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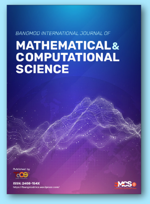

THE SOLUTION EXISTENCE OF INTEGRAL BOUNDARY VALUE PROBLEM INVOLVING NONLINEAR IMPLICIT CAPUTO FRACTIONAL DIFFERENTIAL EQUATIONS IN BANACH SPACES

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Abstract In this paper, we study and consider the following fractional integral boundary value problems:

$$
\begin{aligned}
{}^{c}D^{\alpha}u(t) &= f(t,u(t)), \qquad t \in J = [0,T] \\
u^{(k)}(0) &= \eta_k, \qquad u(T) = \tau \int_0^T u(s)ds
\end{aligned}
$$

where $n-1<\alpha<n$, $n\in\mathbb{N}$, $\eta_k\in\mathbb{R}$, $k=0,1,...,n-2$ and ${}^{c}D^{\alpha}$ is the Caputo fractional derivatives, $f: J\times C([0,T],E)\to E$ and $\tau<\frac{n}{T}$. By using the Darbo's fixed point theorem, we study the existence of this problem. An example is include to show the applicability of our results.

MSC: 74H10, 54H25
Keywords: Caputo fractional derivative; Integral boundary value problems; Existence of solution; Fixed point theorem

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1. Introduction

Fractional differential equations have received much attention recently. They arise in many engineering and scientific disciplines as the modelling of systems and processes in viscoelasticity, electrochemistry, control, porousmedia, electromagnetic, etc.(See [1–5].) A significant feature of a fractional order differential operator, in contrast to its counterpart in classical calculus, is its non local behaviour. It means that the future state of a dynamical system or process based on the fractional differential operator depends on its current state as well its past states. Therefore, many papers and books on fractional calculus, fractional differential equations and fractional integral equations have appeared. Qualitative theory of differential equations is very useful in applications. So, recently much attention has been focused on the study of the existence and multiplicity of solutions of positive solutions for boundary and initial value problems of fractional differential equations. There are many techniques to deal with the existence of solutions of fractional differential equations such as fixed point theorems[6–8], upper and lower solutions method [9], fixed point index [7, 10, 11], coincidence theory [12], etc. In [4, 5, 13], the authors considered the existence of solutions of the following initial value problems

$$
\begin{aligned}
&D^{\alpha}u(t) = f(t, u(t), D^{\beta}u(t)), \qquad t \in (0, 1],\\
&u^{(k)}(0) = \eta_k, \qquad k = 0, 1, ..., m-1,
\end{aligned}
$$

where $n-1 < \beta < \alpha < n$, $(n \in \mathbb{N})$, are the real number ${}^cD^{\alpha}, {}^cD^{\beta}$ are the Caputo fractional derivatives and $f \in C([0, 1] \times \mathbb{R})$ and [14]

$$
\begin{aligned}
{}^cD^{\alpha}_{0+}y(t) &= -f(t, y(t), D^{\alpha}y(t))\ t \in (0, 1],\ 1 < \alpha < 2\\
ay(0) - by'(0) &= 0,\ y(1) = \int_0^1 k(s)g(t, y(s))ds + \mu,
\end{aligned}
$$

where ${}^cD^{\alpha}_{0+}$ is the Caputo fractional derivative, $(E, \|\cdot\|)$ is real Banach space, $f : J \times C([0, 1], E) \times E \to E$. $g \in C(E, E)$, $k \in C([0, 1], E)$, $k \neq 0$,
motivated by the above works. In this paper, our object is to improve the situation. We consider

$$
\begin{cases}
{}^cD^{\alpha}u(t) = f(t, u(t), D^{\alpha}u(t)), \qquad t \in J = [0, T]\\
u^{(k)}(0) = \eta_k, \qquad u(T) = \tau \int_0^T u(s)ds,
\end{cases} \tag{1.1}
$$

where $n-1 < \alpha < n$, $n \in \mathbb{N}$, $\eta_k \in \mathbb{R}$, $k = 0, 1, ..., n-2$ and ${}^cD^{\alpha}$ is the Caputo fractional derivatives, $f : J \times C([0, T], E) \times E \to E$ and $\tau < \frac{n}{T}$. By using the Darbo's fixed point theorem, we study the existence of this problem. An example is include to show the applicability of our results.

The rest of the paper is organized as follows: In Section 2, we present some known results and introduce some conditions to be used in the next section. The main results formulated and proved in Section 3, also an example is presented to demonstrate the applications for guarantee of the main results.

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2. Background materials

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper.
Let $(E, \| \cdot \|)$ be a Banach space. We denoted by $C(J,E)$ the space of E-valued continuous function on J with the usual supremum norm

$$\| y \|_\infty = \sup\{\| y(t) \| : t \in J\} \text{ for every } y \in C(J,E).$$

Also a measurable function $y : J \to E$ is Bochner integrable if and only if $\| y \|$ is Lebesgue measure.
Let $L^1(J,E)$ denote the Banach space of measurable functions $y : J \to E$ which are Bochner integrable normed by

$$\| y \|_{L^1} = \int_1^T \| y(t) \| \, dt.$$

For properties of the Bochner integrable.

Definition 2.1. ([1, 3]). Let $u : (0,\infty) \to \mathbb{R}$ be a function and $\alpha > 0$. The Riemann-Liouville fractional integral of orders α of x is defined by

$$I^\alpha_{0+} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds$$

provided that the integral exists. The Caputo fractional derivative of order α of u is defined by

$${}^cD^\alpha_{0+} u(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} u^{(n)}(s) ds$$

provided that the right side is point wise defined on $(0,\infty)$, where $n = [\alpha]+1$, $n-1 < \alpha < n$, and Γ denotes the gamma function. If $\alpha = n$, then ${}^cD^\alpha_{0+} u(t) = u^{(n)}(t)$.

Lemma 2.2. ([1, 3]). Let $n > \alpha > n-1$. If $u \in C^n([a,b])$, then

$$I^\alpha({}^cD^\alpha u)(t) = u(t) + c_0 + c_1 t + c_2 t^2 + \cdots + C_{n-1} t^{n-1}$$

for some $c_i \in \mathbb{R}$, $i = 0,1,2,\cdots,n-1$, where n is the smallest integer greater than or equal to α.

Moreover, for a given set V of functions $v : J \to E$, let us denoted by $V(t) = \{v(t) : v \in V\}$, $t \in J$ and $V(J) = \{v(t) : v \in V, t \in J\}$. Next we give the definition of the concept of measure of noncompactness and some auxiliary result, see for more details [4, 8, 9] and the references therein.

Definition 2.3. ([15, 16]). Let E be a Banach space and Ω_E the bounded subsets of E. The Kuratowski measure of noncompactness is the map $\alpha : \Omega_E \to [0,\infty]$ defined by $\alpha(B) = \inf\{\epsilon > 0 : B \subseteq \bigcup_{i=1}^n B_i \text{ and } \operatorname{diam}(B_i) \leq \epsilon\}$, here

$$\operatorname{diam}(B_i) = \sup\{\| x-y \| : x, y \in B_i\}.$$

The Kuratowski measure of noncompactness satifies the following properties.

Lemma 2.4. ([15–17]). Let A and B are bounded sets,
(a) $\alpha(B) = 0 \leftrightarrow \bar{B}$ is compact (B is relatively compact), where $\bar{B}$ denoted the closure of B,
(b) nonsingularity; α is equal to zero on every one element set,
(c) $\alpha(B) = \alpha(\bar{B}) = \alpha(convB)$, where $convB$ is the convex hull of B,
(d) monotonicity; $A \subset B \to \alpha(A) \subset \alpha(B)$,
(e) algebraic semi-additively; $\alpha(A + B) \leq \alpha(A) + \alpha(B)$, where $A + B = \{x + y : x \in A, y \in B\}$,
(f) semi-homogencity; $\alpha(\lambda A) =\mid \lambda \mid \alpha(B)$, $\lambda \in \mathbb{R}$, where $\lambda B = \{\lambda x : x \in B\}$,
(g) semi-additivity; $\alpha(A \cup B) = \max\{\alpha(A), \alpha(B)\}$.
(h) invariance under translation; $\alpha(B + x_0) = \alpha(B)$ for any $x_0 \in E$.

For our purpose we will only need the following fixed point theorem and important lemma.

Theorem 2.5. *(Darbo's fixed point theorem)*([18]).
Let X be a Banach space and C be bounded, closed, convex and nonempty subset of X. Suppose a continuous mapping $N : C \to C$ is such that for all closed subsets D of C,

$$\alpha(N(D)) \leq k\alpha(D),$$

where $0 \leq k \leq 1$. Then N has a fixed point in C.

Lemma 2.6. ([19]) If $V \subset C(J, E)$ is a bounded and equicontionuous set, then
(a) the function $t \to \alpha(V(t))$ is continuous on J, and $\alpha_c(V) = \sup_{1\leq t\leq T} \alpha(v(t))$.
(b) $\alpha(\int_1^T x(s)ds : x \in V) \leq \int_1^T \alpha(V(s))ds$, where $V(s) = \{x(s); x \in V\}$, $s \in J$.

Lemma 2.7. *(Ascoli-Arzela)*([19]) Let $A \subset C(J, E)$, A is relatively compact (i.e. , $\bar{A}$ is compact) if;
(a) A is uniformly bounded, i.e., there exists $M > 0$ such that $\| f(t) \| \leq M$ for every $f \in A$ and $t \in J$.
(b) A is equicontinuous i.e., for every $\epsilon > 0$, there exists $\delta > 0$ such that for each $t, \bar{t} \in J$, $\mid t - \bar{t} \mid \leq \delta$ implies $\| f(t) - f(\bar{t}) \| \leq \epsilon$, for every $f \in A$.
(c) The set $\{f(t) : f \in A, t \in J\}$ is relatively compact in E.

3. Main Results

In this section we investigate the existence of solutions for the integral boundary value problem of nonlinear fractional differential equation. (1.1).

Definition 3.1. A function $u \in C(J, E)$ is said to be solution of (1.1), if u satisfies the equation ${}^cD^{\alpha}_{0+}u(t) = f(t, u(t), {}^cD^{\alpha}_{0+}u(t))$ on J, and the conditions $u^{(k)}(0) = \eta_k$, $u(T) = \tau \int_0^T u(s)ds$, $k = 0, 1, 2, \cdots, n-2$.

To prove the existence of solution to (1.1), we need the following auxiliary lemma.

Lemma 3.2. Let $u \in C([0, T])$ the linear fractional boundary value problem (BVP)

$$ {}^cD^{\alpha}u(t) = y(t), \qquad t \in J = [0, T],$$

$$u^{(k)}(0) = \eta_k, u(T) = \tau \int_0^T u(s)ds,$$

where $n-1<\alpha<n,\ n\in\mathbb{N},\ \eta_k\in\mathbb{R},\ k=0,1,...,n-2$ has a unique solution

$$\begin{aligned} u(t) &= \int_0^T G(t,s)y(s)ds \\ &+\sum_{k=0}^{n-2}\frac{\eta_k(T^{n-1}t^k-T^kt^{n-1}(n-T\tau)(k+1)+(T^{n-1}t^{k+1})n-T^kt^n(k+1))}{k!T^{n-1}(n-T\tau)(k+1)} \end{aligned}$$

where

$$G(t,s)=\begin{cases}\frac{((t-s)^{\alpha-1}(n-T\tau)Tn-1\alpha+t^{n-1}[n\tau(T-s-T\alpha)-\alpha(n-T\tau)](T-s)^{\alpha-1})}{(n-T\tau)T^{n-1}\alpha\Gamma(\alpha)} & \text{if } 0<s<t<T \\ \frac{t^{n-1}[n\tau(T-s-T\alpha)-\alpha(n-T\tau)](T-s)^{\alpha-1}}{(n-T\tau)T^{n-1}\alpha\Gamma(\alpha)} & \text{if } 0<t<s<T.\end{cases}$$

Proof. From Lemmma (2.2), we have

$$u(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}y(s)ds+c_0+c_1t+c_2t^2+\cdots+c_{n-1}t^{n-1}.$$

By the boundary condition in BVP (1.1), we have

$$c_0=\eta_0, c_1=\eta_1, c_2=\frac{\eta_2}{2!},\cdots,c_{n-2}=\frac{\eta_{n-2}}{(n-2)!}$$

so,

$$u(t)=\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}y(s)ds+\sum_{k=0}^{n-2}\frac{\eta_kt^k}{k!}+c_{n-1}t^{n-1}$$

and

$$c_{n-1}=\frac{1}{T^{n-1}}\Big(\tau\int_0^T u(s)ds-\frac{1}{\Gamma(\alpha)}\int_0^T(T-s)^{\alpha-1}y(s)ds-\sum_{k=0}^{n-2}\frac{\eta_kt^k}{k!}\Big).$$

Thus,

$$\begin{aligned} u(t) &= \frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}y(s)ds \\ &+\sum_{k=0}^{n-2}\frac{\eta_kt^k}{k!}\big(T^{n-1}t^k-T^kt^{n-1}\big) \\ &+\frac{t^{n-1}}{T^{n-1}\Gamma(\alpha)}\Big(\int_0^T\big(\tau\Gamma(\alpha)u(s)-(T-s)^{\alpha-1}y(s)\big)ds\Big) \\ \int_0^T u(t)dt &= \frac{1}{\Gamma(\alpha)}\int_0^T\int_0^t(t-s)^{\alpha-1}y(s)dsdt \\ &+\int_0^T\sum_{k=0}^{n-2}\frac{\eta_kt^k}{k!}\big(T^{n-1}t^k-T^kt^{n-1}\big)dt \\ &+\int_0^T\frac{t^{n-1}}{T^{n-1}\Gamma(\alpha)}\Big(\int_0^T\big(\tau\Gamma(\alpha)u(s)-(T-s)^{\alpha-1}y(s)\big)ds\Big)dt \end{aligned}$$

Let $\int_0^T u(s)ds = A$, we have

$$\begin{aligned}\int_0^T u(t)dt &= \frac{1}{\Gamma(\alpha)}\int_0^T \frac{(T-s)^{\alpha-1}}{\alpha}y(s)ds \\ &+\sum_{k=0}^{n-2}\frac{\eta_k}{k!T^{n-1}}\Big(\frac{T^{n-1}t^{k+1}}{k+1}-\frac{T^k t^n}{n}\Big)+\frac{T\tau A}{n}-\frac{T}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}y(s)ds \\ A &= \frac{n}{n-T\tau}\Big(\frac{1}{\Gamma(\alpha)}\int_0^T \frac{(T-s)^{\alpha}}{\alpha}y(s)ds-\frac{T}{\Gamma(\alpha)}\int_0^T (T-s)^{\alpha-1}ds\Big) \\ &+\frac{n}{n-T\tau}\sum_{k=0}^{n-2}\frac{\eta_k}{k!T^{n-1}}\Big(\frac{T^{n-1}t^{k+1}}{k+1}-\frac{T^k t^n}{n}\Big)\end{aligned}$$

so,

$$\begin{aligned}&u(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}y(s)ds \\ &+\frac{t^{n-1}}{(n-T\tau)T^{n-1}\alpha\Gamma(\alpha)}\int_0^T \Big(n\tau(T-s-T\alpha)-\alpha(n-T\tau)\Big)(T-s)^{\alpha-1}y(s)ds \\ &+\sum_{k=0}^{n-2}\frac{\eta_k}{k!T^{n-1}}\Big(\frac{(T^{n-1}t^k-T^k t^{n-1})(n-T\tau)(k+1)+(T^{n-1}t^{k+1}n-T^k t^n(k+1))}{(n-T\tau)(k+1)}\Big).\end{aligned}$$

Hence,

$$\begin{aligned}&u(t)=\int_0^T G(t,s)y(s)ds \\ &+\sum_{k=0}^{n-2}\frac{\eta_k}{k!T^{n-1}}\Big(\frac{(T^{n-1}t^k-T^k t^{n-1})(n-T\tau)(k+1)+(T^{n-1}t^{k+1}n-T^k t^n(k+1))}{(n-T\tau)(k+1)}\Big)\end{aligned}$$

∎

Remark 3.3. Obviously, the $G(t,s)$ function satisfies the following properties
(a) $G(t,s)>0,\ t,s\in[0,T]$.
(b) $G(t,s)\le \frac{T^{\alpha+n-2}(n-T\tau)\alpha+T^{n-1}n\tau}{(n-T\tau)T^{n-1}\alpha\Gamma(\alpha)},\ 0\le t,s\le T$.

First, we list the following hypotheses,
(H1) The function $f: J\times E\times E\to E$ are continuous.
(H2) There exists constants $K>0$ and $0<L<1$ such that

$$\| f(t,u,v)-f(t,\bar{u},\bar{v}) \|\le K\| u-\bar{u}\| +L\| v-\bar{v}\|$$

for any $u,\bar{u},v,\bar{v}\in E$ and $t\in J$. We are now in a position to state and prove our existence result for the problem (1.1) based on concept of measures of noncompactness and Darbo's fixed point theorem.

Theorem 3.4. *Suppose that (H1)-(H2) hold. If*

$$\frac{\Big(T^{\alpha+n-2}(n-T\tau)\alpha+T^{n-1}n\tau\Big)KT}{(n-T\tau)T^{n-1}\alpha\Gamma(\alpha)(1-L)}<1 \tag{3.1}$$

then, the BVP (1.1) has at least one solution on J.

Proof. Transform the problem (1.1) into a fixed point problem. Consider the operator $F : C(J, E) \to C(J, E)$ defined by

$$Fu(t) = \int_0^T G(t,s)y(s)ds$$
$$+\sum_{k=0}^{n-2} \frac{\eta_k}{k!T^{n-1}}\Big(\frac{(T^{n-1}t^k - T^k t^{n-1})(n-T\tau)(k+1) + (T^{n-1}t^{k+1}n - T^k t^n(k+1))}{(n-T\tau)(k+1)}\Big)$$

where $y \in C(J, E)$ be such that $y(t) = f(t, u(t), y(t))$. Clearly, if operator F has a fixed point if and only if the (1.1) has a solution. We shall show that F satisfies the assumption of Darbo's fixed point theorem. The proof will be give in several claims.

Claim 1: F is continuous.

Let $\{u_n\}$ be a sequence such that $u_n \to u$ in $C(J, E)$. If $t \in J = [0, T]$, we have

$$\| F(u_n)(t) - F(u)(t) \| \leq \int_0^T | G(t,s) | \| y_n(s) - y(s) \| ds \tag{3.2}$$

where $y_n, y \in C(J, E)$, such that $y_n(t) = f(t, u_n(t), y_n(t)), y(t) = f(t, u(t), y(t))$. By (H2), we have

$$\begin{aligned} \| y_n(t) - y(t) \| &= \| f(t, u_n(t), y_n(t)) - f(t, u(t), y(t)) \| \\ &\leq K \| u_n - u \| + L \| y_n(t) - y(t) \| \\ \| y_n(t) - y(t) \| &\leq \frac{K}{1-L} \| u_n - u \| . \end{aligned}$$

Since $u_n \to u$, then we get $y_n(t) \to y(t)$ as $n \to \infty$ for each $t \in J$. And let $\mu > 0$ such that for each $t \in J$, we have $\| y_n(t) \| \leq \mu, \| y(t) \| \leq \mu$. Then we have

$$\begin{aligned} G(t,s) \| y_n(s) - y(s) \| &\leq \frac{T^{\alpha+n-2}(n-T\tau)\alpha + T^{n-1}n\tau}{(n-T\tau)T^{n-1}\alpha\Gamma(\alpha)}(\| y_n(s) \| + \| y(s) \|) \\ &\leq \frac{2\mu[T^{\alpha+n-2}(n-T\tau)\alpha + T^{n-1}n\tau]}{(n-T\tau)T^{n-1}\alpha\Gamma(\alpha)} \end{aligned}$$

For each $t \in J$, the function $\frac{2\mu[T^{\alpha+n-2}(n-T\tau)\alpha+T^{n-1}n\tau]}{(n-T\tau)T^{n-1}\alpha\Gamma(\alpha)}$ is integrable on $[0, t]$, then the Lebesgue Dominaled convergence theorem and (3.2) imply that

$$\| F(y_n)(t) - Fy(t) \| \to 0, \ as \ n \to \infty$$

and hence,

$$\| F(y_n)(t) - Fy(t) \|_{[0,T]} \to 0, \ as \ n \to \infty.$$

Consequently, F is continuous. Let the constant R such that

$$\frac{\Delta f^* T + | \Omega | (1-L)}{1 - L - \Delta KT} \leq R$$

where $f^* = \sup_{t\in J} \| f(t, 0, 0) \|$, $\Delta := \frac{T^{\alpha+n-2}(n-T\tau)\alpha+T^{n-1}n\tau}{(n-T\tau)T^{n-1}\alpha\Gamma(\alpha)}$,

$\Omega := \sum_{k=0}^{n-2} \frac{\eta_k}{k!T^{n-1}}\Big(\frac{(T^{n-1}t^k - T^k t^{n-1})(n-T\tau)(k+1) + (T^{n-1}t^{k+1}n - T^k t^n(k+1))}{(n-T\tau)(k+1)}\Big)$ define $D_R = \{u \in C(J, E) :\| u \| \leq R\}$. It is clear that D_R is bounded, closed and convex subset of $C(J, E)$.

Claim 2: $F(D_R) \subset D_R$.
Let $u \in D_R$, we show that $Fu \in D_R$. If $t \in [0, T]$ then $\parallel Fu(t) \parallel$ that is, we have

$$\begin{aligned} \parallel Fu(t) \parallel &= \max_{t\in[0,T]} \mid \int_0^T G(t,s)y(s)ds + \Omega \mid \\ &\leq \int_0^T \Delta \parallel y(s) \parallel ds + \mid \Omega \mid . \end{aligned}$$

By (H2), we have for each $t \in J$

$$\begin{aligned} \parallel y(t) \parallel &= \parallel f(t,u,y(t)) - f(t,0,0) + f(t,0,0) \parallel \\ &\leq \parallel f(t,u,y(t)) - f(t,0,0) \parallel + \parallel f(t,0,0) \parallel \\ &\leq K \parallel u \parallel + L \parallel y(t) \parallel + f^* \end{aligned}$$

where $f^* = \parallel f(t,0,0) \parallel, \parallel u \parallel \leq R,$

$$\parallel y(t) \parallel \leq \frac{KR + f^*}{1-L} := A$$

thus,

$$\parallel Fu(t) \parallel \leq \Delta \frac{(KR + f^*)T}{1-L} + \mid \Omega \mid \leq R.$$

Form which it follows that for each $t \in [0, T]$, we have $\parallel Fu(t) \parallel \leq R$, which implies that $F(D_R) \subset D_R$.
Claim 3: $F(D_R)$ is bounded and equicontinuous.
By Claim 2, we have $F(D_R) = \{F(u) : u \in D_R\} \subset D_R$. Thus for each $u \in D_R$, we have $\parallel Tu \parallel_{[0,T]} \leq R$, which means that $T(D_R)$ is bounded. Let $t_1, t_2 \in [0, T], t_1 < t_2$ and $u \in D_R$. Then,

$$\begin{aligned} \parallel Tu(t_2) - Tu(t_1) \parallel &\leq \parallel \int_0^T \Big(G(t_2,s) - G(t_1,s)\Big) y(s)ds \\ &\quad + \sum_{k=0}^{n-2} \frac{\eta_k \Big((n - T\tau)(k+1)(t_2^k - t_1^k) + n(t_2^{k+1} - t_1^{k+1})\Big)}{k!(n - T\tau)(k+1)} \parallel \\ &\leq A \int_0^T \Big(G(t_2,s) - G(t_1,s)\Big) ds \\ &\quad + \sum_{k=0}^{n-2} \frac{\mid \eta_k \mid \Big((n - T\tau)(k+1)(t_2^k - t_1^k) + n(t_2^{k+1} - t_1^{k+1})\Big)}{k!(n - T\tau)(k+1)} \end{aligned}$$

As $t_2 \to t_1$, the right hand side tends to zero. Hence $F(D_R)$ is equicontinuous.
Claim 4: The operator $F : D_R \to D_R$ is a strict set contraction. Let $U \subset D_R$, if $t \in J$, we have

$$\begin{aligned} \alpha(Fu(t)) &= \alpha((Fu)(t), u \in U) \\ &\leq \alpha\Big(\int_0^T G(t,s)y(s)ds + \Omega, u \in U\Big) \end{aligned}$$

then, by Lemma 2.4 implies that for each $s \in J$

$$\begin{aligned}\alpha(\{y(s), u \in U\}) &= \alpha(\{f(s, u(s), y(s)), u \in U\}) \\ &\leq K\alpha\{u(s), u \in U\} + L\alpha\{y(s), u \in U\} \\ \alpha(\{y(s), u \in U\}) &\leq \frac{K}{1-L}\alpha\{u(s), u \in U\}.\end{aligned}$$

Then,

$$\begin{aligned}\alpha(FU(t)) &\leq \frac{K}{1-L}\int_0^T G(t,s)ds\{\alpha(u(s)), u \in U\} \\ &\leq \frac{\Delta KT}{1-L}\alpha(U).\end{aligned}$$

Therefore,

$$\alpha_c(TU) \leq \frac{\Delta KT}{1-L}\alpha_c(U).$$

So, the oprator F is set contraction. As a consequence of Theorem Darbo's we deduce that F has fixed point which is solution to the problem (1.1) ∎

Example 3.5. Consider the following fractional boundary value problems

$$\begin{cases} {}^cD_{0+}^{\frac{7}{2}}u(t) = \frac{u}{100}\cos t + \frac{t^2}{10}\,{}^cD_{0+}^{\frac{7}{2}}u(t) + 1, \ t \in [0, e], \\ u(0) = -1, \ u'(0) = 0, \ u''(0) = 1, \ u(e) = \frac{1}{3}\int_0^e u(s)ds, \end{cases} \tag{3.3}$$

where

$$f(t, u, v) = \frac{u}{100}\cos t + \frac{t^2}{10}\,{}^cD_{0+}^{\frac{7}{2}}u(t) + 1.$$

For any $u, \bar{u}, v, \bar{v} \in E$ and $t \in [0, e]$

$$\begin{aligned}\| f(t, u, v) - f(t, \bar{u}, \bar{v}) \| &\leq \frac{1}{100} | \cos t | \| u - \bar{u} \| + \frac{| t^2 |}{10} \| v - \bar{v} \| \\ &\leq \frac{1}{100} \| u - \bar{u} \| + \frac{e^2}{10} \| v - \bar{v} \| .\end{aligned}$$

Hence condition (H1) and (H2) are satisfied with $K = \frac{1}{5}, L = \frac{e^2}{10}$ and condition

$$\Big(\frac{T^{\alpha+n-2}(n - T\tau)\alpha + T^{n-1}n\tau}{(n - T\tau)T^{n-1}\alpha\Gamma(\alpha)(1 - L)}\Big)KT < 1$$

are satisfied with $\alpha = \frac{7}{2}$, $\tau = \frac{1}{3}$. It follows from (3.1) that the problem (3.3) has at least one solution.

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