BANGMOD INTERNATIONAL JOURNAL OF MATHEMATICAL & COMPUTATIONAL SCIENCE
Volume. 7 Number. 1 & 2 (2021)
Pages 53–73
ISSN: 2408-154X
https://bangmodmcs.wordpress.com/

AN ITERATIVE METHOD WITH INERTIAL EFFECT FOR SOLVING MULTIPLE-SETS SPLIT FEASIBILITY PROBLEM

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Received: 10 November 2021 / Accepted: 23 December 2021

Abstract An iterative method with inertial extrapolation term for approximating the solution of multiple-sets split feasibility problem in the infinite-dimensional Hilbert spaces is presented. In a recent paper, Ogbuisi and Mewomo [1] introduced an iterative algorithm involving an inertial term and a step size independent of the operator norm for approximating a solution to split variational inequality problem in a real Hilbert space. We extend the algorithm introduced by Ogbuisi and Mewomo [1] for solving multiple-set split feasibility problem, and we propose a self-adaptive technique to choose the stepsizes such that the implementation of our algorithm does not need prior information about the operator norm. We prove a weak convergence theorem to the proposed algorithm under some suitable conditions. Finally, we give some numerical examples to illustrate the efficiency and implementation of our method compared to some existing results.

MSC: 47H09, 65J15, 65K05, 65K10, 49J52
Keywords: Multiple-sets split feasibility problem; Sublevel set; Inertial term; weak convergence

1. Introduction

Let H_1 and H_2 be two real Hilbert spaces. Let $A : H_1 \to H_2$ be a bounded linear operator. Let C and Q be two nonempty, closed and convex subsets of H_1 and H_2, respectively. The *split feasibility problem (SFP)* introduced by Censor and Elfving [2] is formulated as follows:

$$\text{find a point } p^* \in C \text{ such that } Ap^* \in Q. \tag{1.1}$$

The SFP was first introduced in 1994 in finite-dimensional Hilbert spaces for modeling inverse problems which arise from phase retrievals and in medical image reconstruction. The SFP has broad theoretical applications in many fields such as approximation theory [3], control [4], and so on, and the references therein. It also plays a fundamental role in signal processing, intensity-modulated radiation therapy etc., see, e.g., [5–8]. Several iterative methods including the popular and most celebrated *CQ-method* by Byrne [8] have been introduced for solving the SFP, see [2, 6, 8–15] and the references therein.

Due to it's practical applications, the SFP has received a great attention by many researchers, and several generalizations of the SFP have been studied, like, the multiple-set SFP (MSSFP) [16], the SFP with multiple output sets [17], the split common null point problem (SCNPP) [18], the split common fixed point problem (SCFPP) [19], the split variational inequality problem (SVIP) [20], just to mention a few.

Let H_1 and H_2 be two real Hilbert spaces. Given operators $f : H_1 \to H_1$ and $g : H_2 \to H_2$, a bounded linear operator $A : H_1 \to H_2$, and nonempty, closed and convex subsets $C \subset H_1$ and $Q \subset H_2$, the *split variational inequality problem (SVIP)* [20] is formulated as follows:

$$\text{find a point } p^* \in C \text{ such that } \langle f(p^*), x - p^* \rangle \geq 0 \ \forall x \in C \tag{1.2}$$

$$\text{and such that}$$

$$\text{the point } y^* = Ap^* \in Q \text{ and solves } \langle g(y^*), y - y^* \rangle \geq 0 \ \forall y \in Q. \tag{1.3}$$

Recently, Ogbuisi and Mewomo [1] introduced the following iterative algorithm involving an inertial term and a step size independent of the operator norm for approximating solutions of the above SVIP in a real Hilbert space: Let $x_0, x_1 \in H_1$, then the sequence $\{x_n\}$ is generated by

$$\begin{cases} y_n = x_n + \beta_n(x_n - x_{n-1}), \\ z_n = P_C(I - \rho f)(y_n + \tau_n A^*(P_Q(I - \rho g) - I)Ay_n), \\ x_{n+1} = (1 - \alpha_n)y_n + \alpha_n z_n, \ n \geq 1, \end{cases} \tag{1.4}$$

where $\{\beta_n\}$ and $\{\alpha_n\}$ are a non-decreasing and non-increasing sequences $(0, 1)$, respectively, $f : H_1 \to H_1$ and $g : H_2 \to H_2$ are δ- and σ-inverse strongly monotone mappings,

Published online: 31 December 2021

Published by Center of Excellence in Theoretical and Computational Science (TaCS-CoE)

Please cite this article as: G. H. Taddele, et al., An iterative method with inertial effect for solving multiple-set split feasibility problem, Bangmod Int. J. Math. & Comp. Sci., Vol. 7 No. 1 & 2 (2021) 53–73.

respectively, $0 < \rho < \min\{\delta, \sigma\}$, the step size τ_n is chosen in such a way that, for some $\epsilon > 0$, $\tau_n \in \left(\epsilon, \frac{\|(P_Q(I-\rho g)-I)Ay_n\|^2}{\|A^*(P_Q(I-\rho g)-I)Ay_n\|^2} - \epsilon\right)$ if $P_Q(I-\rho g)-I)Ay_n \neq 0$, and $\tau_n = \tau$ otherwise (τ being any nonnegative real number), I stands for the identity mapping in H_1 and H_2, and P_C and P_Q are the metric projections of H_1 and H_2 onto C and Q, respectively. It was proved that, under some suitable conditions the sequence $\{x_n\}$ generate by (1.4) converges weakly to a solution point of the SVIP.

In this paper, we study the *multiple-sets split feasibility problem (MSSFP)* which is a general way to characterize various inverse problems arising in many real-world application problems, such as medical image reconstruction and intensity-modulated radiation therapy. *The MSSFP, which was introduced by Censor et al. [16] requires finding a point closet to a family of closed convex sets in one space such that its image under a linear transformation will be closest to another family of closed convex sets in the image space.*

Let H_1 and H_2 be two real Hilbert spaces. Let $A : H_1 \to H_2$ be a bounded linear operator with adjoint operator A^*. Let $\{C_i\}_{i=1}^r$ and $\{Q_j\}_{j=1}^s$ (where r and s are positive integers) be two finite family of nonempty, closed and convex subsets of H_1 and H_2, respectively, the *MSSFP* is formulated as follows:

$$\text{find a point } p^* \in C = \bigcap_{i=1}^{r} C_i \text{ such that } Ap^* \in Q = \bigcap_{j=1}^{s} Q_j. \tag{1.5}$$

The MSSFP (1.5) with particular case where $r = s = 1$ is the SFP (1.1). Denote by Ω the set of solutions for the MSSFP (1.5). For solving the MSSFP (1.5), several iterative methods have been invented, see, e.g., [21–31] and references therein. Initiated by the self-adaptive strategy given by He et al. [32] to solve the variational inequalities, Zhang et al. [33] suggested a self-adaptive projection method for solving the MSSFP, which has no need to estimate the spectral radius of the matrix A^*A. Inspired by Tseng's modified forward-backward splitting method for finding a zero of the sum of two maximal monotone mappings [34], recently, Zhao et al. [35], first proposed a modification for the CQ Algorithm for solving the SFP. Then, they gave a relaxation scheme for this modification by replacing the orthogonal projections onto the sets C and Q by projections onto two half-spaces C_n and Q_n, respectively for solving the SFP. This relaxed algorithm can be implemented easily since it computes projections onto half-spaces and has no need to know a prior the spectral radius of the matrix A^*A. They also extend these modified algorithms to solve the MSSFP.

In this paper, we extend the algorithm (scheme (1.4) for the SVIP) introduced by Ogbuisi and Mewomo [1] for solving MSSFP, and we propose an iterative algorithm with inertial extrapolation term for approximating the solution of the MSSFP in the framework of infinite-dimensional Hilbert spaces, and we develop a self-adaptive technique to choose the stepsizes such that the implementation of our algorithm does not need any prior information about the operator norm. We prove a weak convergence theorem to the proposed algorithm under some conditions.

The remaining part of this paper is organized as follows. We recall some basic definitions and lemmas in the next section. In Section 3, we present the description of the proposed algorithm, and weak convergence of the iterative algorithm for the MSSFP is proved. In the last section, we give a numerical example to illustrate the implementation and efficiency of our proposed method compared to some existing algorithms.

2. Preliminaries

Let H be a real Hilbert space. Let C be a nonempty closed convex subset of H. The *metric projection* on C is a mapping $P_C : H \to C$ defined by

$$P_C(x) = \arg\min\{\|y - x\| : y \in C\}, \quad x \in H.$$

Lemma 2.1. *Let C be a closed convex subset of H. Given $x \in H$ and a point $z \in C$, $z = P_C(x)$ if and only if*

$$\langle x - z, y - z\rangle \leq 0, \quad \forall y \in C.$$

Definition 2.2. The mapping $T : H \to H$ is said to be *firmly nonexpansive* if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 - ||(I - T)x - (I - T)y\|^2, \quad \forall x, y \in H,$$

which is equivalent to

$$\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y\rangle, \quad \forall x, y \in H.$$

If T is firmly nonexpansive, $I - T$ is also firmly nonexpansive. The metric projection P_C on a closed convex subset C of H is firmly nonexpansive.

Definition 2.3. The *subdifferential* of a convex function $f : H \to \mathbb{R}$ at $x \in H$, denote by $\partial f(x)$, is defined by

$$\partial f(x) = \{\xi \in H : f(z) \geq f(x) + \langle \xi, z - x\rangle, \quad \forall z \in H\}.$$

If $\partial f(x) \neq \emptyset$, f is said to be *subdifferentiable* at x. If the function f is continuously differentiable then $\partial f(x) = \{\nabla f(x)\}$, this is the gradient of f.

Definition 2.4. The function $f : H \to \mathbb{R}$ is called *weakly lower semi-continuous* at x_0 if for a sequence $\{x_n\}$ weakly converging to x_0 one has

$$\liminf_{n\to\infty} f(x_n) \geq f(x_0).$$

A function which is weakly lower semi-continuous at each point of H is called *weakly lower semi-continuous* on H.

Lemma 2.5. ([36]) *Let H_1 and H_2 be real Hilbert spaces and $f : H_1 \to \mathbb{R}$ be given by $f(x) = \frac{1}{2}\|(I - P_Q)Ax\|^2$ where Q is a closed convex subset of H_2 and $A : H_1 \to H_2$ is a bounded linear operator. Then*

(i): *the function f is convex and weakly lower semi-continuous on H_1;*
(ii): $\nabla f(x) = A^*(I - P_Q)Ax$, *for $x \in H_1$;*
(iii): ∇f *is $\|A\|^2$-Lipschitz, i.e.,* $\|\nabla f(x) - \nabla f(y)\| \leq \|A\|^2\|x - y\|, \quad \forall x, y \in H_1$.

Lemma 2.6. ([37]) *Let H be a real Hilbert space. Then, for all $x, y \in H$ and $\alpha \in \mathbb{R}$,, we have*

(i): $\|\alpha x + (1 - \alpha)y\|^2 = \alpha\|x\|^2 + (1 - \alpha)\|y\|^2 - \alpha(1 - \alpha)\|x - y\|^2$;
(ii): $\|x + y\|^2 = \|x\|^2 + \|y\|^2 + 2\langle x, y\rangle$;
(iii): $\|x - y\|^2 = \|x\|^2 + \|y\|^2 - 2\langle x, y\rangle$.

Lemma 2.7. ([38]) *Let $\{c_n\}$ and $\{\alpha_n\}$ be a sequences of nonnegative real numbers, $\{\beta_n\}$ be a sequences of real numbers such that*

$$c_{n+1} \leq (1 - \alpha_n)c_n + \beta_n, \quad n \geq 1,$$

where $0 < \alpha_n < 1$.

(i): *If $\beta_n \leq \alpha_n L$ for some $L \geq 0$, then $\{c_n\}$ is a bounded sequence.*

(ii): *If* $\sum \alpha_n = \infty$ *and* $\limsup_{n\to\infty} \frac{\beta_n}{\alpha_n} \leq 0$, *then* $c_n \to 0$ *as* $n \to \infty$.

Definition 2.8. Let $\{\Gamma_n\}$ be a real sequence. Then, $\{\Gamma_n\}$ decrease at infinity if there exists $n_0 \in \mathbb{N}$ such that $\Gamma_{n+1} \leq \Gamma_n$ for $n \geq n_0$. In other words, the sequence $\{\Gamma_n\}$ does not decrease at infinity, if there exists a subsequence $\{\Gamma_{n_t}\}_{t\geq 1}$ of $\{\Gamma_n\}$ such that $\Gamma_{n_t} < \Gamma_{n_t+1}$ for all $t \geq 1$.

Lemma 2.9. ([39]) *Let* $\{\Gamma_n\}$ *be a sequence of real numbers that does not decrease at infinity. Also consider the sequence of integers* $\{\varphi(n)\}_{n\geq n_0}$ *defined by*

$$\varphi(n) = \max\{k \in \mathbb{N} : k \leq n, \Gamma_k \leq \Gamma_{k+1}\}.$$

Then $\{\varphi(n)\}_{n\geq n_0}$ *is a nondecreasing sequence verifying* $\lim_{n\to\infty} \varphi(n) = \infty$, *and for all* $n \geq n_0$, *the following two estimates hold:*

$$\Gamma_{\varphi(n)} \leq \Gamma_{\varphi(n)+1} \text{ and } \Gamma_n \leq \Gamma_{\varphi(n)+1}.$$

Lemma 2.10. ([40]) *Let* $\{\zeta_n\}$ *and* $\{\pi_n\}$ *be nonnegative sequences of real numbers satisfying* $\sum_{n=1}^{\infty} \zeta_n \leq \infty$ *and* $\pi_{n+1} \leq \pi_n + \zeta_n, n = 1, 2, \ldots$. *Then,* $\{\pi_n\}$ *is a convergent sequence.*

Lemma 2.11. ([41]) *Assume* $\phi_n \in [0, \infty)$ *and* $\gamma_n \in [0, \infty)$ *satisfy*

(i): $\phi_{n+1} - \phi_n \leq \beta_n(\phi_n - \phi_{n-1}) + \gamma_n$,

(ii): $\sum_{n=1}^{\infty} \gamma_n < \infty$,

(iii): $\{\beta_n\} \subset [0, \beta]$, *where* $\beta \in [0, 1)$.

Then, the sequence $\{\phi_n\}$ *is convergent with* $\sum_{n=1}^{\infty} [\phi_{n+1} - \phi_n]_+ < \infty$, *where* $[t]_+ := max\{t, 0\}$, *for any* $t \in \mathbb{R}$.

Lemma 2.12. ([42, 43]) *Let* H_1 *be a real Hilbert space and* $\{x_n\}$ *a sequence in* H_1 *such that there exists a nonempty closed set* $\Gamma \subset H_1$ *satisfying:*

(i): *For every* $x^* \in \Gamma$, $\lim_{n\to\infty} \|x_n - x^*\|$ *exists.*

(ii): *Any weak-cluster point of the sequence* $\{x_n\}$ *belongs in* Γ.

Then, there exists a point $\hat{x} \in \Gamma$ *such that* $\{x_n\}$ *weakly converges to* $\hat{x}$.

In this paper, the symbol " $\rightharpoonup$ " and " $\to$ " stands for the weak and strong convergence, respectively.

3. The Iterative Algorithm

In this section, we present convergence result using the inertial extrapolation method for solving the MSSFP (1.5), which is the main result of this work.

We are interested to solve the MSSFP in which the involved sets $\{C_i\}_{i=1}^r$ and $\{Q_j\}_{j=1}^s$ are given as sublevel sets of convex functions, i.e.,

$$C_i = \{x \in H_1 : c_i(x) \leq 0\} \text{ and } Q_j = \{y \in H_2 : q_j(y) \leq 0\}, \tag{3.1}$$

where $c_i : H_1 \to \mathbb{R}$ and $q_j : H_2 \to \mathbb{R}$ are convex functions for all $i \in \{1, \ldots, r\}$ and for all $j \in \{1, \ldots, s\}$. We assume that both c_i and q_j are subdifferentiable on H_1 and H_2, respectively, and that ∂c_i and ∂q_j are bounded operators. Now, we define the following half-spaces at point x_n by

$$C_{i,n} = \{x \in H_1 : c_i(x_n) \leq \langle \xi_{i,n}, x_n - x\rangle\}, \tag{3.2}$$

where $\xi_{i,n} \in \partial c_i(x_n)$, and

$$Q_{j,n} = \{y \in H_2 : q_j(Ax_n) \leq \langle \varepsilon_{j,n}, Ax_n - y\rangle\}, \tag{3.3}$$

where $\varepsilon_{j,n} \in \partial q_j(Ax_n)$.

For approximating a solution of the MSSFP, we present an iterative algorithm with extrapolated point assuming C_i and Q_j are given as sublevel sets of convex functions (3.1) where the projection onto half-spaces (3.2) and (3.3) is computed in parallel and prior knowledge of the operator norm is not required. We introduce the extended form of the way of selecting stepsize used by Lopez et al. [15] and we analyze the weak convergence of our proposed algorithm. For this purpose, we define the following settings: for $x_n \in H_1$,

(1): for each $i \in \{1, \dots, r\}$ and $n \geq 1$, define

$$p_{i,n}(x_n) = \frac{1}{2}\|(I - P_{C_{i,n}})x_n\|^2 \text{ and } \nabla p_{i,n}(x_n) = (I - P_{C_{i,n}})x_n,$$

(2): $p(x_n)$ and $\nabla p(x_n)$ are defined as $p(x_n) = p_{i_x,n}(x_n)$ and so $\nabla p(x_n) = \nabla p_{i_x,n}(x_n)$ where $i_x \in \{1, \dots, r\}$ is such that for each $n \geq 1$,

$$i_{x_n} \in \arg\max\{p_{i,n}(x_n) : i \in \{1, \dots, r\}\},$$

(3): for each $j \in \{1, \dots, s\}$ and $n \geq 1$, define

$$h_{j,n}(x_n) = \frac{1}{2}\|(I - P_{Q_{j,n}})Ax_n\|^2 \text{ and } \nabla h_{j,n}(x_n) = A^*(I - P_{Q_{j,n}})Ax_n.$$

We note that $p_{i,n}$ and $h_{j,n}$ are convex, weakly lower semi-continuous, and differentiable functions ([36]). Assuming the solution set Ω of the MSSFP (1.5) is nonempty, the suggested algorithm is given as follows.

Algorithm 1: Inertial Relaxed Algorithm for solving MSSFP

Let $\{\beta_n\}$, $\{\gamma_{j,n}\}_{j=1}^s$, $\{\rho_n\}$, $\{\alpha_n\}$ be real sequences. Choose initial points x_0, $x_1 \in H_1$, assume that the current iterate $\{x_n\}$ has been constructed, and $\nabla p(y_n) + \nabla h_{j,n}(y_n) \neq 0$. Then, compute $\{x_{n+1}\}$ via the rule

$$\begin{cases} y_n = x_n + \beta_n(x_n - x_{n-1}), \\ z_n = y_n - \sum\limits_{j=1}^{s} \Big\{\gamma_{j,n}\tau_{j,n}(\nabla h_{j,n}(y_n) + \nabla p(y_n))\Big\}, \\ x_{n+1} = (1 - \alpha_n)y_n + \alpha_n z_n, \end{cases} \tag{3.4}$$

where $\tau_{j,n} = \rho_n \frac{h_{j,n}(y_n)+p(y_n)}{\theta_j^2(y_n)}$, and $\theta_j(y_n) = \max\{1, \|\nabla p(y_n) + \nabla h_{j,n}(y_n)\|\}$.

The term $\beta_n(x_n - x_{n-1})$ appeared in Algorithm 1 is the inertial term with an extrapolation factor β_n. It is remarkable that the inertial terminology greatly improves the performance of the algorithm and has a nice convergence properties [44–46].

Remark 3.1. In Algorithm 1 above, if $\nabla h_{j,n}(y_n) = \nabla p(y_n) = 0$ and $y_n = x_n$, $j \in \{1, \dots, s\}$, then $x_n \in \Omega$ and the iterative process stops. Otherwise, set $n := n + 1$ and repeat the iteration.

Theorem 3.2. *Let $\{\alpha_n\}$ be a non-increasing real sequence in $(0, 1)$ and $\{\beta_n\}$ be a non-decreasing real sequence in $(0, 1)$. If the parameters $\{\beta_n\}$, $\{\gamma_{j,n}\}_{j=1}^s$, $\{\rho_n\}$, $\{\alpha_n\}$ in Algorithm 1 satisfy the following conditions:*

(C1): $0 < \alpha \leq \alpha_n \leq \frac{1}{2}$;

(C2): $0 \leq \beta_n \leq \frac{1-m}{3} < \frac{1}{3}$, *for some* $m \in (0, 1)$;

(C3): $0 < \liminf_{n\to\infty} \gamma_{j,n} \le \limsup_{n\to\infty} \gamma_{j,n} < 1$, $\forall j \in \{1, \dots, s\}$, *and* $\sum_{j=1}^{s} \gamma_{j,n} = 1$;

(C4): $0 < \rho_n < 4$ *and* $\liminf_{n\to\infty} \rho_n(4-\rho_n) > 0$;

then the sequence $\{x_n\}$ *generated by Algorithm 1 converges weakly to* $p^* \in \Omega$.

Proof. **Claim 1:** For every $p^* \in \Omega$, $\lim_{n\to\infty} \|x_n - p^*\|$ exists.
Let $p^* \in \Omega$. Since $I - P_{C_{i,n}}$ and $I - P_{Q_{j,n}}$ are firmly nonexpansive, and since p^* verifies (1.5), we have for all $x \in H_1$

$$\begin{aligned} \langle \nabla p_{i,n}(x), x - p^* \rangle &= \langle (I - P_{C_{i,n}})x, x - p^* \rangle \\ &\ge \|(I - P_{C_{i,n}})x\|^2 = 2p_{i,n}(x) \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} \langle \nabla h_{j,n}(x), x - p^* \rangle &= \langle A^*(I - P_{Q_{j,n}})Ax, x - p^* \rangle \\ &= \langle (I - P_{Q_{j,n}})Ax, Ax - Ap^* \rangle \\ &\ge \|(I - P_{Q_{j,n}})Ax\|^2 = 2h_{j,n}(x). \end{aligned} \tag{3.6}$$

Now from the definition of y_n, we get

$$\begin{aligned} \|y_n - p^*\| &= \|x_n + \beta_n(x_n - x_{n-1}) - p^*\| \\ &\le \|x_n - p^*\| + \beta_n \|x_n - x_{n-1}\|. \end{aligned} \tag{3.7}$$

From (3.4) and Lemma 2.6 (ii), we obtain

$$\begin{aligned} \|z_n - p^*\|^2 &= \Big\| y_n - \sum_{j=1}^{s} \Big\{ \gamma_{j,n}\tau_{j,n}(\nabla h_{j,n}(y_n) + \nabla p(y_n)) \Big\} - p^* \Big\|^2 \\ &\quad - 2\Big\langle \sum_{j=1}^{s} \Big\{ \gamma_{j,n}\tau_{j,n}(\nabla h_{j,n}(y_n) + \nabla p(y_n)) \Big\}, y_n - p^* \Big\rangle \\ &\le \|y_n - p^*\|^2 + \Big\| \sum_{j=1}^{s} \Big\{ \gamma_{j,n}\tau_{j,n}(\nabla h_{j,n}(y_n) + \nabla p(y_n)) \Big\} \Big\|^2. \end{aligned} \tag{3.8}$$

Using convexity of $\|.\|^2$, definition of $\tau_{j,n}$, and condition (C3), we have

$$\begin{aligned} &\Big\| \sum_{j=1}^{s} \Big\{ \gamma_{j,n}\tau_{j,n}(\nabla h_{j,n}(y_n) + \nabla p(y_n)) \Big\} \Big\|^2 \\ \le\ & \sum_{j=1}^{s} \Big\{ \gamma_{j,n} \Big(\rho_n \frac{h_{j,n}(y_n) + p(y_n)}{\theta_j^2(y_n)} \Big)^2 \big\| \nabla h_{j,n}(y_n) + \nabla p(y_n) \big\|^2 \Big\} \\ \le\ & \sum_{j=1}^{s} \Big\{ \gamma_{j,n} \Big(\rho_n \frac{h_{j,n}(y_n) + p(y_n)}{\theta_j^2(y_n)} \Big)^2 \theta_j^2(y_n) \Big\} \\ =\ & \rho_n^2 \sum_{j=1}^{s} \gamma_{j,n} \frac{(h_{j,n}(y_n) + p(y_n))^2}{\theta_j^2(y_n)}. \end{aligned} \tag{3.9}$$

From (3.5) and (3.6), we have

$$\begin{aligned}
&\Big\langle \sum_{j=1}^{s}\Big\{\gamma_{j,n}\rho_n\frac{h_{j,n}(y_n)+p(y_n)}{\theta_j^2(y_n)}(\nabla h_{j,n}(y_n)+\nabla p(y_n))\Big\}, y_n-p^*\Big\rangle\\
= &\sum_{j=1}^{s}\Big\{\gamma_{j,n}\rho_n\frac{h_{j,n}(y_n)+p(y_n)}{\theta_j^2(y_n)}\Big\langle\nabla h_{j,n}(y_n)+\nabla p(y_n), y_n-p^*\Big\rangle\Big\}\\
= &\sum_{j=1}^{s}\Big\{\gamma_{j,n}\rho_n\frac{h_{j,n}(y_n)+p(y_n)}{\theta_j^2(y_n)}(\langle\nabla h_{j,n}(y_n), y_n-p^*\rangle+\langle\nabla p(y_n), y_n-p^*\rangle)\Big\}\\
\geq &\sum_{j=1}^{s}\Big\{\gamma_{j,n}\rho_n\frac{h_{j,n}(y_n)+p(y_n)}{\theta_j^2(y_n)}(2h_{j,n}(y_n)+2p(y_n))\Big\}\\
\geq &\ 2\rho_n\sum_{j=1}^{s}\gamma_{j,n}\frac{(h_{j,n}(y_n)+p(y_n))^2}{\theta_j^2(y_n)}. \qquad (3.10)
\end{aligned}$$

From (3.8), (3.9) and (3.10), we obtain

$$\begin{aligned}
\|z_n-p^*\|^2 &\leq \|y_n-p^*\|^2+\rho_n^2\sum_{j=1}^{s}\gamma_{j,n}\frac{(h_{j,n}(y_n)+p(y_n))^2}{\theta_j^2(y_n)}\\
&- 4\rho_n\sum_{j=1}^{s}\gamma_{j,n}\frac{(h_{j,n}(y_n)+p(y_n))^2}{\theta_j^2(y_n)}\\
&= \|y_n-p^*\|^2-\rho_n(4-\rho_n)\sum_{j=1}^{s}\gamma_{j,n}\frac{(h_{j,n}(y_n)+p(y_n))^2}{\theta_j^2(y_n)}. \qquad (3.11)
\end{aligned}$$

From (3.11) and (C4), we have

$$\|z_n-p^*\|\leq\|y_n-p^*\|. \qquad (3.12)$$

From the definition of x_{n+1}, we have that

$$x_{n+1}-p^*=y_n-p^*-\alpha_n(y_n-z_n).$$

By Lemma 2.6 (iii) and using (3.12), we have that

$$\begin{aligned}
-2\alpha_n\langle y_n-p^*, y_n-z_n\rangle &= -\alpha_n\|y_n-p^*\|^2-\alpha_n\|y_n-z_n\|^2+\alpha_n\|z_n-p^*\|^2\\
&\leq -\alpha_n\|y_n-p^*\|^2-\alpha_n\|y_n-z_n\|^2+\alpha_n\|y_n-p^*\|^2\\
&= -\alpha_n\|y_n-z_n\|^2.
\end{aligned}$$

Therefore, since $y_n - z_n = \frac{1}{\alpha_n}(y_n - x_{n+1})$, we get

$$\begin{aligned}
\|x_{n+1} - p^*\|^2 &= \|(y_n - p^*) - \alpha_n(y_n - z_n)\|^2 \\
&= \|y_n - p^*\|^2 + \alpha_n^2\|(y_n - z_n)\|^2 - 2\alpha_n\langle y_n - p^*, y_n - z_n\rangle \\
&\leq \|y_n - p^*\|^2 + \alpha_n^2\|y_n - z_n\|^2 - \alpha_n\|y_n - z_n\|^2 \\
&= \|y_n - p^*\|^2 - \alpha_n(1 - \alpha_n)\|y_n - z_n\|^2 \\
&= \|y_n - p^*\|^2 - \alpha_n(1 - \alpha_n)\|\frac{1}{\alpha_n}(y_n - x_{n+1})\|^2 \\
&= \|y_n - p^*\|^2 - \frac{1 - \alpha_n}{\alpha_n}\|y_n - x_{n+1}\|^2 \\
&= \|y_n - p^*\|^2 - \Big(\frac{1 - \alpha_n}{\alpha_n} - 1\Big)\|y_n - x_{n+1}\|^2 + \|y_n - x_{n+1}\|^2.
\end{aligned} \tag{3.13}$$

Hence, we have from (3.13) that

$$\|x_{n+1} - p^*\|^2 + \Big(\frac{1 - \alpha_n}{\alpha_n} - 1\Big)\|y_n - x_{n+1}\| \leq \|y_n - p^*\|^2 - \|y_n - x_{n+1}\|^2. \tag{3.14}$$

Now, using definition of y_n, we get from (3.14) that

$$\begin{aligned}
&\|x_{n+1} - p^*\|^2 + \Big(\frac{1 - \alpha_n}{\alpha_n} - 1\Big)\|y_n - x_{n+1}\| \\
\leq\ & \|(x_n - p^*) + \beta_n(x_n - x_{n-1})\|^2 - \|x_n + \beta_n(x_n - x_{n-1}) - x_{n+1}\|^2 \\
=\ & \Big(\|x_n - p^*\|^2 + \|\beta_n(x_n - x_{n-1})\|^2 + 2\langle x_n - p^*, \beta_n(x_n - x_{n-1})\rangle\Big) - \\
& \Big(\|x_n - x_{n+1}\|^2 + 2\beta_n\langle x_n - x_{n+1}, x_n - x_{n-1}\rangle + \|\beta_n(x_n - x_{n-1})\|^2\Big) \\
=\ & \Big(\|x_n - p^*\|^2 + 2\beta_n\langle x_n - p^*, x_n - x_{n-1}\rangle + \|\beta_n(x_n - x_{n-1})\|^2\Big) - \\
& \Big(\|x_n - x_{n+1}\|^2 + 2\beta_n\langle x_n - x_{n+1}, x_n - x_{n-1}\rangle + \|\beta_n(x_n - x_{n-1})\|^2\Big) \\
=\ & \Big(\|x_n - p^*\|^2 + 2\beta_n\langle x_n - p^*, x_n - x_{n-1}\rangle + \beta_n^2\|x_n - x_{n-1}\|^2\Big) - \\
& \Big(\|x_n - x_{n+1}\|^2 + 2\beta_n\langle x_n - x_{n+1}, x_n - x_{n-1}\rangle + \|\beta_n(x_n - x_{n-1})\|^2\Big) \\
=\ & \|x_n - p^*\|^2 + 2\beta_n\langle x_n - p^*, x_n - x_{n-1}\rangle - \|x_n - x_{n+1}\|^2 - \\
& 2\beta_n\langle x_n - x_{n+1}, x_n - x_{n-1}\rangle.
\end{aligned} \tag{3.15}$$

From (3.15), we have

$$\begin{aligned}
&\|x_{n+1} - p^*\|^2 + \Big(\frac{1}{\alpha_n}(1 - \alpha_n) - 1\Big)\|x_{n+1} - y_n\|^2 \\
&\leq \|x_n - p^*\|^2 + 2\beta_n\langle x_n - p^*, x_n - x_{n-1}\rangle - \|x_n - x_{n+1}\|^2 \\
&\qquad -2\beta_n\langle x_n - x_{n+1}, x_n - x_{n-1}\rangle.
\end{aligned} \tag{3.16}$$

Using the fact that $2cd \geq -c^2 - d^2, \forall c, d \in \mathbb{R}$, we get

$$2\langle x_n - x_{n+1}, x_n - x_{n-1}\rangle \geq -\|x_n - x_{n+1}\|^2 - \|x_n - x_{n-1}\|^2,$$

which by (3.16) implies that

$$\begin{aligned}&\|x_{n+1}-p^*\|^2-\|x_n-p^*\|^2-\beta_n\|x_n-x_{n-1}\|^2+(1-\beta_n)\|x_n-x_{n+1}\|^2\\&+\Big(\tfrac{1}{\alpha_n}(1-\alpha_n)-1\Big)\|x_{n+1}-y_n\|^2\le 2\beta_n\langle x_n-p^*,x_n-x_{n-1}\rangle.\end{aligned}\tag{3.17}$$

Using Lemma 2.6 (iii), we have from (3.17) that

$$\begin{aligned}&\|x_{n+1}-p^*\|^2-\|x_n-p^*\|^2-\beta_n\|x_n-x_{n-1}\|^2+(1-\beta_n)\|x_n-x_{n+1}\|^2\\&+\Big(\tfrac{1}{\alpha_n}(1-\alpha_n)-1\Big)\|x_{n+1}-y_n\|^2\\&\le\beta_n(-\|x_{n-1}-p^*\|^2+\|x_n-p^*\|^2+\|x_n-x_{n-1}\|^2).\end{aligned}\tag{3.18}$$

Adding $\beta_{n+1}\|x_{n+1}-x_n\|^2-\beta_{n+1}\|x_{n+1}-x_n\|^2$ to one side of (3.18), we have

$$\begin{aligned}&\|x_{n+1}-p^*\|^2-\|x_n-p^*\|^2+2\beta_{n+1}\|x_{n+1}-x_n\|^2-2\beta_n\|x_n-x_{n-1}\|^2\\&+\beta_n(\|x_{n-1}-p^*\|^2-\|x_n-p^*\|^2)+(1-\beta_n-2\beta_{n+1})\|x_n-x_{n+1}\|^2\\&+\Big(\tfrac{1}{\alpha_n}(1-\alpha_n)-1\Big)\|x_{n+1}-y_n\|^2\le 0.\end{aligned}\tag{3.19}$$

Since $\{\beta_n\}$ is non-decreasing and $\{\alpha_n\}$ is non-increasing, we obtain

$$\begin{aligned}&\|x_{n+1}-p^*\|^2-\|x_n-p^*\|^2+2\beta_{n+1}\|x_{n+1}-x_n\|^2\\&-2\beta_n\|x_n-x_{n-1}\|^2-\beta_n\|x_n-p^*\|^2\\&-\beta_{n-1}\|x_{n-1}-p^*\|^2+(1-3\beta_{n+1})\|x_n-x_{n+1}\|^2\\&+\Big(\tfrac{1}{\alpha_n}(1-\alpha_n)-1\Big)\|x_{n+1}-y_n\|^2\le 0.\end{aligned}\tag{3.20}$$

Now, let

$$\Psi_n:=\|x_n-p^*\|^2+2\beta_n\|x_n-x_{n-1}\|^2-\beta_{n-1}\|x_{n-1}-p^*\|^2.$$

Then, from (3.20) and using condition (C2), we have that

$$\begin{aligned}\Psi_{n+1}-\Psi_n &\le -(1-3\beta_{n+1})\|x_n-x_{n+1}\|^2-\Big(\frac{1}{\alpha_n}(1-\alpha_n)-1\Big)\|x_{n+1}-y_n\|^2\\&\le 0.\end{aligned}\tag{3.21}$$

Therefore, the sequence $\{\Psi_n\}$ is non-increasing. Let $\phi_n=\|x_n-p^*\|^2$. Since $\beta_n<\frac{1}{3}$, we have

$$-\frac{1}{3}\phi_{n-1}\le\phi_n-\frac{1}{3}\phi_{n-1}\le\Psi_n\le\Psi_1,\forall n\ge 1.\tag{3.22}$$

This implies that

$$\phi_n\le\Big(\frac{1}{3}\Big)^n\phi_0+\Psi_1\sum_{k=0}^{n-1}\Big(\frac{1}{3}\Big)^k\le\Big(\frac{1}{3}\Big)^n\phi_0+\frac{\Psi_1}{1-\frac{1}{3}},\forall n\ge 1.$$

We observe that

$$\begin{aligned}\Psi_n &= \|x_n-p^*\|^2+2\beta_n\|x_n-x_{n-1}\|^2-\beta_{n-1}\|x_{n-1}-p^*\|^2\\&\ge \|x_n-p^*\|^2+2\beta_n\|x_n-x_{n-1}\|^2-\frac{1}{3}\|x_{n-1}-p^*\|^2.\end{aligned}$$

Thus, $\Psi_1\ge 0$. It follows from (3.21) and (3.22) that

$$\sum_{k=1}^{n}(1-3\beta_{k+1})\|x_k-x_{k+1}\|^2\le\Psi_1-\Psi_{n+1}\le\Psi_1+\frac{1}{3}\phi_n\le\Big(\frac{1}{3}\Big)^{n+1}\phi_0+\frac{3\Psi_1}{2}.$$

This implies that $\sum_{n=1}^{\infty}(1-3\beta_{n+1})\|x_n - x_{n+1}\|^2 \leq \infty$. By (C2), we have that

$$\sum_{n=1}^{\infty} \|x_n - x_{n+1}\|^2 \leq \infty. \tag{3.23}$$

Furthermore, from (3.19), we get that

$$\begin{aligned}&\|x_{n+1}-p^*\|^2 - \|x_n - p^*\|^2 - 2\beta_n\|x_n - x_{n-1}\|^2 \\ &+\beta_n(\|x_{n-1}-p^*\|^2 - \|x_n - p^*\|^2) + (1-\beta_n)\|x_n - x_{n+1}\|^2 \leq 0.\end{aligned}$$

This implies that

$$\begin{aligned}&\|x_{n+1}-p^*\|^2 - \|x_n - p^*\|^2 + \beta_n(\|x_{n-1}-p^*\|^2 - \|x_n - p^*\|^2) \\ &\leq 2\beta_n\|x_n - x_{n-1}\|^2 - (1-\beta_n)\|x_n - x_{n+1}\|^2.\end{aligned}$$

Hence,

$$\begin{aligned}&\|x_{n+1}-p^*\|^2 \leq \|x_n - p^*\|^2 + \beta_n(\|x_n-p^*\|^2 - \|x_{n-1} - p^*\|^2) \\ &+2\beta_n\|x_n - x_{n-1}\|^2 - (1-\beta_n)\|x_n - x_{n+1}\|^2.\end{aligned} \tag{3.24}$$

Using Lemma 2.10 in (3.24), we have that $\lim_{n\to\infty} \|x_n - p^*\|$ exists.

Claim 2: Any weak-cluster point of the sequence $\{x_n\}$ belongs in Ω.

Now, for $p^* \in \Omega$, using Lemma 2.6 (i) in the definition of x_{n+1}, we have

$$\begin{aligned}\|x_{n+1}-p^*\|^2 &= \|(1-\alpha_n)y_n + \alpha_n z_n - p^*\|^2 \\ &= \|(1-\alpha_n)(y_n - p^*) + \alpha_n)(z_n - p^*)\|^2 \\ &= (1-\alpha_n)\|y_n - p^*\|^2 + \alpha_n\|z_n - p^*\|^2 - \alpha_n(1-\alpha_n)\|z_n - y_n\|^2 \\ &\leq \|y_n - p^*\|^2 - \alpha_n(1-\alpha_n)\|z_n - y_n\|^2.\end{aligned} \tag{3.25}$$

Also, using Lemma 2.6 (ii) in the definition of y_n, we have

$$\begin{aligned}\|y_n - p^*\|^2 &= \|x_n + \beta_n(x_n - x_{n-1}) - p^*\|^2 \\ &= \|x_n - p^* - \beta_n(x_{n-1} - x_n)\|^2 \\ &\leq \|x_n - p^*\|^2 - 2\beta_n\langle x_n - p^*, x_{n-1} - x_n\rangle + \beta_n^2\|x_{n-1} - x_n\|^2.\end{aligned} \tag{3.26}$$

Substituting (3.26) into (3.25) and using the fact that $\beta_n^2 \leq \beta_n, \beta_n \in [0,1)$, we have

$$\begin{aligned}\|x_{n+1}-p^*\|^2 \leq\ & \|x_n - p^*\|^2 - 2\beta_n\langle x_n - p^*, x_{n-1} - x_n\rangle + \beta_n^2\|x_{n-1} - x_n\|^2 \\ -\ & \alpha_n(1-\alpha_n)\|z_n - y_n\|^2.\end{aligned} \tag{3.27}$$

Again, using Lemma 2.6 (i) in (3.27), we have

$$\begin{aligned}\|x_{n+1}-p^*\|^2 &\leq \beta_n(\|x_n - p^*\|^2 - \|x_{n-1} - x_n\|^2) + 2\beta_n\|x_{n-1} - x_n\|^2 \\ &- \alpha_n(1-\alpha_n)\|z_n - y_n\|^2 \\ &\leq \beta_n(\|x_n - p^*\|^2 - \|x_{n-1} - x_n\|^2) + 2\beta_n\|x_{n-1} - x_n\|^2.\end{aligned} \tag{3.28}$$

Using Lemma 2.10 in (3.28), we have that

$$\sum_{n=1}^{\infty}[\|x_{n+1}-p^*\|^2 - \|x_n - p^*\|^2]_+ < \infty. \tag{3.29}$$

By condition (C1) (noting that $\beta_n \in [0,1)$), we have from (3.28) that

$$\begin{aligned}\sum_{n=1}^{\infty}\alpha_n(1-\alpha_n)\|z_n-y_n\|^2 &\leq \sum_{n=1}^{\infty}[\|x_n-p^*\|^2-\|x_{n+1}-p^*\|^2] \\ &+ \sum_{n=1}^{\infty}\beta_n(\|x_n-p^*\|^2-\|x_{n-1}-p^*\|^2) \\ &+ 2\sum_{n=1}^{\infty}\beta_n\|x_{n-1}-x_n\|^2. \end{aligned} \tag{3.30}$$

Using (3.23) and (3.29) in (3.30), we obtain

$$\sum_{n=1}^{\infty}\alpha_n(1-\alpha_n)\|z_n-y_n\|^2<\infty.$$

By the condition (C1), we have

$$\lim_{n\to\infty}\|z_n-y_n\|=0. \tag{3.31}$$

From (3.23), we have that $\lim_{n\to\infty}\|x_n-x_{n-1}\|=0$, which furthermore implies that $\lim_{n\to\infty}\beta_n\|x_n-x_{n-1}\|=0$. This implies from Algorithm 1 that $\lim_{n\to\infty}\|y_n-x_n\|=0$. Also from (3.29), we have that $\lim_{n\to\infty}[\|x_{n+1}-p^*\|^2-\|x_n-p^*\|^2]=0$. From (3.11) and (3.31), we have

$$\begin{aligned}&\rho_n(4-\rho_n)\sum_{j=1}^{s}\gamma_{j,n}\frac{(h_{j,n}(y_n)+p(y_n))^2}{\theta_j^2(y_n)} \\ \leq\ & \|y_n-p^*\|^2-\|z_n-p^*\|^2 \\ =\ & (\|y_n-p^*\|-\|z_n-p^*\|)(\|y_n-p^*\|+\|z_n-p^*\|) \\ \leq\ & \|y_n-z_n\|(\|y_n-p^*\|+\|z_n-p^*\|)\to 0,\quad n\to\infty. \end{aligned} \tag{3.32}$$

Hence, we obtain

$$\rho_n(4-\rho_n)\sum_{j=1}^{s}\gamma_{j,n}\frac{(h_{j,n}(y_n)+p(y_n))^2}{\theta_j^2(y_n)}\to 0,\quad n\to\infty. \tag{3.33}$$

Using condition (C3), from (3.33), we have

$$\frac{(h_{j,n}(y_n)+p(y_n))^2}{\theta_j^2(y_n)}\to 0,\quad n\to\infty \tag{3.34}$$

for all $j\in\{1,\dots,s\}$. Now, using the definition of y_n and (C2), we have

$$\|x_n-y_n\|=\|x_n-x_n-\beta_n(x_n-x_{n-1})\|=\beta_n\|x_n-x_{n-1}\|\to 0,\quad n\to\infty. \tag{3.35}$$

By (3.24) and (3.25), we get

$$\|x_n-z_n\|\leq\|x_n-y_n\|+\|y_n-z_n\|\to 0,\quad n\to\infty. \tag{3.36}$$

Using the definition of x_{n+1}, (C1) and noting that $\{y_n\}$ and $\{z_n\}$ are bounded, we have

$$\|x_{n+1}-z_n\|=(1-\alpha_n)\|y_n-z_n\|\to 0,\quad n\to\infty. \tag{3.37}$$

(3.26) and (3.27) gives

$$\|x_{n+1}-x_n\|\leq\|x_{n+1}-z_n\|+\|z_n-x_n\|\to 0,\quad n\to\infty. \tag{3.38}$$

For each $i \in \{1,\dots,r\}$ and for each $j \in \{1,\dots,s\}$, $\nabla h_{j,n}(.)$ and $\nabla p_{i,n}(.)$ are Lipschitz continuous with constant $\|A\|^2$ and 1, respectively. Since the sequence $\{z_n\}$ is bounded and

$$\|\nabla h_{j,n}(y_n)\| = \|\nabla h_{j,n}(y_n) - \nabla h_{j,n}(p^*)\| \le \|A\|^2\|y_n - p^*\|,\ \forall j \in \{1,\dots,s\},$$

$$\|\nabla p_{i,n}(y_n)\| = \|\nabla p_{i,n}(y_n) - \nabla p_{i,n}(p^*)\| \le \|y_n - p^*\|,\ \forall i \in \{1,\dots,r\},$$

we have the sequences $\{\|\nabla p_{i,n}(y_n)\|\}_{n=1}^{\infty}$ and $\{\|\nabla h_{j,n}(y_n)\|\}_{n=1}^{\infty}$ are bounded. Hence, we have $\{\theta_j(y_n)\}_{n=1}^{\infty}$ is bounded and hence $\{\theta_j(y_{n_k})\}_{k=1}^{\infty}$ is bounded. As a result by (3.34), we have

$$\lim_{k\to\infty} h_{j,n_k}(y_{n_k}) = \lim_{k\to\infty} p_{n_k}(y_{n_k}) = 0,\ \ \forall j \in \{1,\dots,s\}. \tag{3.39}$$

From the definition of $p_{n_k}(y_{n_k})$, we can have

$$p_{i,n_k}(y_{n_k}) \le p_{n_k}(y_{n_k}),\ \ \forall i \in \{1,\dots,r\}. \tag{3.40}$$

Therefore, (3.39) and (3.40) gives

$$\lim_{k\to\infty} h_{j,n_k}(y_{n_k}) = \lim_{k\to\infty} p_{i,n_k}(y_{n_k}) = 0,\ \ \forall i \in \{1,\dots,r\}, \forall j \in \{1,\dots,s\}.$$

That is, for all $i \in \{1,\dots,r\}$, $j \in \{1,\dots,s\}$, we have

$$\lim_{k\to\infty} \|(I - P_{Q_{j,n_k}})Ay_{n_k}\| = \lim_{k\to\infty} \|(I - P_{C_{i,n_k}})y_{n_k}\| = 0. \tag{3.41}$$

Therefore, since $\{y_n\}$ is bounded and from the boundedness assumption of the subdifferential operator ∂q_j, the sequence $\{\varepsilon_{j,n}\}_{n=1}^{\infty}$ is bounded. In view of this and (3.41), for all $j \in \{1,\dots,s\}$ we have

$$\begin{aligned} q_j(Ay_{n_k}) &\le \langle \varepsilon_{j,n_k}, Az_{n_k} - P_{Q_{j,n_k}}(Ay_{n_k})\rangle \\ &\le \|\varepsilon_{j,n_k}\|\|(I - P_{Q_{j,n_k}})Ay_{n_k}\| \to 0,\ \ k \to \infty. \end{aligned} \tag{3.42}$$

Similarly, from the boundedness of $\{\xi_{i,n}\}_{n=1}^{\infty}$ and (3.41), for all $i \in \{1,\dots,r\}$, we obtain

$$\begin{aligned} c_i(y_{n_k}) &\le \langle \xi_{i,n_k}, y_{n_k} - P_{C_{i,n_k}}(y_{n_k})\rangle \\ &\le \|\xi_{i,n_k}\|\|(I - P_{C_{i,n_k}})y_{n_k}\| \to 0,\ \ k \to \infty. \end{aligned} \tag{3.43}$$

Let x^* be a weak cluster point of $\{x_n\}$. Thus, by Lemma 2.12 there exists a subsequence $\{x_{n_k}\}$ which weakly converges to x^*. Since $\lim_{n\to\infty} \|y_n - x_n\| = 0$ and since $x_{n_k} \rightharpoonup x^*$, $k \to \infty$, we have $y_{n_k} \rightharpoonup x^*, k \to \infty$ which implies $Ay_{n_k} \rightharpoonup Ax^*$, $k \to \infty$.
The weak lower semi-continuity of $q_j(.)$ and (3.42) implies that

$$q_j(Ax^*) \le \liminf_{k\to\infty} q_j(Ay_{n_k}) \le \limsup_{k\to\infty} q_j(Ay_{n_k}) \le 0,\ \ \forall j \in \{1,\dots,s\}.$$

That is, $Ax^* \in Q_j$ for all $j \in \{1,\dots,s\}$.
Likewise, the weak lower semi-continuity of $c_i(.)$ and (3.43) implies that

$$c_i(x^*) \le \liminf_{k\to\infty} c_i(y_{n_k}) \le 0,\ \ \forall i \in \{1,\dots,r\}.$$

That is, $x^* \in C_i$ for all $i \in \{1,\dots,r\}$. Thus, $x^* \in \Omega$. Hence, every weak-cluster point of the sequence $\{x_n\}$ belongs to Ω. Therefore, by Lemma 2.12, there exists a point $p^* \in \Omega$ such that $\{x_n\}$ weakly converges to p^*. This completes the proof. ∎

For the particular case, where r=s=1, we note the following corollary regarding the SFP (1.1), which is an immediate consequence of Theorem 3.2.

Corollary 3.3. *Let $\{\alpha_n\}$ be a non-increasing real sequence in $(0,1)$ and $\{\beta_n\}$ be a non-decreasing real sequence in $(0,1)$, and $\{\rho_n\}$ is real parameter sequence. Consider the iterative algorithm*

$$\begin{cases} x_0, x_1 \in H_1, \\ y_n = x_n + \beta_n(x_n - x_{n-1}), \\ z_n = y_n - \tau_n(\nabla h_n(y_n) + \nabla p(y_n)), \\ x_{n+1} = (1-\alpha_n)y_n + \alpha_n z_n, \end{cases} \tag{3.44}$$

where $\tau_n = \rho_n \frac{h_n(y_n)+p(y_n)}{\theta^2(y_n)}$ with $\theta(y_n) = \max\{1, \|\nabla p(y_n) + \nabla h_n(y_n)\|\}$. If the parameters $\{\beta_n\}$, $\{\rho_n\}$, $\{\alpha_n\}$ in the iterative algorithm (3.44) satisfy the following conditions:
(C1): $0 < \alpha \le \alpha_n \le \frac{1}{2}$;
(C2): $0 \le \beta_n \le \frac{1-m}{3} < \frac{1}{3}$, *for some* $m \in (0,1)$;
(C3): $0 < \rho_n < 4$ *and* $\liminf_{n\to\infty} \rho_n(4-\rho_n) > 0$.
Then, the sequence $\{x_n\}$ generated by (3.44) converges weakly to $p^* \in \bar{\Omega} = \{p^* \in C : Ap^* \in Q\}$.

4. Numerical Results

In this section, we present some numerical experiments to illustrate the implementation and efficiency of our proposed method and we compare it with [47, Algorithm 3.1] (say He et al. Alg.), [27, Algorithm 3.1](say Suantai et al. Alg.), and [48, Scheme (18)](say Tang et al. Alg.) by solving sa MSSFP problem. The numerical results are completed on a standard TOSHIBA laptop with Intel(R) Core(TM) i5-2450M CPU@2.5GHz 2.5GHz with memory 4GB. The code is implemented in MATLAB R2020a. In our numerical experiments, $Iter.(n)$ stands for the number of iterations and $CPU(s)$ is Elapsed time in seconds.

Example 4.1. Consider two Hilbert spaces $H_1 = \mathbb{R}^N$, $H_2 = \mathbb{R}^M$. The goal is to find a point $p^* \in \mathbb{R}^N$ such that

$$p^* \in \bigcap_{i=1}^{r} C_i \text{ such that } Ap^* \in \bigcap_{j=1}^{s} Q_j, \tag{4.1}$$

where $A : \mathbb{R}^N \to \mathbb{R}^M$ is a linear bounded operator whose representing elements are randomly generated in the closed interval $[-0.5, 0.5]$, and the closed convex subsets C_i $(i = 1, 2, \dots, r)$ of $\mathbb{R}^N$ are given by

$$C_i = \left\{ x \in \mathbb{R}^N : \sum_{k=i}^{N} 10^{\left(\frac{k-1}{N-1}\right)} x_k^2 \le 1 \right\},$$

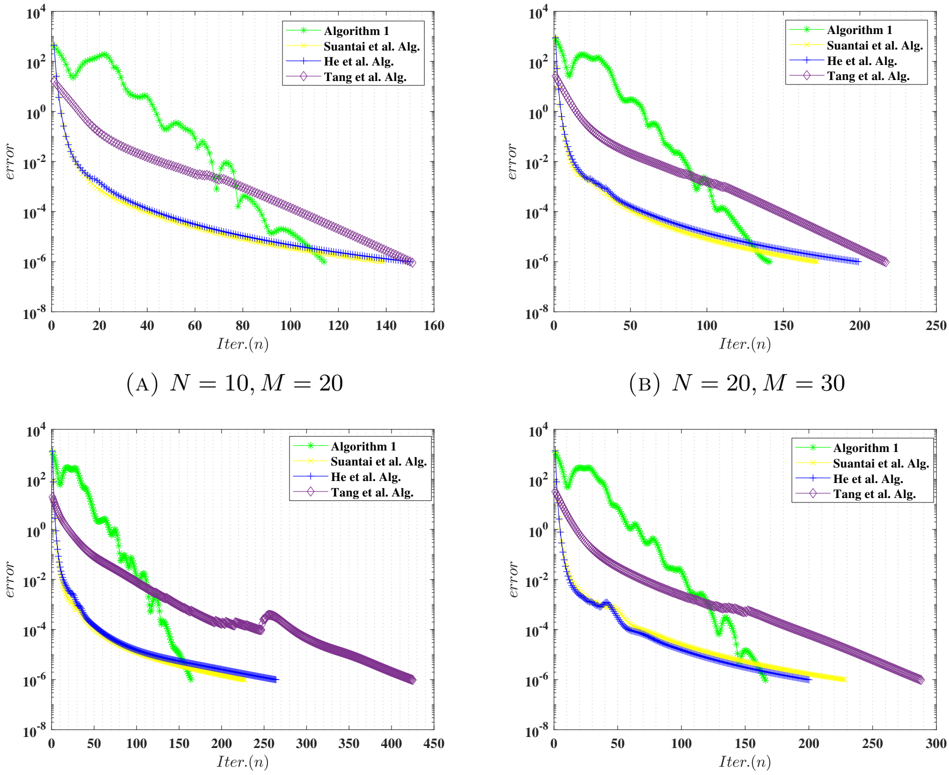
where k is positive integer and $i = 1, 2, \dots, r = N$, and the closed convex subsets Q_j $(j = 1, 2, \dots, s)$ of $\mathbb{R}^M$ are given by

$$Q_j = \left\{ y \in \mathbb{R}^M : \sum_{k=j}^{M} 10^{\left(\frac{k-1}{M-1}\right)} y_k^2 \le 1 \right\},$$

where k is positive integer and $j = 1, 2, \dots, s = M$. Obviously, C_i and Q_j are both ellipsoids [49]. So such a MSSFP can be solved by the proposed algorithms. In this example, we study the numerical behaviour of our proposed Algorithm 1, He et al. Alg., Suantai et al. Alg., and Tang et al. Alg. by solving (4.1) for different choices of the dimensions N and M. For the sake of convenience, we denote $e_1 = (1, 1, \dots, 1)^T \in \mathbb{R}^N$. For Algorithm 1, we take $x_0 = 3e_1$, $x_1 = 10e_1$, $\theta_n = 0.9$, $\rho_n = \frac{n}{2n+1}$, $\alpha_n = \frac{1}{12} + \frac{1}{6n}$, $\gamma_{j,n} = \frac{j}{\sum_{m=1}^{M} m}$ for $j = 1, 2, \dots, s = M$. For He et al. Alg., we take $u = rand(N, 1)$, $x_0 = 3e_1$, $\rho_n = \frac{n}{2n+1}$, $\alpha_n = \frac{1}{n+1}$, $\iota_i = \frac{i}{\sum_{m=1}^{N} m + \sum_{m=1}^{M} m}$ for $i = 1, 2, \dots, r = N$, and $\lambda_j = \frac{j}{\sum_{m=1}^{N} m + \sum_{m=1}^{M} m}$ for $j = 1, 2, \dots, s = M$. For Suantai et al. Alg., we take $u = rand(N, 1)$, $x_0 = 3e_1$, $x_1 = 10e_1$, $\beta = 0.9$, $\beta_n = \bar{\beta}_n$, $\omega_n = \frac{1}{(n+1)^2}$, $\rho_n = \frac{n}{2n+1}$, $\alpha_n = \frac{1}{n+1}$, $\iota_i = \frac{i}{\sum_{m=1}^{N} m + \sum_{m=1}^{M} m}$ for $i = 1, 2, \dots, r = N$, and $\lambda_j = \frac{j}{\sum_{m=1}^{N} m + \sum_{m=1}^{M} m}$ for $j = 1, 2, \dots, s = M$. For Tang et al. Alg., we take $x_0 = 3e_1$, $\rho_1^k = 0.09 = \rho_2^k$, $\alpha_i = \frac{i}{\sum_{m=1}^{N} m + \Sigma_{m=1}^{M} m}$ for $i = 1, 2, \dots, r = N$, and $\beta_j = \frac{j}{\sum_{m=1}^{N} m + \sum_{m=1}^{M} m}$ for $j = 1, 2, \dots, s = M$. In the implementation, we take $error = \|x_{n+1} - x_n\|^2 < 10^{-6}$ as the stopping criterion. The numerical results of the compared algorithms in terms of the number of iterations ($Iter.(n)$) and the time of execution in seconds ($CPU(s)$), are described in Table 1 and Figure 1. In Figure 1, we give *error* versus the $Iter.(n)$ for different choices of N and M. It is readily seen from Table 1 and Figure 1 that our proposed Algorithm 1 has a better performance than the compared algorithms.

Table 1. Comparison of Algorithm 1 with Suantai et al. Alg., He et al. Alg., and Tang et al. Alg.

	Algorithm 1		Suantai et al. Alg.		He et al. Alg.		Tang et al. Alg.	
(N, M)	$Iter.(n)$	$CPU(s)$	$Iter.(n)$	$CPU(s)$	$Iter.(n)$	$CPU(s)$	$Iter.(n)$	$CPU(s)$
(10, 20)	114	0.278821	139	0.278319	150	0.391167	151	0.233414
(20, 30)	141	0.445766	172	0.47222	199	0.562628	217	0.519087
(30, 20)	164	0.458431	227	0.535639	264	0.606438	425	0.959047
(30, 40)	166	0.801689	228	0.946397	200	0.72992	288	1.144844



(A) $N = 10, M = 20$

(B) $N = 20, M = 30$

(C) $N = 30, M = 20$

(D) $N = 30, M = 40$

FIGURE 1. Comparison of Algorithm 1 with Suantai et al. Alg., He et al. Alg., and Tang et al. Alg. for different choices of N and M

CONCLUSIONS

In this work, we studied the multiple-sets split feasibility problem in the framework of real Hilbert spaces. A self-adaptive inertial relaxed technique that does not need prior information about the operator norm is proposed to solve MSSFP. A weak convergence theorem to the proposed algorithm is established and proved under some suitable conditions. Our numerical experiments showed that the proposed method is easily implementable and better performance that the compared algorithms interms of iteration numbers and elapsed time taken.

COMPETING INTERESTS

The authors declare that they have no competing interests.

FUNDING

This research was funded by the Center of Excellence in Theoretical and Computational Science (TaCS-CoE), Faculty of Science, KMUTT. The first author was supported by the

Petchra Pra Jom Klao Ph.D. Research Scholarship from King Mongkut's University of Technology Thonburi with Grant No. 37/2561. Moreover, this project is supported by Thailand Science Research and Innovation (TSRI) Basic Research Fund: Fiscal year 2021 under project number 64A306000005.

Acknowledgements

The authors acknowledge the financial support provided by the Center of Excellence in Theoretical and Computational Science (TaCS-CoE), KMUTT. Guash Haile Taddele is supported by the Petchra Pra Jom Klao Ph.D. Research Scholarship from King Mongkut's University of Technology Thonburi (Grant No.37/2561). Moreover, this project is supported by Thailand Science Research and Innovation (TSRI) Basic Research Fund: Fiscal year 2021 under project number 64A306000005.

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