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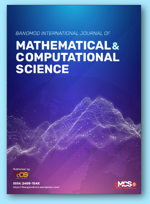

AN INERTIAL ITERATIVE SCHEME FOR SOLVING SPLIT VARIATIONAL INCLUSION PROBLEMS IN REAL HILBERT SPACES

Pawicha Phairatchatniyom [1], **Habib ur Rehman**[2], **Jamilu Abubakar**[2], **Poom Kumam**[1,2,*], **Juan Martínez-Moreno**[3]

[1] *Fixed Point Research Laboratory, Fixed Point Theory and Applications Research Group, Center of Excellence in Theoretical and Computational Science (TaCS-CoE), Faculty of Science, King Mongkuts University of Technology Thonburi (KMUTT), 126 Pracha Uthit Rd., Bangmod, Thung Khru, 10140, Bangkok, Thailand*

[2] *Center of Excellence in Theoretical and Computational Science (TaCS-CoE), Faculty of Science, King Mongkuts University of Technology Thonburi (KMUTT), 126 Pracha Uthit Rd., Bangmod, Thung Khru, 10140, Bangkok, Thailand*

[3] *Department of Mathematics, University of Jaén, Campus las Lagunillas s/n, 23071, Jaén, Spain*

E-mails: pawicha.phairat@mail.kmutt.ac.th, hrehman.hed@gmail.com, jamilu.abubakar@mail.kmutt.ac.th, poom.kumam@mail.kmutt.ac.th, jmmoreno@ujaen.es

**Corresponding author.*

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Abstract In this paper, we present a new inertial iterative algorithm for solving split variational inclusion problems in real Hilbert spaces. The strong convergence is proved under standard conditions. In applications, the proposed iterative algorithm is applicable to the iterative methods for solving split feasibility and split minimization problems. Additionally, we present numerical experiments with some examples to illustrate the convergence performance of the proposed algorithm in comparisons with some existing approaches in the literature. Finally, we concluded that the proposed algorithm is faster and more efficient than some existing ones.

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1. Introduction

Let H_1 and H_2 be two real Hilbert spaces with inner product $\langle \cdot, \cdot \rangle$ and the induced norm $\| \cdot \|$. Let $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ be two multivalued maximal monotone mappings. Let $A : H_1 \to H_2$ be a bounded linear operator. Based on the result of Censor et al. [7], Moudafi [20] firstly introduced the following *split variational inclusion problem*:

$$\text{Find} \quad x^* \in H_1 \quad \text{such that} \quad 0 \in B_1(x^*) \tag{1.1}$$

$$\text{and} \quad Ax^* \in H_2 \quad \text{such that} \quad 0 \in B_2(Ax^*). \tag{1.2}$$

The set of all solutions to (1.1) is denoted by $B_1^{-1}(0) = \{x^* \in H_1 : 0 \in B_1(x^*)\}$, while the set of all solutions to (1.2) is denoted by $B_2^{-1}(0) = \{Ax^* \in H_2 : 0 \in B_2(Ax^*)\}$. Furthermore, the set of all solutions to the problem (1.1) - (1.2) is denoted by Ω, that is, $\Omega = \{x^* \in H_1 : x^* \in B_1^{-1}(0) \text{ and } Ax^* \in B_2^{-1}(0)\}$. It is known that the problem (1.1) - (1.2) contains several special cases, including the split variational inequality problem, the split feasibility problem, the split zeroes problem, and the split common fixed point problem, among others. Furthermore, the problem (1.1) - (1.2) can be written as fixed point equations: $x^* = J_\lambda^{B_1}(x^*)$ and $Ax^* = J_\lambda^{B_2}(Ax^*)$, where $\lambda > 0$ is a parameter, and $J_\lambda^{B_1}$ and $J_\lambda^{B_2}$ are the resolvent operators defined by $J_\lambda^{B_i}(x) = (I + \lambda B_i)^{-1}(x)$ for $i = \{1, 2\}$, where I is the identity operator on H_1 and H_2, respectively. Note that the problem (1.1) is a classical *variational inclusion problem*:

$$\text{Find} \quad x^* \in H_1 \quad \text{such that} \quad 0 \in B_1(x^*). \tag{1.3}$$

A well known classical method for solving the problem (1.3) is called a *proximal point method* which was first introduced by Martinet [18] and further extended by Rockafellar [24]. To improve the convergence rate of sequence, Alvarez and Attouch [2] applied an idea of the *heavy ball method* introduced by poyak [23] into the proximal point method for solving (1.3) as follows: Choose arbitrary $x_0, x_1 \in H_1$ and set $\lambda_n > 0$, compute $\{x_n\}$ as follows:

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \\ x_{n+1} = J_{\lambda_n}^{B_1}(w_n), \end{cases} \tag{1.4}$$

where $w_n = x_n + \theta_n(x_n - x_{n-1})$ represents the so called inertial step. It was shown that, if $\{\theta_n\} \subseteq [0, 1)$ satisfies $\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\|^2 < \infty$, then the sequence $\{x_n\}$ generated by (1.4) converges weakly to a zero of B_1.

The inertial effect is usually incorporated into an algorithm for the purpose of speeding up the iteration process. Several studies have shown that iterative algorithms for solving nonlinear problems that incorporate the inertial step have better numerical performance in terms of number of iterations and CPU time compared to their non-inertial counterparts.

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For instance, it was shown via numerical experiments that (1.4) and other associated methods, such as [11, 15], have significantly improved the effectiveness of their non-inertial algorithms, that is, when $\theta_n = 0$. Interested readers can also refer to the inertial iterative methods in [21, 22] for solving the SVIP (1.1) - (1.2).

On the other hand, Byrne et al. [5] introduced the following iterative algorithm, based on the proximal point algorithm, in order to solve the problem (1.1) - (1.2): Choose an arbitrary $x_1 \in H_1$ and set $\lambda > 0$, compute $\{x_n\}$ as follows:

$$x_{n+1} = J_\lambda^{B_1}\left(x_n - \gamma A^*\left(I - J_\lambda^{B_2}\right)Ax_n\right), \tag{1.5}$$

where $\gamma \in (0, \frac{2}{\|A\|^2})$ and A^* denotes the adjoint of A. They showed the weak and strong convergence theorems of the iterative method (1.5).

Many authors have been inspired by the work of Byrne et al. [5] and have developed the iterative method for solving the problem (1.1) - (1.2). In this regard, one can refer to [1, 8, 9, 12, 25, 26] and references therein.

Very recently, Tan and Cho [27] proposed and studied an iterative method for solving the fixed point problem of nonexpansive mappings. The method comprises of inertial step and Manntype [17] method. Let $C \subset H_1$ be a nonempty closed and convex set and $T : C \to C$ be a nonexpansive mapping. Their proposed algorithm is as follows: Choose an arbitrary $x_1, x_2 \in H_1$, $\{\delta_n\} \subseteq [a, b] \subset (0, 1]$. Then compute the following sequences:

$$\begin{cases} w_n = x_n + \theta_n(x_n - x_{n-1}), \\ y_n = (1 - \beta_n)w_n, \\ x_{n+1} = (1 - \delta_n)y_n + \delta_n T y_n, \end{cases} \tag{1.6}$$

where $\{\beta_n\} \subset (0, 1)$ satisfies $\lim_{n\to\infty} \beta_n = 0$ and $\sum_{n=1}^{\infty} \beta_n = \infty$. It was shown that, if a sequence $\{\theta_n\}$ satisfies $\lim\limits_{n\to\infty} \dfrac{\theta_n}{\beta_n}\|x_n - x_{n-1}\| = 0$, then the sequence $\{x_n\}$ generated by (1.6) converges strongly to a fixed point of T.

Motivated and inspired by the above results, in this article, we give a positive answer to the following natural questions:

(1) Can we design a strong convergence of inertial iterative method involving an inertial step as defined in (1.6) together with (1.5) for solving the problem (1.1) - (1.2)?
(2) Does the incorporated modified inertial step have better numerical performance than the usual inertial algorithms for solving the problem (1.1) - (1.2)?

In this paper, we propose a new strong convergence for an inertial iterative algorithm for solving the problem (1.1) - (1.2). We construct and show a strong convergence theorem for the proposed method in generic Hilbert spaces under certain conditions. Furthermore, we derive algorithms for solving the split feasibility and split minimization problems from the proposed algorithm.

The remaining part of this paper is organized as follows: In the next section, we recall some basic tools, definitions, and useful lemmas that are needed in order to show the convergence analysis of the proposed method. In Section 3, we present our proposed algorithm and it's convergence analysis. In Section 4, we apply our results to the split feasibility and split minimization problems. In section 5, we perform some numerical experiments to ensure the validity and efficiency of our proposed algorithm. In the last section, we also compare our method with some existing results in the literature. Furthermore, in the same section, we apply our method to recovering sparse and noisy data.

2. Preliminaries

In this section, let H be a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$, the norm $\|\cdot\|$, and the identity operator I. Let C be a closed and convex subset of H. The notion $x_n \rightharpoonup x$ is denoted by the weak convergence of $\{x_n\}$ to x, while $x_n \to x$ is denoted by strong convergence of $\{x_n\}$ to x.
It is known that for any $x, y \in H$ the following are satisfied:

(1) $\langle x, y\rangle| \leq \|x\|\|y\|$; (Cauchy-Schwarz inequality)
(2) $\|x+y\|^2 \leq \|x\|^2 + \langle y, x+y\rangle$;
(3) $\|x \pm y\|^2 = \|x\|^2 + \|y\|^2 \pm 2\langle x, y\rangle$.

Definition 2.1. Let H be a real Hilbert space and $T : H \to H$ be a mapping. Then T is said to be:

(1) *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in H$;
(2) *firmly nonexpansive* if $\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y\rangle$ for all $x, y \in H$.

Note that firmly nonexpansive is nonexpansive.

Recall that for every $x \in H$, there exists a unique nearest point in C denoted by $P_C x$ such that $\|x - P_C x\| \leq \|x - y\|$ for all $y \in C$. The operator P_C of H onto C is called *metric projection* and is characterized by

$$\langle x - P_C x, y - P_C x\rangle \leq 0, \tag{2.1}$$

for all $x \in H$ and $y \in C$. Furthermore, the operator P_C is firmly nonexpansive.

Let $B : H \to 2^H$ be a multivalued mapping with domain $Dom(B) := \{x \in H : Bx \neq \emptyset\}$. Then B is called *maximal monotone* if it is *monotone*, that is,

$$\langle x - y, u - v\rangle \geq 0,$$

for any $x, y \in H$, $u \in Bx$, $v \in By$, and its *graph* $G(B) := \{(x, u) \in H \times H : u \in Bx\}$ is not properly contained in the graph of any other monotone mapping. For any maximal monotone B with parameter $\lambda > 0$, the *resolvent* operator is defined by

$$J_\lambda^B(x) := (I + \lambda B)^{-1}(x)$$

for all $x \in H$, where I is the identity operator on H. Further, we recall that the *subdifferential*, denoted by ∂f, of a proper lower semi-continuous and convex function $f : H \to (-\infty, \infty]$ is defined by $\partial f(x) := \{z \in H : f(x) - f(y) \leq \langle z, x - y\rangle, \forall y \in H\}$ for all $x \in H$. The *indicator function* of a nonempty closed convex set C is defined by

$$i_C(x) := \begin{cases} 0, & x \in C; \\ \infty, & x \neq C. \end{cases}$$

The *normal cone* of C at $x \in H$ is defined by $N_C(x) := \{z \in H : \langle z, y - x\rangle \leq 0\}$ for all $y \in C$. Note that the indicator function of C is a proper lower semi-continuous and convex function on H and its subdifferential is a maximal monotone, and is normal cone, that is, for each $x \in C$

$$\begin{aligned} \partial i_C(x) &= \{z \in H : \partial i_C(x) - \partial i_C(y) + \langle z, y - x\rangle \leq 0\} \\ &= \{z \in H : \langle z, y - x\rangle \leq 0\} = N_C(x), \end{aligned}$$

for all $y \in C$. It follows from the face that ∂i_C is maximal monotone mapping, so for all $x \in H$ and $\lambda > 0$ we have

$$\begin{aligned} x^* = J_\lambda^{\partial i_C}(x) &\Leftrightarrow x \in x^* + \lambda \partial i_C(x^*) \Leftrightarrow x - x^* \in \lambda N_C(x^*) \\ &\Leftrightarrow \langle x - x^*, y - x^* \rangle \leq 0 \ (\forall y \in C) \\ &\Leftrightarrow x^* = P_C x. \end{aligned} \tag{2.2}$$

Next, we recall some lemmas useful for the remainder of this paper.

Lemma 2.2. *[5, 8] Let H be a real Hilbert space and $T : H \to H$ be a nonexpansive mapping. If $x_n \rightharpoonup x$ and $(I - T) \to 0$, then $Tx = x$.*

Lemma 2.3. *[9] Let H be a real Hilbert space, $B : H \to 2^H$ be a multivalued maximal monotone, and J_λ^B be a resolvent operator of B with $\lambda > 0$. Therefore,*

(1) $(I - J_\lambda^B)$ is single-valued and firmly nonexpansive mapping;
(2) $Dom(J_\lambda^B) = H$ and $Fix(J_\lambda^B) = \{x \in Dom(B) : 0 \in Bx\}$.

Lemma 2.4. *[8] Let H_1 and H_2 be two real Hilbert spaces, $A : H_1 \to H_2$ be a linear and bounded operator with with its adjoint A^*, and $B_2 : H_2 \to 2^{H_2}$ be a multivalued maximal monotone mapping. Let $\lambda_n > 0$ and $\{\gamma_n\}$ be a sequence of positive real numbers. Then*

$$\begin{aligned} &\left\| \left(I - \gamma_n A^* \left(I - J_{\lambda_n}^{B_2}\right) A\right) x - \left(I - \gamma_n A^* \left(I - J_{\lambda_n}^{B_2}\right) A\right) y \right\|^2 \\ &\quad \leq \|x - p\|^2 - \gamma_n (2 - \gamma_n \|A\|^2) \left\| (I - J_{\lambda_n}^{B_2}) Ax - (I - J_{\lambda_n}^{B_2}) Ay \right\|^2 \end{aligned}$$

for all $x, y \in H_1$.

Lemma 2.5. *[30] Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \leq (1 - \beta_n) s_n + \beta_n t_n + v_n, n \geq 0,$$

where $\{\beta_n\} \subset [0, 1]$, $\{t_n\} \subset (-\infty, \infty)$, and $\{w_n\} \subset [0, \infty)$ such that $\lim_{n\to\infty} \beta_n = 0$, $\sum_{n=0}^{\infty} \beta_n = \infty$, $\limsup_{n\to\infty} t_n \leq 0$, and $\sum_{n=0}^{\infty} v_n < \infty$. Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.6. *[16] Let $\{s_n\}$ be a sequence of nonnegative real numbers such that there exists a subsequence $\{s_{n_j}\}$ of $\{s_n\}$ such that $s_{n_j} < s_{n_j+1}$ for all $j \in \mathbb{N}$. Then there exists a nondecreasing sequence of integers $\{\tau(n)\}$ defined by $\tau(n) := \max\{k \leq n : s_k < s_{k+1}\}$ such that $\lim_{n\to\infty} \tau(n) = \infty$ and the following two properties hold by all (sufficiently large) numbers $n \in \mathbb{N}$:*

$$s_{\tau(n)} \leq s_{\tau(n)+1} \text{ and } s_n \leq s_{\tau(n)+1}.$$

3. Main Results

In this section, we show our suggested inertial iterative technique for solving the SVIP (1.1) - (1.2) and we prove that the sequence established by the proposed algorithm has a strong convergence theorem.

Remark 3.1. It can be seen from Algorithm 1 as follows:

(1) The inertial parameter θ_n in Algorithm 1 is easy to implement in numerical computation because the value of $\|x_n - x_{n-1}\|$ is known before choosing θ_n. One can choose θ_n by using the idea of the results in [1, 13, 27, 29].

Algorithm 1 Modified inertial iterative algorithm for (1.1) - (1.2)

Initialization: Choose arbitrary initial points $x_0, x_1 \in H_1$ and set $\alpha > 0$, $\lambda > 0$ and $\{\gamma_n\} \subset [\gamma_*, \gamma^*] \subset \frac{1}{L}$, where $L = \|A\|^2$. Moreover, choose $\{\beta_n\} \subset (0,1)$ such that

$$\lim_{n\to\infty} \beta_n = 0 \text{ and } \sum_{n=1}^{\infty} \beta_n = \infty. \tag{3.1}$$

Iterative Steps: Calculate $\{x_{n+1}\}$ as follows:

Step 1: Compute $w_n = (1-\beta_n)\big[x_n + \theta_n(x_n - x_{n-1})\big]$, where $\{\theta_n\} \subset [0,1)$ satisfying

$$\theta_n = \begin{cases} \min\left\{\theta, \frac{\epsilon_n}{\|x_n - x_{n-1}\|}\right\} & \text{if } \quad x_n \neq x_{n-1}, \\ \theta & \text{otherwise,} \end{cases} \tag{3.2}$$

while $\epsilon_n = \circ(\beta_n)$ is a positive sequence, i.e., $\lim_{n\to\infty} \dfrac{\epsilon_n}{\beta_n} = 0$.

Step 2: Compute $x_{n+1} = J_\lambda^{B_1}(w_n + \gamma_n A^*(J_\lambda^{B_2} - I)Aw_n)$.

Set $n := n+1$ and go to Step 1.

(2) It can be seen from the expression (3.2) in Algorithm 1 that $\theta_n \|x_n - x_{n-1}\| \leq \epsilon_n$ and this implies that $\sum_{n=1}^{\infty} \theta_n \|x_n - x_{n-1}\| < \infty$. Therefore,

$$\lim_{n\to\infty} \frac{\theta_n}{\beta_n}\|x_n - x_{n-1}\| \leq \lim_{n\to\infty} \frac{\epsilon_n}{\beta_n} = 0. \tag{3.3}$$

We now present our main strong convergence theorem of the sequence generated by the proposed algorithm.

Theorem 3.2. *Let H_1 and H_2 be two real Hilbert spaces. Let $B_1 : H_1 \to 2^{H_1}$ and $B_2 : H_2 \to 2^{H_2}$ be two maximal monotone mappings. Let $A : H_1 \to H_2$ be a linear bounded operator with its adjoint A^*. Let Ω be a solution set of the problem (1.1) - (1.2) and assume that $\Omega \neq \emptyset$. Then sequence $\{x_n\}$ generated by Algorithm 1 converges strongly to a point $x^* \in \Omega$, where $x^* = P_\Omega(0)$.*

Proof. Let $x^* \in \Omega$. Thus $x^* = J_\lambda^{B_1} x^*$ and $Ax^* = J_\lambda^{B_2}(Ax^*)$. By the definition of $\{w_n\}$, we get

$$\begin{aligned} \|w_n - x^*\| &= \|(1-\beta_n)[x_n + \theta_n(x_n - x_{n-1})] - x^*\| \\ &\leq (1-\beta_n)\|x_n - x^*\| + (1-\beta_n)\theta_n \|x_n - x_{n-1}\| + \beta_n \|x^*\| \\ &= (1-\beta_n)\|x_n - x^*\| + \beta_n\left[(1-\beta_n)\frac{\theta_n}{\beta_n}\|x_n - x_{n-1}\| + \|x^*\|\right]. \end{aligned} \tag{3.4}$$

It can be deduced from (3.3) that there exists a constant $K_1 > 0$ such that $(1-\beta_n)\frac{\theta_n}{\beta_n}\|x_n - x_{n-1}\| \leq K_1$ for all $n \in \mathbb{N}$. Thus, we obtain from (3.4) that

$$\|w_n - x^*\| \leq (1-\beta_n)\|x_n - x^*\| + \beta_n K_2, \tag{3.5}$$

where $K_2 := K_1 + \|x^*\|$. From Lemma 2.4 and definition of $\{\gamma_n\}$ we have

$$\|x_{n+1} - x^*\|^2 \leq \|w_n - x^*\|^2 - \gamma_n(1-\gamma_n L)\left\|(J_\lambda^{B_2} - I)Aw_n\right\|^2 \tag{3.6}$$

$$\leq \|w_n - x^*\|^2. \tag{3.7}$$

This implies that

$$\|x_{n+1} - x^*\| \leq \|w_n - x^*\|. \tag{3.8}$$

Next, by combining (3.5) with (3.8) and by induction we obtain

$$\begin{aligned}\|x_{n+1} - x^*\| &\leq (1-\beta_n)\|x_n - x^*\| + \beta_n K_2 \\ &\leq \max\left\{\|x_n - x^*\|, K_2\right\} \leq \ldots \leq \max\left\{\|x_0 - x^*\|, K_2\right\}.\end{aligned}$$

This shows that sequence $\{x_n\}$ is bounded. Consequently, $\{w_n\}$ is also bounded. By the definition of $\{w_n\}$ we have

$$\begin{aligned}\|w_n - x^*\|^2 &= \|(1-\beta_n)[x_n + \theta_n(x_n - x_{n-1})] - x^*\|^2 \\ &\leq \|(1-\beta_n)(x_n - x^*) + (1-\beta_n)\theta_n(x_n - x_{n-1})\|^2 + 2\beta_n\langle -x^*, w_n - x^*\rangle \\ &= (1-\beta_n)^2\|x_n - x^*\|^2 + (1-\beta_n)^2\theta_n^2\|x_n - x_{n-1}\|^2 \\ &\quad + 2\theta_n(1-\beta_n)\langle x_n - x^*, x_n - x_{n-1}\rangle + 2\beta_n\langle -x^*, w_n - x_{n+1}\rangle \\ &\quad + 2\beta_n\langle -x^*, x_{n+1} - x^*\rangle \\ &\leq (1-\beta_n)\|x_n - x^*\|^2 + 2\beta_n\langle -x^*, x_{n+1} - x^*\rangle \\ &\quad + \beta_n\Bigg[(1-\beta_n)^2\frac{\theta_n^2}{\beta_n}\|x_n - x_{n-1}\|^2 + 2\|x^*\|\|w_n - x_{n+1}\| \\ &\quad + 2(1-\beta_n)\frac{\theta_n}{\beta_n}\|x_n - x_{n-1}\|\|x_n - x^*\|\Bigg] \qquad (3.9) \\ &\leq (1-\beta_n)\|x_n - x^*\|^2 + 2\beta_n\langle -x^*, x_{n+1} - x^*\rangle \\ &\quad + \beta_n\Big[K_3 + 2\|x^*\|\|w_n - x_{n+1}\|\Big] \qquad (3.10)\end{aligned}$$

for some $K_3 > 0$. Combining (3.7) together with (3.10) we obtain

$$\begin{aligned}\|x_{n+1} - x^*\|^2 \leq (1-\beta_n)\|x_n - x^*\|^2 + 2\beta_n\langle -x^*, x_{n+1} - x^*\rangle \\ + \beta_n\Big[K_3 + 2\|x^*\|\|w_n - x_{n+1}\|\Big].\end{aligned} \tag{3.11}$$

Next, we consider the following two cases.

Case 1. Assume that the sequence $\{\|x_n - x^*\|\}$ is a monotonically decreasing sequence, that is, there exists a natural number N such that $\|x_{n+1} - x^*\| \leq \|x_n - x^*\|$ for all $n \geq N$. Then $\{\|x_n - x^*\|\}$ is convergent and $\|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \to 0$ as $n \to \infty$. Let $\sigma_n := x_n + \theta_n(x_n - x_{n-1})$. Then

$$\begin{aligned}\|\sigma_n - x^*\|^2 &= \|x_n + \theta_n(x_n - x_{n-1}) - x^*\|^2 \\ &\leq \|x_n - x^*\|^2 + \beta_n\left[\frac{\theta_n^2}{\beta_n}\|x_n - x_{n-1}\|^2 + 2\frac{\theta_n}{\beta_n}\|x_n - x_{n-1}\|\|x_n - x^*\|\right].\end{aligned}$$

Since $\{x_n\}$ is bounded, we observe that $\{\sigma_n\}$ is also bounded. Due to the condition (3.1), and (3.3), there exists $K_4 > 0$ such that $\|\sigma_n - x^*\|^2 \leq \|x_n - x^*\|^2 + \beta_n K_4$. This implies that

$$\begin{aligned}\|w_n - x^*\|^2 &= \|\sigma_n - x^* - \beta_n\sigma_n\|^2 = \|\sigma_n - x^*\|^2 + \beta_n^2\|\sigma_n\|^2 - 2\beta_n\langle\sigma_n - x^*, \sigma_n\rangle \\ &\leq \|x_n - x^*\|^2 + \beta_n K_4 + \beta_n^2\|\sigma_n\|^2 - 2\beta_n\langle\sigma_n - x^*, \sigma_n\rangle \\ &\leq \|x_n - x^*\|^2 + \beta_n K_5, \qquad (3.12)\end{aligned}$$

for some $K_5 > 0$. Combining the expressions (3.6) and (3.12) we have

$$\|x_{n+1} - x^*\|^2 \leq \|x_n - x^*\|^2 + \beta_n K_5 - \gamma_n(1-\gamma_n L)\left\|(J_\lambda^{B_2} - I)Aw_n\right\|^2.$$

This implies that

$$\gamma_n(1-\gamma_n L)\left\|(J_\lambda^{B_2} - I)Aw_n\right\|^2 \leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \beta_n K_5. \tag{3.13}$$

We obtain from inequality above together with $\{\gamma_n\} \subset (0, \frac{1}{L})$ and (3.1) that

$$\lim_{n\to\infty}\left\|(J_\lambda^{B_2} - I)Aw_n\right\| = 0. \tag{3.14}$$

This implies that

$$\left\|A^*(J_\lambda^{B_2} - I)Aw_n\right\| \leq \|A^*\|\left\|(J_\lambda^{B_2} - I)Aw_n\right\| \to 0 \text{ as } n \to \infty. \tag{3.15}$$

Using Lemma 2.3 (1) we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \left\|J_\lambda^{B_1}\left(w_n + \gamma_n A^*(J_\lambda^{B_2} - I)Aw_n\right) - x^*\right\|^2 \\
&\leq \left\langle x_{n+1} - x^*, w_n + \gamma_n A^*(J_\lambda^{B_2} - I)Aw_n - x^*\right\rangle \\
&= \frac{1}{2}\|x_{n+1} - x^*\|^2 + \frac{1}{2}\left\|w_n - x^* + \gamma_n A^*(J_\lambda^{B_2} - I)Aw_n\right\|^2 \\
&\quad - \frac{1}{2}\left\|x_{n+1} - w_n - \gamma_n A^*(J_\lambda^{B_2} - I)Aw_n\right\|^2 \\
&= \frac{1}{2}\|x_{n+1} - x^*\|^2 + \frac{1}{2}\Big[\|w_n - x^*\|^2 + \gamma_n^2\left\|A^*(J_\lambda^{B_2} - I)Aw_n\right\|^2 \\
&\quad + 2\left\langle w_n - x^*, \gamma_n A^*(J_\lambda^{B_2} - I)Aw_n\right\rangle\Big] \\
&\quad - \frac{1}{2}\Big[\|x_{n+1} - w_n\|^2 + \gamma_n^2\left\|A^*(J_\lambda^{B_2} - I)Aw_n\right\|^2 \\
&\quad - 2\gamma_n\left\langle x_{n+1} - w_n, A^*(J_\lambda^{B_2} - I)Aw_n\right\rangle\Big] \\
&= \frac{1}{2}\|x_{n+1} - x^*\|^2 + \frac{1}{2}\|w_n - x^*\|^2 - \frac{1}{2}\|x_{n+1} - w_n\|^2 \\
&\quad + 2\gamma_n\left\langle x_{n+1} - x^*, A^*(J_\lambda^{B_2} - I)Aw_n\right\rangle
\end{aligned}$$

which implies that

$$\begin{aligned}
\|x_{n+1} - w_n\|^2 &\leq \|w_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\
&\quad + 2\gamma_n\left\langle x_{n+1} - x^*, A^*(J_\lambda^{B_2} - I)Aw_n\right\rangle.
\end{aligned} \tag{3.16}$$

From (3.1), (3.15), (3.12), and (3.16) we obtain

$$\begin{aligned}
\|x_{n+1} - w_n\|^2 &\leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \beta_n K_4 \\
&\quad + 2\gamma_n\|x_{n+1} - x^*\|\left\|A^*(J_\lambda^{B_2} - I)Aw_n\right\| \to 0 \text{ as } n \to \infty.
\end{aligned} \tag{3.17}$$

Further, we also obtain

$$\begin{aligned}\left\|x_{n+1} - J_\lambda^{B_1} w_n\right\| &= \left\|J_\lambda^{B_1}\left(w_n + \gamma_n A^*\left(J_\lambda^{B_2} - I\right) A w_n\right) - J_\lambda^{B_1} w_n\right\| \\ &\leq \gamma_n \left\|A^*\left(J_\lambda^{B_2} - I\right) A w_n\right\| \to 0 \text{ as } n \to \infty. \end{aligned} \tag{3.18}$$

and

$$\begin{aligned}\|w_n - x_n\| &= \|(1-\beta_n)[x_n + \theta_n(x_n - x_{n-1})] - x_n\| \\ &\leq (1-\beta_n)\theta_n\|x_n - x_{n-1}\| + \beta_n\|x_n\| \\ &= \beta_n\left[(1-\beta_n)\frac{\theta_n}{\beta_n}\|x_n - x_{n-1}\| + \|x_n\|\right] \to 0 \text{ as } n \to \infty. \end{aligned} \tag{3.19}$$

Therefore, by (3.17) and (3.18), we have

$$\lim_{n\to\infty}\left\|w_n - J_\lambda^{B_1} w_n\right\| = 0 \tag{3.20}$$

and it follows from (3.17) and (3.19) that

$$\|x_{n+1} - x_n\| \leq \|x_{n+1} - w_n\| + \|w_n - x_n\| \to 0 \text{ as } n \to \infty. \tag{3.21}$$

Since sequence $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup z$ for some $z \in H_1$ and

$$\limsup_{n\to\infty}\langle -x^*, x_n - x^*\rangle = \lim_{i\to\infty}\langle -x^*, x_{n_i} - x^*\rangle.$$

By (3.19), we have $u_{n_i} \rightharpoonup z$. Therefore, from Lemma 2.2, Lemma 2.3, and (3.20) we have $z = J_\lambda^{B_1} z$. Since A is a linear operator, it follows that $Au_{n_i} \rightharpoonup Az$. Similarly, by Lemma 2.2, Lemma 2.3, and (3.14) we have $Az = J_\lambda^{B_2} Az$. Therefore, $z \in \Omega$.
Since $x^* = P_\Omega(0)$, thus

$$\limsup_{n\to\infty}\langle -x^*, x_n - x^*\rangle = \lim_{i\to\infty}\langle -x^*, x_{n_i} - x^*\rangle = \langle -x^*, x^* - z\rangle \leq 0.$$

By using (3.1), (3.3), (3.11), (3.17), and Lemma 2.5, we have that $\lim_{n\to\infty}\|x_n - x^*\| = 0$.

Case 2. Assume that there exists a subsequence $\{\|x_{n_j} - x^*\|\}$ of $\{\|x_n - x^*\|\}$ such that $\|x_{n_j} - x^*\| < \|x_{n_j+1} - x^*\|$ for all $j \geq 0$. Applying Lemma 2.6, there exists a nondecreasing sequence integers $\{\tau(n)\}$ such that $\lim_{n\to\infty}\tau(n) = \infty$ and

$$\|x_{\tau(n)} - x^*\| \leq \|x_{\tau(n)+1} - x^*\| \quad \text{and} \quad \|x_n - x^*\| \leq \|x_{\tau(n)+1} - x^*\|. \tag{3.22}$$

It follows from (3.13) and (3.22) that

$$\begin{aligned}&\gamma_{\tau(n)}(1 - \gamma_{\tau(n)}L)\left\|(J_\lambda^{B_2} - I)Au_{\tau(n)}\right\|^2 \\ &\qquad \leq \|x_{\tau(n)} - x^*\|^2 - \|x_{\tau(n)+1} - x^*\|^2 + \beta_{\tau(n)}K_4 \leq \beta_{\tau(n)}K_4 \to 0 \text{ as } n \to \infty. \end{aligned} \tag{3.23}$$

In a similar manner as in Case 1, we can show that

$$\lim_{n\to\infty}\left\|u_{\tau(n)} - J_\lambda^{B_1} u_{\tau(n)}\right\| = 0,$$

$$\lim_{n\to\infty}\left\|x_{\tau(n)+1} - x_{\tau(n)}\right\| = 0, \tag{3.24}$$

$$\lim_{n\to\infty}\left\|u_{\tau(n)} - x_{\tau(n)}\right\| = 0, \tag{3.25}$$

and the boundlessness of sequence $\{x_{\tau(n)}\}$, we also show that

$$\limsup_{n\to\infty}\langle -x^*, x_{\tau(n)} - x^*\rangle = \langle -x^*, z - x^*\rangle \le 0. \tag{3.26}$$

Furthermore, it follows from (3.9) and (3.22) that

$$\begin{aligned}\beta_{\tau(n)}\|x_{\tau(n)} - x^*\|^2 &\le \|x_{\tau(n)} - x^*\|^2 - \|x_{\tau(n)+1} - x^*\|^2 \\ &\quad + \beta_{\tau(n)}\Big[(1-\beta_{\tau(n)})^2\frac{\alpha_{\tau(n)}^2}{\beta_{\tau(n)}}\|x_{\tau(n)} - x_{\tau(n)-1}\|^2 \\ &\quad + 2(1-\beta_{\tau(n)})\frac{\alpha_{\tau(n)}}{\beta_{\tau(n)}}\|x_{\tau(n)} - x_{\tau(n)-1}\|\|x_{\tau(n)} - x^*\| \\ &\quad + 2\|x^*\|\|u_{\tau(n)} - x_{\tau(n)+1}\| + 2\langle -x^*, x_{\tau(n)+1} - x^*\rangle\Big] \\ &\le \beta_{\tau(n)}\Big[(1-\beta_{\tau(n)})^2\frac{\alpha_{\tau(n)}^2}{\beta_{\tau(n)}}\|x_{\tau(n)} - x_{\tau(n)-1}\|^2 \\ &\quad + 2(1-\beta_{\tau(n)})\frac{\alpha_{\tau(n)}}{\beta_{\tau(n)}}\|x_{\tau(n)} - x_{\tau(n)-1}\|\|x_{\tau(n)} - x^*\| \\ &\quad + 2\|x^*\|\|u_{\tau(n)} - x_{\tau(n)+1}\| + 2\langle -x^*, x_{\tau(n)+1} - x^*\rangle\Big].\end{aligned}$$

Thus,

$$\begin{aligned}\|x_{\tau(n)} - x^*\|^2 &\le (1-\beta_{\tau(n)})^2\frac{\alpha_{\tau(n)}^2}{\beta_{\tau(n)}}\|x_{\tau(n)} - x_{\tau(n)-1}\|^2 \\ &\quad + 2(1-\beta_{\tau(n)})\frac{\alpha_{\tau(n)}}{\beta_{\tau(n)}}\|x_{\tau(n)} - x_{\tau(n)-1}\|\|x_{\tau(n)} - x^*\| \\ &\quad + 2\|x^*\|\|u_{\tau(n)} - x_{\tau(n)+1}\| + 2\langle -x^*, x_{\tau(n)} - x^*\rangle.\end{aligned} \tag{3.27}$$

By (3.3), (3.25), (3.26), and (3.27), we have $\limsup_{n\to\infty}\|x_{\tau(n)} - x^*\| = 0$. This implies that

$$\lim_{n\to\infty}\|x_{\tau(n)} - x^*\| = 0. \tag{3.28}$$

Hence, from (3.24) and (3.28) we obtain

$$\|x_{\tau(n)+1} - x^*\| \le \|x_{\tau(n)+1} - x_{\tau(n)}\| + \|x_{\tau(n)} - x^*\| \to 0 \text{ as } n \to \infty. \tag{3.29}$$

It follows from (3.22) and (3.29) that $\lim_{n\to\infty}\|x_n - x^*\| = 0$. Therefore, the proof is completed. ∎

4. Applications

In this section, we apply the proposed method to derive algorithms for solving two main problems; split feasibility problem and split minimization problem.

4.1. Split Feasibility Problem

Let H_1 and H_2 be two real Hilbert spaces, $C \subset H_1$ and $Q \subset H_2$ be nonempty closed and convex sets, and $A : H_1 \to H_2$ be a bounded linear operator. The *split feasibility problem* introduced by Censor and Elfving [6] deals with the following problem:

$$\text{Find} \quad x^* \in C \quad \text{such that} \quad Ax^* \in Q. \tag{4.1}$$

The set Θ is used to denote the solution set of the problem (4.1). This problem has been successfully applied to a variety of real-world problems, particularly in signal processing and picture reconstruction, with significant progress made in intensity modulated treatment. Many iterative algorithms have been established for the problem (4.1), one can refer to, for example, [1, 3, 4, 10, 14, 29, 31] and references therein.

Applying the derivations (2.2) and Theorem 4.1 we obtain the following result for solving the problem (4.1).

Algorithm 2 Modified inertial iterative algorithm for (4.1)

Initialization: Choose arbitrary starting points $x_0, x_1 \in H_1$ and set $\alpha > 0$, $\lambda > 0$ and $\{\gamma_n\} \subset [\gamma_*, \gamma^*] \subset \frac{1}{L}$, where $L = \|A\|^2$. Moreover, choose $\{\beta_n\} \subset (0, 1)$ defined in (3.1).
Iterative Steps: Calculate $\{x_{n+1}\}$ as follows:

$$\begin{cases} w_n = (1 - \beta_n)\big[x_n + \theta_n(x_n - x_{n-1})\big], \\ x_{n+1} = P_C\left(w_n - \gamma_n A^*(I - P_Q)Aw_n\right), \end{cases}$$

where sequence $\{\theta_n\}$ is defined in (3.2).

Theorem 4.1. *Let H_1 and H_2 be two real Hilbert spaces, C and Q be nonempty closed convex subsets, and $A : H_1 \to H_2$ be a bounded linear operator with its adjoint A^*. be the adjoint of A. Let Θ be a solution set of the problem (4.1) and assume that $\Theta \neq \emptyset$. Then sequence $\{x_n\}$ generated by Algorithm 2 converges strongly to a point $x^* \in \Theta$, where $x^* = P_\Theta(0)$.*

4.2. Split Minimization Problem

Let H_1 and H_2 be real Hilbert spaces, and $f : H_1 \to \mathbb{R}\cup\{+\infty\}$ and $g : H_2 \to \mathbb{R}\cup\{+\infty\}$ be proper lower semi-continuous and convex functions. It follows from the face that the subdifferential of f and g, ∂f and ∂g, respectively, are maximal monotone mappings. In this section, we consider the following the *split minimization problem*:

$$x^* \in \arg\min f \quad \text{and} \quad Ax^* \in \arg\min g, \tag{4.2}$$

where A is a bounded linear operator. The set Γ is used to denote the solution set of the problem (4.2). In other words, this problem is finding a minimizer x^* of f in Hilbert space H_1 such that its image under linear bounded operator Ax^* minimizes g in Hilbert space H_2. Many iterative algorithms has been established for the problem (4.2), one can refer to, for example, [10, 28] and references therein.

Letting $B_1 = \partial f$ and $B_2 = \partial g$ in the problem (1.1) - (1.2) and applying Theorem 4.1 we obtain the result for solving the problem (4.2).

Theorem 4.2. *Let H_1 and H_2 be two real Hilbert spaces. Let $\partial f : H_1 \to 2^{H_1}$ and $\partial g : H_2 \to 2^{H_2}$ be two maximal monotone mappings. Let $A : H_1 \to H_2$ be a bounded linear operator with its adjoint A^*. Let Γ be a solution set of the problem (4.2) and assume that $\Gamma \neq \emptyset$. Then sequence $\{x_n\}$ generated by Algorithm 3 converges strongly to a point $x^* \in \Gamma$, where $x^* = P_\Gamma(0)$.*

5. Numerical Experiments

In this section, we present some numerical experiments to test the computational performance and potential applicability of our proposed Algorithm 1 in comparison with

Algorithm 3 Modified inertial iterative algorithm for (4.2)

Initialization: Choose arbitrary starting points $x_0, x_1 \in H_1$ and set $\alpha > 0$, $\lambda > 0$ and $\{\gamma_n\} \subset [\gamma_* \gamma^*] \subset \frac{1}{L}$, where $L = \|A\|^2$. Moreover, choose $\{\beta_n\} \subset (0,1)$ defined in (3.1).
Iterative Steps: Calculate $\{x_{n+1}\}$ as follows:

$$\begin{cases} w_n = (1-\beta_n)\left[x_n + \theta_n(x_n - x_{n-1})\right], \\ x_{n+1} = J_\lambda^{\partial f}\left(w_n - \gamma_n A^*\left(I - J_\lambda^{\partial g}\right)Aw_n\right), \end{cases}$$

where sequence $\{\theta_n\}$ is defined in (3.2).

some existing algorithms for solving the considered split variational inequality problem. We present two numerical examples and we apply the proposed algorithm in signal processing, in particular we solve the problem of recovering a sparse and noisy signal from a limited number. All algorithms were tested by the MATLAB program (version R2021a) on a macOS (1.60 GHz Dual-Core, Intel Core i5, CPU @ 2133 MHz, Ram 8.00 GB).

For both examples, the number of iterations and CPU time in seconds are denoted by Iter and CPU, respectively. We denote Byrne's Alg. (1.5) by the compared algorithm proposed in [5] and denote Chuang's Alg. by the algorithm proposed in [10, Algorithm 3.1].

Example 5.1. [13] Let $A, A_1, A_2 : R^m \to R^m$ be matrices generated from a normal distribution with mean zero and unit variance. Mappings $B_1, B_2 : R^m \to R^m$ are defined by

$$B_1(x) = A_1^* A_1(x) \quad \text{and} \quad B_2(y) = A_2^* A_2(y).$$

Find a point $x^* = (x_1^*, \ldots, x_m^*)^T \in R^m$ such that $B_1(x^*) = (0, \ldots, 0)^T$ and $B_2(Ax^*) = (0, ..., 0)^T$. Note that $x^* = (0, \ldots, 0)^T \in \mathbb{R}^m$ is a the minimum norm solution.

In this example, we divide our comparison into three experiments, which are as follows.

Experiment 1. In order to determine the effectiveness of the inertial term $(1-\beta_n)$, we analyze the proposed Algorithm 1 with and without the inertial term $(1-\beta_n)$. For this experiment the initial points x_0 and x_1 are chosen randomly and $x_0 = x_1 = [1, \ldots, 1]^T$. We set $m = 1000, \theta = \frac{1}{100}, \lambda = 1, \gamma_n = \frac{0.9}{L}$, $\beta_n = \frac{1}{n+2}$, and $\epsilon_n = \beta_n^3$ as the control parameters. The procedure will terminate when $\|x_n - x^*\| \leq TOL$, where TOL is $10^{-2}, 10^{-4}, 10^{-6}$, 10^{-8}, respectively.

It can be seen from Table 1 that Algorithm 1 with inertial term $(1-\beta_n)$, denoted by Alg. 1 (W), have greatly improved more than algorithm without inertial term $(1-\beta_n)$, denoted by Alg. 1 (WO), in terms of number of iterations and CPU time.

Experiment 2. We illustrate the convergence behavior of the proposed algorithm 1 with respect to the inertial parameter θ_n and the parameter β_n. We consider different values of the parameters β_n. For this experiment the initial points x_0 and x_1 are chosen randomly and set $m = 1000, \theta = \frac{1}{100}, \lambda = 1, \gamma_n = \frac{0.9}{L}$, $\theta_n = \frac{n}{2n+2}$, and $\epsilon_n = \beta_n^3$ as the control parameters. We set $\|x_n - x^*\| \leq 10^{-5}$ as the stopping criteria.

As shown in Table 2 and Figure 1, the performance of Algorithm 1 has greatly improved performance with different values of β_n.

Experiment 2. We compare our proposed Algorithm 1 with Byrne's Alg. (1.5) and Chuang's Alg. For the comparison the initial points x_0 and x_1 are chosen randomly, $m = 1000$, $\lambda_n = \lambda = 1$ and $\gamma_n = \frac{0.3}{L}$ for all algorithms. We choose $\alpha = \frac{1}{2}$, $\beta_n = \frac{1}{n+2}$

TABLE 1. The performance between the proposed Algorithm 1 with and without $(1-\beta_n)$.

TOL	x_0 and x_1 are chosen randomly				$x_0 = x_1 = [1, \ldots, 1]^T$			
	Alg. 1 (W)		Alg. 1 (WO)		Alg. 1 (W)		Alg. 1 (WO)	
	Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU
10^{-2}	9	1.2248	19	3.0434	10	1.3866	24	4.5949
10^{-4}	17	2.8467	41	8.1389	19	3.0083	45	8.2932
10^{-6}	23	4.2276	52	10.0477	27	4.5611	61	11.1340
10^{-8}	36	6.9059	77	14.1564	39	7.4917	84	17.1669

TABLE 2. Performance of the proposed Algorithm 1 with different values of parameter β_n.

θ_n	β_n							
	$\frac{1}{5n+3}$		$\frac{1}{n+2}$		$\frac{\ln n}{n}$		$\frac{1}{\sqrt{n+2}}$	
	Iter	CPU	Iter	CPU	Iter	CPU	Iter	CPU
$\frac{n}{2n+2}$	43	8.6495	37	7.0866	24	5.7033	20	3.6564

and $\epsilon_n = \beta_n^3$ for Algorithm 1, $k = 5$, $\delta = 0.3$ and $\gamma_n = \frac{\delta}{L}$ for Chuang's Alg. Finally, we set $\|x_n - x^*\|_2 \leq TOL$ as the stopping criteria for all the algorithms, where TOL is $10^{-2}, 10^{-3}, 10^{-4}, 10^{-5}$, respectively.

Table 3 with Figure 2 illustrate that proposed Algorithm 1 is the best effective method of convergence rate. Because the Chuang's Alg. makes use of the linesearch process, it is significantly faster than the Byrne's Alg. (1.5) when it comes to stopping criteria 10^{-2} and 10^{-3}. However, it performs more slowly than the Byrne Algorithm when it comes to the stopping criteria 10^{-4} and 10^{-5}.

TABLE 3. The performance of Algorithm 1, Byrne's Alg. (1.5) , and Chuang's Alg. with different values of stopping criterion TOL.

TOL	**Algorithm 1**		**Byrne's Alg. (1.5)**		**Chuang's Alg.**	
	Iter	CPU	Iter	CPU	Iter	CPU
10^{-2}	28	7.0783	41	7.1841	46	27.5944
10^{-3}	54	13.4705	84	17.9171	84	55.8905
10^{-4}	77	21.3791	113	26.8859	116	86.7047
10^{-5}	94	25.9012	136	34.2200	116	87.2640

Next example, we will look at the problem of recovering a sparse and noisy signal.

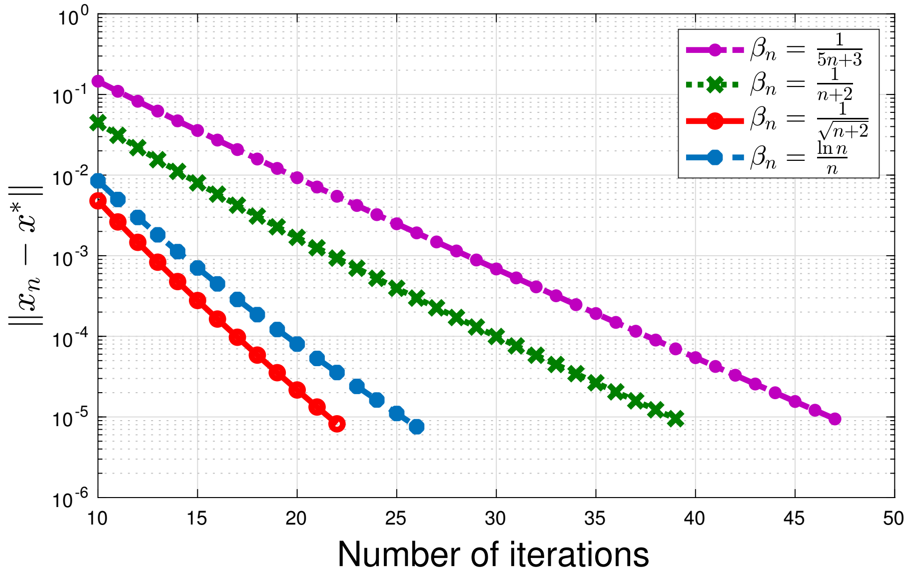


FIGURE 1. Performance of the proposed Algorithm 1 with different values of parameter β_n.

Example 5.2. [28, 29] Let $A : \mathbb{R}^n \to \mathbb{R}^m$ $(m < n)$ be a bounded linear operator. A *compressed sensing problem* can be written as the following underdetermined linear equation system:

$$u = Av + \tau, \tag{5.1}$$

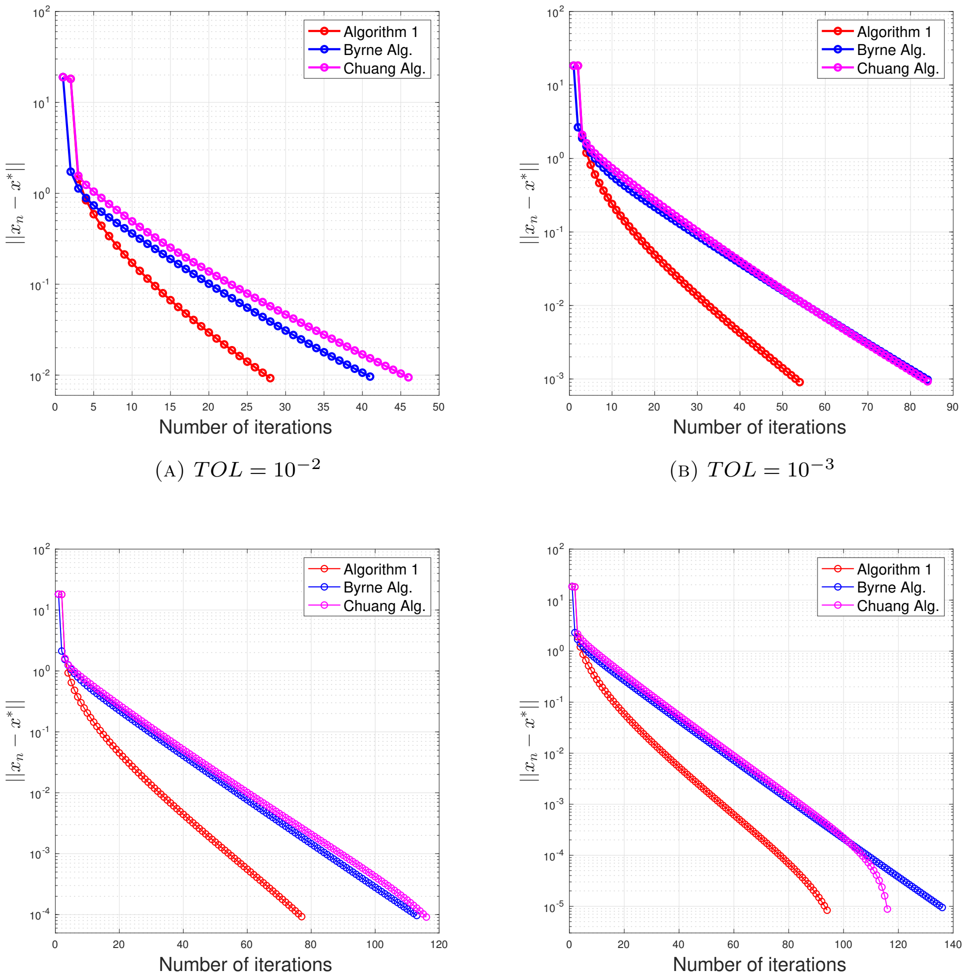
where $v \in \mathbb{R}^N$ represents K-sparse signal to be recovered $(i << n)$ and the vector $u \in \mathbb{R}^m$ is the observed data with additive noisy τ. The problem (5.1) is equivalent to the following minimization problem, or least absolute shrinkage and selection operator problem (LASSO):

$$\begin{aligned} &\min_{x \in \mathbb{R}^n} \frac{1}{2}\|Av - u\|_2^2, \\ &\quad \text{s.t. } \|v\|_1 \leq t, \end{aligned} \tag{5.2}$$

where t is a positive constant.

To solve the problem (1.1) - (1.2), we now apply our Algorithm 1 to the problem (5.2). Here, $A \in \mathbb{R}^{m\times n}$ is generated from a normal distribution with mean zero and unit variance, original signal v contained i randomly generated ± 1 spikes with nonzero elements, u is generated by Gaussian noise τ of variance 10^{-4}, and the initial points x_0 and x_1 are chosen randomly. For simplicity we choose the following parameters: Iter $=$ 100000, $\gamma_n = \frac{0.5}{L}$ and $t = i - 0.0001$ for all the algorithms, $\theta = \frac{1}{100}$, $\beta_n = \frac{1}{\sqrt{n+2}}$ and $\epsilon_n = \beta_n^3$ for Algorithm 1, and $\delta = 0.5$ and $\gamma_n = \frac{\delta}{L}$ for Chuang's Alg. For all algorithms are measured by mean squared error $E_n := \frac{1}{N}\|x^* - v\|$ and algorithms stop if $E_n \leq 10^{-4}$.

The numerical results of all algorithms are described in Table 4, Figure 3 and 4. Observing that our the performance of proposed Algorithm 1 is better than other algorithms.



(A) $TOL = 10^{-2}$ (B) $TOL = 10^{-3}$

(C) $TOL = 10^{-4}$ (D) $TOL = 10^{-5}$

Figure 2. Performance of Algorithms 1, Byrne's Alg. (1.5), and Chuang's Alg. with different values of the stopping criterion.

6. Conclusion

In this article, we proposed an inertial algorithm for solving split variational inequality problem involving two maximally monotone mappings. We showed a strong convergence of the proposed algorithm under some easy to verify and standard conditions. We presented numerical examples to illustrate and support the proposed theoretical convergence results. Additionally, we showed the potential applicability of the proposed algorithm in signal recovery. Numerical experiments presented suggested that the proposed algorithm is fast, efficient and robust in comparison with some algorithms in the literature.

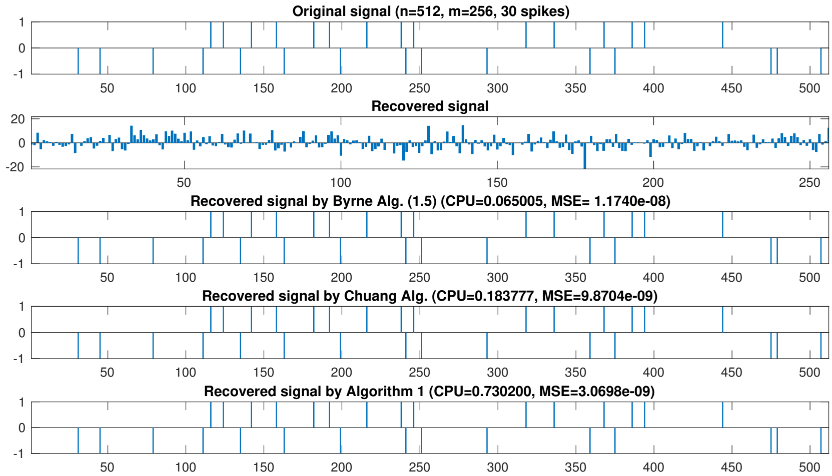


FIGURE 3. The numerical performance of Byrne's Alg. (1.5), and Chuang's Alg. and Algorithm 1 for the recovery of a sparse $i = 30$ signal, $n = 512$, and $m = 256$.

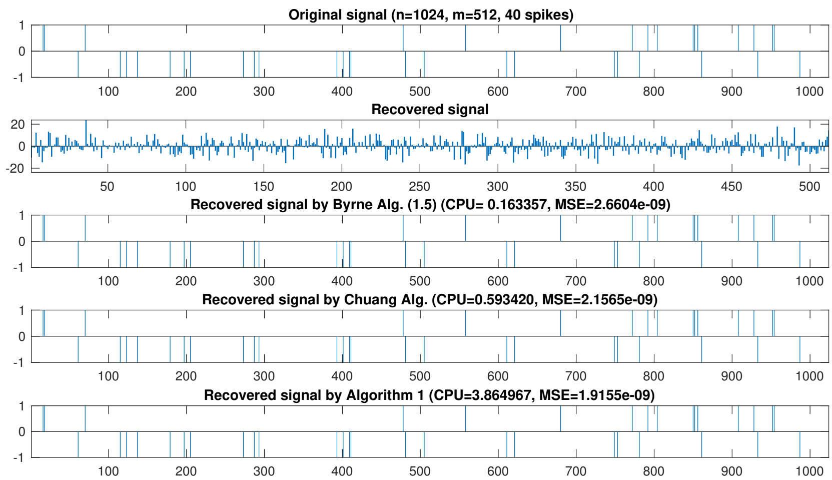


FIGURE 4. The numerical performance of Byrne's Alg. (1.5), and Chuang's Alg. abd Algorithm 1 for the recovery of a sparse $i = 40$ signal, $n = 1024$, and $m = 512$.

TABLE 4. Recovered signal of all algorithms with the different values of n, m, and i.

Algorithms	$n = 512$, $m = 256$, 30 spikes		$n = 1024$, $m = 512$, 40 spikes	
	MSE	CPU	MSE	CPU
Algorithm 1	3.0698e-09	0.730200	1.9155e-09	3.864967
Byrne's Alg. (1.5)	1.1740e-08	0.065005	2.6604e-09	0.163357
Chuang's Alg.	9.8704e-09	0.183777	2.1565e-09	0.593420

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