2020. Furthermore, we observe the

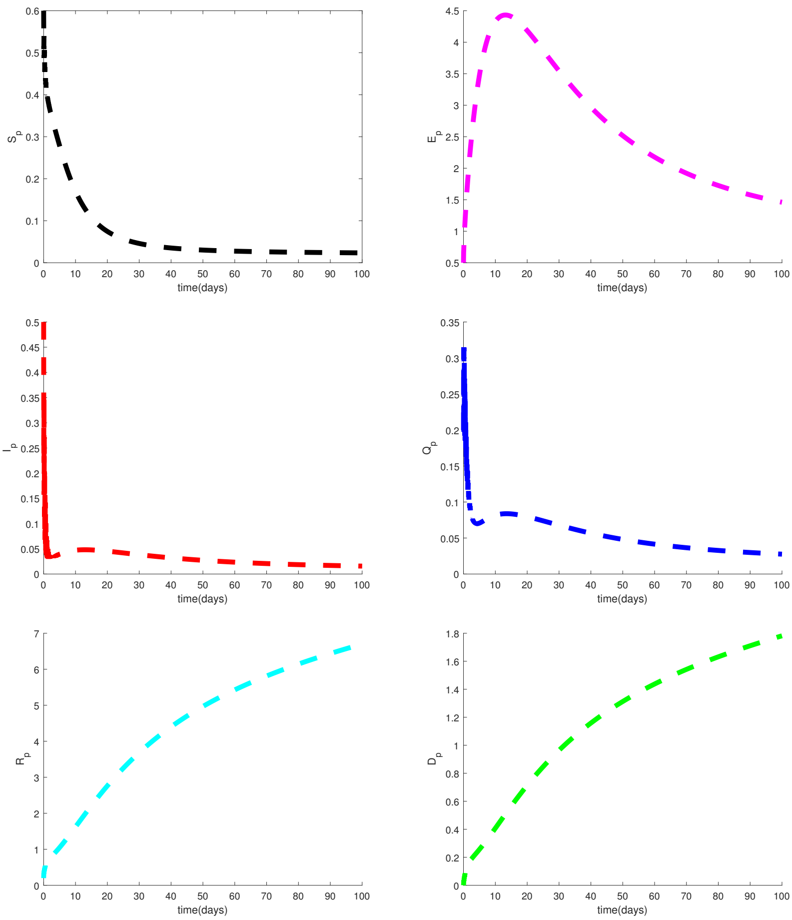
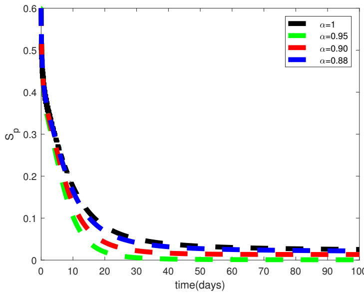
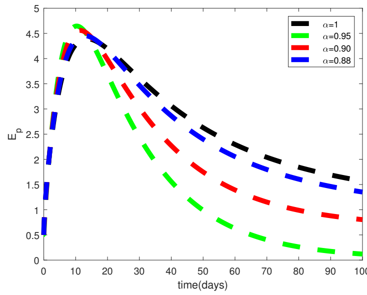


FIGURE 6. Profiles for behavior of each state variable for the ABC version of the fractional model.

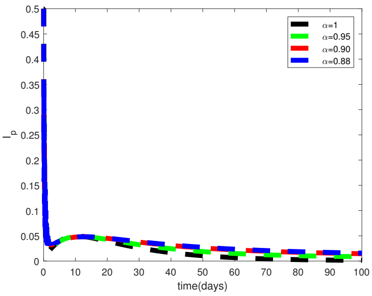
significant reduction in the number of recovered and dead individuals for smaller fractional orders as shown in Figures 7(E)-7(F). In this regard, it will be interesting to see various properties of the dynamic pattern of the COVID-19 model with different fractional-order $(0 < \alpha < 1)$ compared with the integer case $\alpha = 1$.



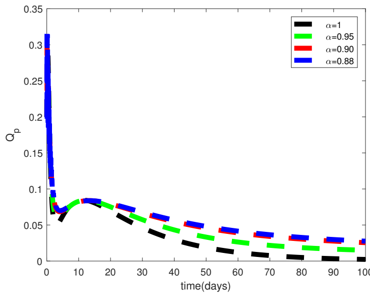
(A) Susceptible individuals with different values of α



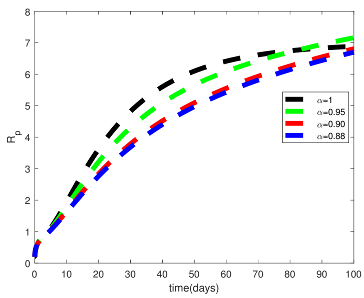
(B) Exposed individuals with different values of α



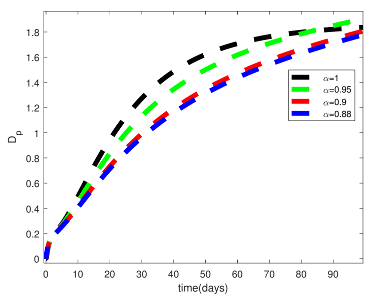
(C) Infected individuals with different values of α



(D) Quarantine individuals with different values of α



(E) Recovered individuals with different values of α



(F) Dead individuals with different values of α

FIGURE 7. Profiles for behavior of each state variable for different values of the fractional order.

8. CONCLUSION

In this paper, considering fractional order derivative due to Atangana and Baleanu we have proposed a mathematical model of novel coronavirus according to the situation of COVID-19 in Pakistan. We presented the existence and uniqueness of the related fractional differential equation of the model utilizing Schaefer's and Banach fixed point theorems respectively. Making use of Sumudu transform and Picard iterative procedure, we presented iterative solutions and proved the stability of iterative method. The proposed model was formulated in the framework of the ABC fractional operator. We also obtained some of the values of the unknown biological parameters of the modified SEIR model, which successfully capture the nCoV-19 pattern for the integer case $\alpha = 1$, based on real Pakistan data and best fitting techniques. In order to solve the proposed fractional model, we presented a numerical scheme and captured different graphical results which lead to a decrease in the infected class due to a decrease in the fractional order parameters.

Acknowledgments

The authors would like to thanks referees for useful comments and suggestions which lead to an extensive improvement of the manuscripts. These works were done while the second author visits Cankaya University, Ankara, Turkey.

Funding

Not Applicable

Availability of data and materials

All data generated or analyzed during this study are included in this published article.

Competing interests

The authors declare that they have no competing interests.

Author contributions

The authors contributed equally to this paper. All authors have read and approved the final version of the manuscript.

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