BANGMOD INTERNATIONAL JOURNAL OF MATHEMATICAL & COMPUTATIONAL SCIENCE
ISSN: 2408-154X (Print)
ISSN: 3057-0557 (Online)
https://bangmodjmcs.com
Volume 12 (2026)
Pages 434–474



A Novel Framework for Global Business Partner Selection Using p, q-Quasi-Rung Orthopair Normal Fuzzy Sets

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Received: 23 July 2025 / Accepted: 29 January 2026

Abstract In the contemporary global economy, organizations increasingly confront complex and interdependent challenges that demand advanced analytical frameworks to maintain sustainable competitiveness. Among these challenges, the selection of an appropriate global business partner constitutes a critical yet intricate decision-making problem, often complicated by uncertainty, ambiguity, and incomplete information. To address these limitations, this study introduces the concept of the p, q-quasi-rung orthopair normal fuzzy set (p, q-QRONFS), which integrates the structural adaptability of p, q-quasi-rung orthopair fuzzy sets with the interpretive strength of normal fuzzy numbers.

On this theoretical foundation, novel weighted averaging and geometric aggregation operators for p, q-quasi-rung orthopair normal fuzzy numbers (p, q-QRONFNs) are developed, explicitly incorporating the decision-makers confidence level to enhance the precision, robustness, and flexibility of the decision process. Subsequently, a multiple attribute decision-making (MADM) model, termed the p, q-QRONFMADM approach, is proposed to facilitate rational evaluations in uncertain environments.

The applicability and effectiveness of the proposed method are illustrated through a comprehensive case study involving global business partner selection. Comparative analyses with existing fuzzy decision-making methods further demonstrate the superiority of the proposed framework in terms of resilience, adaptability, and decision reliability, underscoring its potential as a robust tool for complex real-world decision-making scenarios.

MSC: 03E72; 94D05

Keywords: decision-making, multiple attribute decision-making, normal fuzzy number, p, q-quasi-rung orthopair fuzzy set, q-rung orthopair fuzzy set

Published online: 7 November 2026
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Published by Center of Excellence in Theoretical and Computational Science (TaCS-CoE)

Please cite this article as: P. Yiarayong, A Novel Framework for Global Business Partner Selection Using p, q-Quasi-Rung Orthopair Normal Fuzzy Sets, Bangmod Int. J. Math. & Comp. Sci., Vol. 12 (2026), 434–474. https://doi.org/10.58715/bangmodjmcs.2026.12.19

1. Introduction

In the increasingly interconnected global economy, organizations are continually confronted with complex, dynamic, and interdependent challenges that demand sophisticated analytical frameworks to sustain competitiveness. Among these challenges, the selection of suitable global business partners represents a crucial yet intricate decision-making process, often hindered by ambiguity, uncertainty, and incomplete information. Addressing such complexities requires the integration of advanced decision-support tools capable of managing vagueness and imprecision in human judgment, thereby enhancing the reliability of strategic evaluations.

FS theory has emerged as a powerful mathematical tool for modeling vagueness and ambiguity inherent in human judgment. Since Zadeh [32] introduced the seminal concept of FSs in 1965, researchers have proposed numerous generalizations to refine the capacity of fuzzy systems to represent uncertainty in decision-making contexts. Atanassov [2] advanced this field in 1986 through the introduction of IFSs, which incorporate both MG and NMG to better capture hesitation and uncertainty. Subsequently, Yager [23] proposed PFSs, which relaxed the squared sum constraint of membership and non-membership degrees, followed by the q-ROFSs [24], which further generalized the concept by introducing a parameter q to increase flexibility in modeling uncertainty. Further advancements have yielded more powerful and generalized frameworks. Senapati and Yager [15, 16] developed FFSs to address higher-order uncertainty. Seikh and Mandal [13] introduced the $3,4$-quasi-rung fuzzy set, while Ibrahim and Alshammari [7] proposed the n,m-rung orthopair fuzzy set, both offering refined mechanisms for representing multi-dimensional uncertainty structures. These developments demonstrate the progressive expansion of fuzzy theory toward more expressive and adaptable models suitable for diverse decision-making contexts.

In recent years, FS-based decision-making frameworks [4–6, 8, 9, 12, 17–19, 22, 25–28] have been extensively employed to manage information heterogeneity and uncertainty arising in multi-expert evaluation settings, thereby enhancing the consistency, interpretability, and reliability of collective decision outcomes. Seikh and Mandal [14] presented p,q-quasi-rung orthopair fuzzy aggregation operators, enhancing the accuracy of multi-attribute group decision-making processes. Chu et al. [3] later introduced p,q-cubic quasi-rung orthopair fuzzy sets, extending this structure to accommodate complex and uncertain data. Wang et al. [21] proposed INFSs and developed corresponding aggregation operators namely, the IN-OWA and IN-OWG operators to better capture normally distributed uncertainty. Subsequent advancements included prioritized Bonferroni mean operators for normal intuitionistic fuzzy information by Liu and Liu [11], and dynamic aggregation operators for INFSs by Yang et al. [30], enabling more refined decision analyses under temporal or weighted conditions. Building upon these contributions, Yang et al. [29] in 2019 introduced q-RONFSs, integrating the q-rung orthopair structure with the statistical characteristics of NFNs. Temel et al. [20] further expanded this concept by proposing power Muirhead mean operators for spherical normal fuzzy sets, while Kumar and Gupta [10] introduced confidence-based q-rung orthopair normal fuzzy aggregation operators, allowing decision-makers to incorporate confidence levels into the aggregation process.

Despite these achievements, existing FSs still face limitations in representing intricate relationships between MG and NMG. This restricts their effectiveness in highly uncertain or multidimensional decision-making problems. To overcome these constraints, this study

proposes the p,q-QRONFS, a more generalized framework that introduces two parameters, p and q, to enhance flexibility and representational depth. The proposed model extends the capabilities of q-RONFSs, allowing a broader and more adaptive treatment of uncertainty while encompassing several existing fuzzy models as special cases. The key contributions of this study are summarized as follows:

(1) Model extension: Development of the p,q-QRONFS model as a generalization of q-RONFSs, providing enhanced flexibility in modeling uncertainty.
(2) Operational foundation: Definition of fundamental operational laws and mathematical properties of p,q-QRONFSs.
(3) Aggregation mechanisms: Formulation of new weighted averaging and geometric aggregation operators to facilitate efficient information integration.
(4) Decision-making application: Construction of a p,q-QRONFMADM model and demonstration through a global business partner selection case study.

The remainder of this paper is organized as follows. Section 2 provides the fundamental definitions and concepts of quasi-rung orthopair fuzzy sets. Section 3 introduces the proposed p,q-QRONFS model and its basic operations. Section 4 develops new aggregation operators for p,q-QRONFNs. Section 5 presents the p,q-QRONFMADM approach and an illustrative case study. Finally, Section 6 concludes the paper and outlines potential directions for future research.

2. Preliminaries

In the subsequent sections, we will recall and explore these basic concepts, setting the stage for their application in developing novel approaches to decision-making under uncertainty

The space of an element's MG and NMG was expanded using the class of p,q-QROFSs developed by Seikh and Mandal [14] as follows:

Definition 2.1. [14] A p,q**-QROFS** $\mathbb{A}$ on a fixed set U is an object having the form

$$\mathcal{A} = \{\{x, \mu_A(x), \eta_A(x)\} : a \in U\}, \tag{2.1}$$

where the mappings $\mu_A : U \to [0,1]$ and $\eta_A : U \to [0,1]$ represent the **MG** and the **NMG**, respectively. These functions satisfy the condition $\mu_A^p(x) + \eta_A^q(x) \in [0,1]$ for all $x \in U$.

In 1996, Yang and Ko [31] provide the following definition of NFNs:

Definition 2.2. [31] A **NFN** $\mathcal{A}$ is an object having the form $\mathcal{A} = (a_A, \sigma_A)$. The **MG** of the NFN can be defined as follows

$$\mathcal{A}(x) = e^{-\left(\frac{x-a_A}{\sigma_A}\right)^2}, \tag{2.2}$$

where $a_A, x, \sigma_A \in (-\infty, \infty)$ such that $\sigma_A > 0$.

As a generalization of the concept of IFSs, Wang et al. [21] presented the concept of NIFSs as follows:

Definition 2.3. [21] A **NIFS** $\mathcal{A}$ over a fixed set U is an object having the form

$$\mathcal{A} = \{\{x, (a_A, \sigma_A), \mu_A(x), \eta_A(x)\} : x \in U\}, \tag{2.3}$$

where **MG** and **NMG** functions are presented as

$$\mu_A(x) = \mu_A e^{-\left(\frac{x-a_A}{\sigma_A}\right)^2} \tag{2.4}$$

and

$$\eta_A(x) = 1 - (1-\eta_A)\, e^{-\left(\frac{x-a_A}{\sigma_A}\right)^2}, \tag{2.5}$$

where $\mu_A : U \to [0,1]$, $\eta_A : U \to [0,1]$ and $0 \leq \mu_A(x) + \eta_A(x) \leq 1$ for all $x \in U$.

In 2019, Yang et al. [29] introduced the concept of q-ROFSs to handle more scenarios of vaguness/uncertainty as follows:

Definition 2.4. [29] A **q-RONFS** $\mathcal{A}$ over a fixed set U is an object having the form

$$\mathcal{A} = \{\{x, (a_A, \sigma_A), \mu_A(x), \eta_A(x)\} : x \in U\}, \tag{2.6}$$

where **MG** and **NMG** functions are presented as

$$\mu_A(x) = \mu_A e^{-\left(\frac{x-a_A}{\sigma_A}\right)^2} \tag{2.7}$$

and

$$\eta_A(x) = 1 - (1-\eta_A)\, e^{-\left(\frac{x-a_A}{\sigma_A}\right)^2}, \tag{2.8}$$

where $\mu_A : U \to [0,1]$, $\eta_A : U \to [0,1]$ and $0 \leq (\mu_A(x))^q + (\eta_A(x))^q \leq 1$ for all $x \in U$.

3. p, q-QRONFSs

In this part, we define and examine the concepts of p, q-QRONFS and p, q-QRONFN.

3.1. p, q-QRONFSs

The p, q-QRONFSs are generated using p, q-QROFSs and q-RONFSs. As a result, we can say that the p, q-QRONFS is a generalization of previously stated concepts in fuzzy set theory.

Definition 3.1. A **p, q-QRONFS** $\mathcal{A}$ over a fixed set U is an object having the form

$$\mathcal{A} = \{\{x, (a_A, \sigma_A), \mu_A(x), \eta_A(x)\} : x \in U\}, \tag{3.1}$$

where **MG** and **NMG** functions are presented as

$$\mu_A(x) = \mu_A e^{-\left(\frac{x-a_A}{\sigma_A}\right)^2} \tag{3.2}$$

and

$$\eta_A(x) = 1 - (1-\eta_A)\, e^{-\left(\frac{x-a_A}{\sigma_A}\right)^2}, \tag{3.3}$$

where $\mu_A : U \to [0,1]$, $\eta_A : U \to [0,1]$ and $0 \leq (\mu_A(x))^q + (\eta_A(x))^q \leq 1$ for all $x \in U$.

A comparative evaluation, summarized in Table 1 and illustrated in Figure 1, demonstrates that the proposed p, q-QRONFS framework markedly extends the capabilities of existing FS models. By simultaneously integrating membership, non-membership, and normal fuzzy functions, and incorporating the parameters p and q, the framework achieves greater flexibility and precision in representing uncertainty.

Let $\mathcal{A} = \{\{x, (a_A, \sigma_A), \mu_A(x), \eta_A(x)\} : x \in U\}$ be a p, q-QRONFS on a fixed set U. In particular, if U has only one element, $\mathcal{A}$ is called a p, q-**QRONFN**. For convenience, a p, q-QRONFN is denoted by $\mathcal{A} = ((a_A, \sigma_A), \mu_A, \eta_A)$. Because they are generated by taking

TABLE 1. Comparative overview of p, q-QRONFS and classical set models

Model	MG	NMG	NFN	p	q
p, q-QROFS [14]	✓	✓	×	✓	✓
NFN [31]	✓	×	✓	×	×
NIFS [21]	✓	✓	✓	×	×
q-RONFS [29]	✓	✓	✓	×	✓
p, q-QRONFS (Proposed)	✓	✓	✓	✓	✓

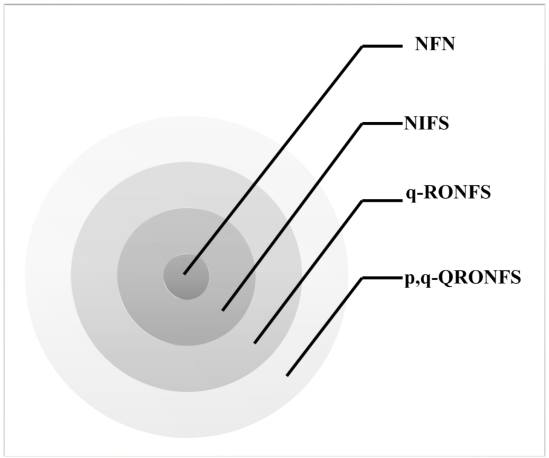

Figure 1. From classical sets to p, q-QRONFSs

$p = q$, the preceding types of orthopair normal fuzzy numbers are special examples of p, q-QRONFNs. The environment of p, q-QRONFNs provides a variety of grade spaces that are larger than those obtained from the various types of q-RONFs, making it a successful tool for modeling many real-life problems. It also allows us to apply different preferences for opposing views by controlling the values of variables p and q. Moreover, we denote by $\mathcal{QR}(U)$ the collection of p, q-QRONFNs on a fixed set U with $\mathbb{U} = ((a_A, \sigma_A), 1, 0)$ and $\emptyset = ((a_A, \sigma_A), 0, 1)$.

The terms less than and equality are defined as follows:

Definition 3.2. Let $\mathcal{A} = ((a_A, \sigma_A), \mu_A, \eta_A)$ and $\mathcal{B} = ((a_B, \sigma_B), \mu_B, \eta_B)$ be two p, q-QRONFNs on a fixed set U, the corresponding operations are defined as follows:

(1) $\mathcal{A} \preceq \mathcal{B}$ if and only if $a_A \leq a_B$, $\sigma_A \geq \sigma_B$ and $\mu_A \leq \mu_B$, $\eta_A \geq \eta_B$;
(2) $\mathcal{A} = \mathcal{B}$ if and only if $\mathcal{A} \preceq \mathcal{B}$ and $\mathcal{B} \preceq \mathcal{A}$.

3.2. Operational laws

Motivated by the operations of NIFNs and q-RONFNs, we will define certain operational laws of p, q-QRONFNs in the following.

Definition 3.3. Let $\mathcal{A} = ((a_A, \sigma_A), \mu_A, \eta_A)$ and $\mathcal{B} = ((a_B, \sigma_B), \mu_B, \eta_B)$ be any p, q-QRONFNs on a fixed set U. For every element $\lambda \in (0, \infty)$, then their operations are defined as follows:

(1) $\mu_A \oplus \mu_B = \sqrt[p]{(\mu_A)^p + (\mu_B)^p - (\mu_A)^p (\mu_B)^p}$;
(2) $\eta_A \oplus \eta_B = \eta_A \eta_B$;
(3) $\mu_A \otimes \mu_B = \mu_A \mu_B$;
(4) $\eta_A \otimes \eta_B = \sqrt[q]{(\eta_A)^q + (\eta_B)^q - (\eta_A)^q (\eta_B)^q}$;
(5) $\mathcal{A} \oplus \mathcal{B} = \left((a_A + a_B, \sigma_A + \sigma_B), \mu_A \oplus \mu_B, \eta_A \oplus \eta_B \right)$;
(6) $\mathcal{A} \otimes \mathcal{B} = \left((a_A a_B, \sigma_A \sigma_B), \mu_A \otimes \mu_B, \eta_A \otimes \eta_B \right)$;
(7) $\lambda\mathcal{A} = \left((\lambda a_A, \lambda \sigma_A), \sqrt[p]{1 - (1 - (\mu_A)^p)^\lambda}, (\eta_A)^\lambda \right)$;
(8) $\mathcal{A}^\lambda = \left(\left((a_A)^\lambda, (\sigma_A)^\lambda \right), (\mu_A)^\lambda, \sqrt[q]{1 - (1 - (\eta_A)^q)^\lambda} \right)$.

The p, q-QRONFN has the following properties, which are easily proved.

Theorem 3.4. *Let $\mathcal{A} = \left((a_A, \sigma_A), \mu_A, \eta_A \right)$, $\mathcal{B} = \left((a_B, \sigma_B), \mu_B, \eta_B \right)$ and $\mathcal{C} = \left((a_C, \sigma_C), \mu_C, \eta_C \right)$ be any p, q-QRONFNs on a fixed set U. For every elements $\lambda, \xi \in (0, \infty)$, the following properties hold.*

(1) $\mathcal{A} \oplus \mathcal{B} = \mathcal{B} \oplus \mathcal{A}$;
(2) $\mathcal{A} \otimes \mathcal{B} = \mathcal{B} \otimes \mathcal{A}$;
(3) $(\mathcal{A} \oplus \mathcal{B}) \oplus \mathcal{C} = \mathcal{A} \oplus (\mathcal{B} \oplus \mathcal{C})$;
(4) $(\mathcal{A} \otimes \mathcal{B}) \otimes \mathcal{C} = \mathcal{A} \otimes (\mathcal{B} \otimes \mathcal{C})$;
(5) $\lambda (\mathcal{A} \oplus \mathcal{B}) = \lambda\mathcal{A} \oplus \lambda\mathcal{B}$;
(6) $(\mathcal{A} \otimes \mathcal{B})^\lambda = \mathcal{A}^\lambda \otimes \mathcal{B}^\lambda$;
(7) $(\lambda + \xi) \mathcal{A} = \lambda\mathcal{A} \oplus \xi\mathcal{A}$;
(8) $\mathcal{A}^{\lambda+\xi} = \mathcal{A}^\lambda \otimes \mathcal{A}^\xi$;
(9) $\left(\mathcal{A}^\lambda\right)^\xi = \mathcal{A}^{\lambda\xi}$.

Proof. For conciseness, we present detailed proofs only for Propositions (1), (5), and (7); the remaining propositions can be established analogously by applying the same underlying mathematical principles.

(1) Let $\mathcal{A} = \left((a_A, \sigma_A), \mu_A, \eta_A \right)$ and $\mathcal{B} = \left((a_B, \sigma_B), \mu_B, \eta_B \right)$ be any p, q-QRONFNs on U. Where from it can be easily proved that

$$\begin{aligned}
\mathcal{A} \oplus \mathcal{B} &= \left((a_A a_B, \sigma_A \sigma_B), \mu_A \oplus \mu_B, \eta_A \oplus \eta_B \right) \\
&= \left((a_B a_A, \sigma_B \sigma_A), \sqrt[p]{(\mu_A)^p + (\mu_B)^p - (\mu_A)^p (\mu_B)^p}, \eta_A \eta_B \right) \\
&= \left((a_B a_A, \sigma_B \sigma_A), \sqrt[p]{(\mu_B)^p + (\mu_A)^p - (\mu_B)^p (\mu_A)^p}, \eta_B \eta_A \right) \\
&= \left((a_B a_A, \sigma_B \sigma_A), \mu_B \oplus \mu_A, \eta_B \eta_A \right) \\
&= \mathcal{B} \oplus \mathcal{A}.
\end{aligned}$$

Therefore we obtain that $\mathcal{A} \oplus \mathcal{B} = \mathcal{B} \oplus \mathcal{A}$.

(5) Let $\mathcal{A} = \left((a_A, \sigma_A), \mu_A, \eta_A \right)$ and $\mathcal{B} = \left((a_B, \sigma_B), \mu_B, \eta_B \right)$ be any p, q-QRONFNs on U. Where from it can be easily proved that

$$\begin{aligned}\lambda\mu_A \oplus \lambda\mu_B &= \left(\sqrt[p]{1-(1-(\mu_A)^p)^\lambda}\right) \oplus \left(\sqrt[p]{1-(1-(\mu_B)^p)^\lambda}\right) \\ &= \sqrt[p]{\begin{array}{l}\left(\sqrt[p]{1-(1-(\mu_A)^p)^\lambda}\right)^p + \\ \left(\sqrt[p]{1-(1-(\mu_B)^p)^\lambda}\right)^p + \\ -\left(\begin{array}{l}\left(\sqrt[p]{1-(1-(\mu_A)^p)^\lambda}\right)^p \\ \left(\sqrt[p]{1-(1-(\mu_B)^p)^\lambda}\right)^p\end{array}\right)\end{array}}\end{aligned}$$

and

$$\begin{aligned}\lambda(\mu_A \oplus \mu_B) &= \lambda\left(\sqrt[p]{(\mu_A)^p + (\mu_B)^p - (\mu_A)^p(\mu_B)^p}\right) \\ &= \sqrt[p]{1-\left(1-\left(\sqrt[p]{(\mu_A)^p + (\mu_B)^p - (\mu_A)^p(\mu_B)^p}\right)^p\right)^\lambda} \\ &= \sqrt[p]{1-(1-((\mu_A)^p + (\mu_B)^p - (\mu_A)^p(\mu_B)^p))^\lambda} \\ &= \sqrt[p]{1-((1-(\mu_A)^p)(1-(\mu_B)^p))^\lambda} \\ &= \sqrt[p]{\begin{array}{l}1-(1-(\mu_A)^p)^\lambda + 1-(1-(\mu_B)^p)^\lambda + \\ -\left(1-(1-(\mu_A)^p)^\lambda\right)\left(1-(1-(\mu_B)^p)^\lambda\right)\end{array}}\end{aligned}$$

and so $\lambda(\mu_A \oplus \mu_B) = \lambda\mu_A \oplus \lambda\mu_B$. It is easy to see that

$$\begin{aligned}\lambda(\mathcal{A} \oplus \mathcal{B}) &= \lambda((a_A + a_B, \sigma_A + \sigma_B), \mu_A \oplus \mu_B, \eta_A \oplus \eta_B) \\ &= \left((\lambda(a_A + a_B), \lambda(\sigma_A + \sigma_B)), \lambda(\mu_A \oplus \mu_B), (\eta_A\eta_B)^\lambda\right) \\ &= \left((\lambda a_A + \lambda a_B, \lambda\sigma_A + \lambda\sigma_B), \lambda\mu_A \oplus \lambda\mu_B, (\eta_A)^\lambda(\eta_B)^\lambda\right) \\ &= \left((\lambda a_A, \lambda\sigma_A), \lambda\mu_A, (\eta_A)^\lambda\right) \oplus \left((\lambda a_B, \lambda\sigma_B), \lambda\mu_B, (\eta_B)^\lambda\right) \\ &= \lambda\mathcal{A} \oplus \lambda\mathcal{B}.\end{aligned}$$

Therefore we obtain that $\lambda(\mathcal{A} \oplus \mathcal{B}) = \lambda\mathcal{A} \oplus \lambda\mathcal{B}$.

(7) Let $\mathcal{A} = ((a_A, \sigma_A), \mu_A, \eta_A)$ be a p,q-QRONFN on U. Where from it can be easily proved that

$$\begin{aligned}(\lambda+\xi)\mu_A &= \sqrt[p]{1-(1-(\mu_A)^p)^{\lambda+\xi}} \\ &= \sqrt[p]{1-(1-(\mu_A)^p)^\lambda(1-(\mu_A)^p)^\xi}\end{aligned}$$

and

$$\begin{aligned}
\lambda\mu_A \oplus \xi\mu_A &= \sqrt[p]{1-(1-(\mu_A)^p)^\lambda} \oplus \sqrt[p]{1-(1-(\mu_A)^p)^\xi} \\
&= \sqrt[p]{\begin{array}{c}\left(\sqrt[p]{1-(1-(\mu_A)^p)^\lambda}\right)^p + \\ \left(\sqrt[p]{1-(1-(\mu_A)^p)^\xi}\right)^p + \\ -\left(\begin{array}{c}\left(\sqrt[p]{1-(1-(\mu_A)^p)^\lambda}\right)^p \\ \left(\sqrt[p]{1-(1-(\mu_A)^p)^\xi}\right)^p\end{array}\right)\end{array}} \\
&= \sqrt[p]{\begin{array}{c}1-(1-(\mu_A)^p)^\lambda + 1-(1-(\mu_A)^p)^\xi + \\ -\left(1-(1-(\mu_A)^p)^\lambda\right)\left(1-(1-(\mu_A)^p)^\xi\right)\end{array}}
\end{aligned}$$

and so $(\lambda+\xi)\mu_A = \lambda\mu_A \oplus \xi\mu_A$. It is easy to verify that

$$\begin{aligned}
(\lambda+\xi)\mathcal{A} &= \left(\left((\lambda+\xi)a_A, (\lambda+\xi)\sigma_A\right), (\lambda+\xi)\mu_A, (\eta_A)^{\lambda+\xi}\right) \\
&= \left(\left(\lambda a_A + \xi a_A, \lambda\sigma_A + \xi\sigma_A\right), \lambda\mu_A + \xi\mu_A, (\eta_A)^\lambda (\eta_A)^\xi\right) \\
&= \left(\left(\lambda a_A, \lambda\sigma_A\right), \lambda\mu_A, (\eta_A)^\lambda\right) \oplus \left(\left(\xi a_A, \xi\sigma_A\right), \xi\mu_A, (\eta_A)^\xi\right) \\
&= \lambda\mathcal{A} \oplus \xi\mathcal{A}.
\end{aligned}$$

Therefore we obtain that $(\lambda+\xi)\mathcal{A} = \lambda\mathcal{A} \oplus \xi\mathcal{A}$. ∎

The specific form of the aggregation result is presented in Theorem 3.5 based on the operational laws of p,q-QRONFNs in Definition 3.3.

Theorem 3.5. *Let $\mathcal{A} = \left(\left(a_A, \sigma_A\right), \mu_A, \eta_A\right)$ and $\mathcal{B} = \left(\left(a_B, \sigma_B\right), \mu_B, \eta_B\right)$ be any p,q-QRONFNs on a fixed set U. For every elements $\lambda \in (0,\infty)$, the following properties hold.*

(1) *$\mathcal{A} \oplus \mathcal{B}$ is a p,q-QRONFN;*
(2) *$\mathcal{A} \otimes \mathcal{B}$ is a p,q-QRONFN;*
(3) *$\lambda\mathcal{A}$ is a p,q-QRONFN;*
(4) *$\mathcal{A}^\lambda$ is a p,q-QRONFN.*

Proof. For conciseness, we present detailed proofs only for Propositions (1), and (3); the remaining propositions can be established analogously by applying the same underlying mathematical principles.

(1) Let $\mathcal{A} = \left(\left(a_A, \sigma_A\right), \mu_A, \eta_A\right)$ and $\mathcal{B} = \left(\left(a_B, \sigma_B\right), \mu_B, \eta_B\right)$ be any p,q-QRONFNs on U. Because $0 \le (\mu_A)^p + (\eta_A)^q \le 1$ and $0 \le (\mu_B)^p + (\eta_B)^q \le 1$, we have $0 \le (\mu_A)^p \le 1-(\eta_A)^q$ and $0 \le (\mu_B)^p \le 1-(\eta_B)^q$, by Definition 3.3, we have that

$$\begin{aligned}
(\mu_A \oplus \mu_B)^p + (\eta_A \oplus \eta_B)^q &= \left(\sqrt[p]{(\mu_A)^p + (\mu_B)^p - (\mu_A)^p (\mu_B)^p}\right)^p + (\eta_A\eta_B)^q \\
&= (\mu_A)^p + (\mu_B)^p - (\mu_A)^p (\mu_B)^p + (\eta_A)^q (\eta_B)^q \\
&\le \begin{array}{c}1-(\eta_A)^q + 1-(\eta_B)^q + \\ -(1-(\eta_A)^q)(1-(\eta_B)^q) + (\eta_A)^q (\eta_B)^q\end{array} \\
&= \begin{array}{c}1-(\eta_A)^q + 1-(\eta_B)^q + \\ -1 + (\eta_A)^q + (\eta_B)^q\end{array} \\
&= 1.
\end{aligned}$$

Thus, we obtain that $\mathcal{A} \oplus \mathcal{B}$ is a p,q-QRONFN.

(3) Let $\mathcal{A} = ((a_A, \sigma_A), \mu_A, \eta_A)$ be a p,q-QRONFN on U. As λ be any positive real number, then $0 \leq (\mu_A)^p + (\eta_A)^q \leq 1$, we have $0 \leq (\eta_A)^q \leq 1 - (\mu_A)^p$, i.e., $0 \leq ((\eta_A)^q)^\lambda \leq (1 - (\mu_A)^p)^\lambda$. Where from it can be easily proved that

$$\begin{aligned} \left(\sqrt[p]{1 - (1 - (\mu_A)^p)^\lambda}\right)^p + \left((\eta_A)^\lambda\right)^q &= 1 - (1 - (\mu_A)^p)^\lambda + \left((\eta_A)^\lambda\right)^q \\ &\leq 1 - (1 - (\mu_A)^p)^\lambda + (1 - (\mu_A)^p)^\lambda \\ &= 1. \end{aligned}$$

Therefore we obtain that $\lambda\mathcal{A}$ is a p,q-QRONFN. ∎

3.3. Score and accuracy functions for p,q-QRONFNs

We need to know how to rank p,q-QRONFNs in order to use them in multi-attribute decision-making issues. To rank a p,q-QRONFN, we must calculate its score and accuracy, as specified in the following definition.

Definition 3.6. Let $\mathcal{A} = ((a_A, \sigma_A), \mu_A, \eta_A)$ be a p,q-QRONFN on a fixed set U. Then the **score function** $\mathcal{SC}_i(\mathcal{A})$ **(accuracy function** $\mathcal{AC}_i(\mathcal{A})$**)** of $\mathcal{A}$ can be defined as follows:

$$\mathcal{SC}_1(\mathcal{A}) = \frac{a_A}{2}\left(2(\mu_A)^p - (\mu_A)^p(\eta_A)^q\right) \tag{3.4}$$

$$\mathcal{SC}_2(\mathcal{A}) = \frac{\sigma_A}{2}\left(2(\mu_A)^p - (\mu_A)^p(\eta_A)^q\right) \tag{3.5}$$

$$\mathcal{AC}_1(\mathcal{A}) = \frac{a_A}{2}\left(2(\eta_A)^q - (\mu_A)^p(\eta_A)^q\right) \tag{3.6}$$

$$\mathcal{AC}_2(\mathcal{A}) = \frac{\sigma_A}{2}\left(2(\eta_A)^q - (\mu_A)^p(\eta_A)^q\right). \tag{3.7}$$

It should be highlighted that the score function (accuracy function) $\mathcal{SC}_i(\mathcal{A})$ $(\mathcal{AC}_i(\mathcal{A}))$ has the following desirable qualities.

Theorem 3.7. *Let $\mathcal{A} = ((a_A, \sigma_A), \mu_A, \eta_A)$ be a p,q-QRONFN on a fixed set U. Then the following properties hold.*

(1) $-a_A \leq \mathcal{SC}_1(\mathcal{A}) \leq a_A$;
(2) $0 \leq \mathcal{SC}_2(\mathcal{A}) \leq \sigma_A$;
(3) $-a_A \leq \mathcal{AC}_1(\mathcal{A}) \leq a_A$;
(4) $0 \leq \mathcal{AC}_2(\mathcal{A}) \leq \sigma_A$;
(5) $\mathcal{SC}_1(\mathbb{U}) = a_A$ *(one property)*;
(6) $\mathcal{SC}_2(\mathbb{U}) = \sigma_A$ *(one property)*;
(7) $\mathcal{SC}_1(\emptyset) = 0$ *(zero property)*;
(8) $\mathcal{SC}_2(\emptyset) = 0$ *(zero property)*;
(9) $\mathcal{AC}_1(\mathbb{U}) = 0$ *(one property)*;
(10) $\mathcal{AC}_2(\mathbb{U}) = 0$ *(one property)*;
(11) $\mathcal{AC}_1(\emptyset) = a_A$ *(zero property)*;
(12) m $\mathcal{AC}_2(\emptyset) = \sigma_A$ *(zero property)*.

Proof. For the sake of conciseness, a detailed proof is presented only for Proposition (1), whereas the proofs of the remaining propositions can be derived in an analogous manner by employing the same fundamental mathematical principles.

Let $\mathcal{A} = ((a_A, \sigma_A), \mu_A, \eta_A)$ be a p,q-QRONFN defined on a fixed set U. Since $0 \leq (\eta_A)^q \leq 1$, it follows that $0 \leq \frac{(\eta_A)^q}{2} \leq \frac{1}{2}$, which implies $\frac{1}{2} \leq 1 - \frac{(\eta_A)^q}{2} \leq 1$. Furthermore, since $0 \leq (\mu_A)^p \leq 1$, multiplying both sides of the above inequality by $(\mu_A)^p$ (which is nonnegative) yields $0 \leq (\mu_A)^p \left(1 - \frac{(\eta_A)^q}{2}\right) \leq 1$. Hence, we obtain $0 \leq \frac{1}{2}\left(2(\mu_A)^p - (\mu_A)^p (\eta_A)^q\right) \leq 1$, which can be equivalently expressed as $-a_A \leq \frac{a_A}{2}\left(2(\mu_A)^p - (\mu_A)^p (\eta_A)^q\right) \leq a_A$. Therefore, the inequality is proven. ∎

4. Aggregation operators for the p,q quasi-rung orthopair normal fuzzy information

This section is dedicated to presenting and exploring several innovative sorts of operations on p,q-QRONFNs. Then, we develop new types of aggregation operators with respect to p,q-QRONFNs and illustrate the relationships between them.

4.1. p,q-QRONFWA operators

The p,q quasi-rung orthopair normal fuzzy weighted averaging operator is provided below based on the p,q-QRONFN operating rules We will now look at some aggregation operators that are based on the operational rules of p,q-QRONFNs. Thus, it is evident from the preceding description that p,q quasi-rung orthopair normal fuzzy weighted averaging operators are a generalization of a q rung orthopair normal fuzzy weighted averaging operator.

Definition 4.1. Let $\Delta = \{\mathcal{A}_i = ((a_{A_i}, \sigma_{A_i}), \mu_{A_i}, \eta_{A_i}) : i = 1, 2, 3, \ldots, n\}$ be a collection of p,q-QRONFNs on a fixed set U and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)^T$ be the weight vector of Δ indicates the importance degree of Δ satisfying $0 \leq \lambda_1, \lambda_2, \ldots, \lambda_n \leq 1$ and $\sum_{i=1}^{n} \lambda_i = 1$, and let **$p,q$-QRONFWA** $\mathcal{WA} : (\mathcal{QR}(U))^n \to \mathcal{QR}(U)$ if

$$\mathcal{WA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \bigoplus_{i=1}^{n} \lambda_i \mathcal{A}_i \tag{4.1}$$

then the function p,q-QRONFWA is called the **p,q-QRONFWA operator** .

The aggregation result indicated in Theorem 4.2 can be obtained using the operating rules of the p,q-QRONFNs on a fixed set U.

Theorem 4.2. *Let $\Delta = \{\mathcal{A}_i = ((a_{A_i}, \sigma_{A_i}), \mu_{A_i}, \eta_{A_i}) : i = 1, 2, 3, \ldots, n\}$ be a collection of p,q-QRONFNs on a fixed set U. If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)^T$ is the weight vector of Δ where λ_j indicates the importance degree of Δ, satisfying $0 \leq \lambda_1, \lambda_2, \ldots, \lambda_n \leq 1$ and $\sum_{i=1}^{n} \lambda_i = 1$,*

then

$$\mathcal{WA}\left(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n\right)=\left(\begin{array}{c}\left(\sum_{i=1}^{n}\lambda_i a_{A_i},\sum_{i=1}^{n}\lambda_i\sigma_{A_i}\right),\\ \sqrt[p]{1-\prod_{i=1}^{n}\left(1-\left(\mu_{A_i}\right)^p\right)^{\lambda_i}},\prod_{i=1}^{n}\left(\eta_{A_i}\right)^{\lambda_i}\end{array}\right). \tag{4.2}$$

Proof. To prove the theorem, we use mathematical induction on n. Therefore we have the following.

Step 1. Now for $n=2$, we get

$$\mathcal{WA}\left(\mathcal{A}_1,\mathcal{A}_2\right)=\left(\begin{array}{c}\left(\lambda_1 a_{A_1},\lambda_1\sigma_{A_1}\right),\\ \sqrt[p]{1-\left(1-\left(\mu_{A_1}\right)^p\right)^{\lambda_1}},\left(\eta_{A_1}\right)^{\lambda_1}\end{array}\right)\oplus\left(\begin{array}{c}\left(\lambda_2 a_{A_2},\lambda_2\sigma_{A_2}\right),\\ \sqrt[p]{1-\left(1-\left(\mu_{A_2}\right)^p\right)^{\lambda_2}},\left(\eta_{A_2}\right)^{\lambda_2}\end{array}\right)$$

$$=\left(\begin{array}{c}\left(\lambda_1 a_{A_1}+\lambda_2 a_{A_2},\lambda_1\sigma_{A_1}+\lambda_2\sigma_{A_2}\right),\\ \sqrt[p]{\begin{array}{c}\left(\sqrt[p]{1-\left(1-\left(\mu_{A_1}\right)^p\right)^{\lambda_1}}\right)^p+\\ \left(\sqrt[p]{1-\left(1-\left(\mu_{A_2}\right)^p\right)^{\lambda_2}}\right)^p+\\ -\left(\begin{array}{c}\left(\sqrt[p]{1-\left(1-\left(\mu_{A_1}\right)^p\right)^{\lambda_1}}\right)^p\\ \left(\sqrt[p]{1-\left(1-\left(\mu_{A_2}\right)^p\right)^{\lambda_2}}\right)^p\end{array}\right)\end{array}},\\ \left(\eta_{A_1}\right)^{\lambda_1}\left(\eta_{A_2}\right)^{\lambda_2}\end{array}\right)$$

$$=\left(\begin{array}{c}\left(\sum_{i=1}^{2}\lambda_i a_{A_i},\sum_{i=1}^{2}\lambda_i\sigma_{A_i}\right),\\ \sqrt[p]{\begin{array}{c}1-\left(1-\left(\mu_{A_1}\right)^p\right)^{\lambda_1}+1-\left(1-\left(\mu_{A_2}\right)^p\right)^{\lambda_2}+\\ -\left(1-\left(1-\left(\mu_{A_1}\right)^p\right)^{\lambda_1}\right)\left(1-\left(1-\left(\mu_{A_2}\right)^p\right)^{\lambda_2}\right)\end{array}},\\ \left(\eta_{A_1}\right)^{\lambda_1}\left(\eta_{A_2}\right)^{\lambda_2}\end{array}\right)$$

$$=\left(\begin{array}{c}\left(\sum_{i=1}^{2}\lambda_i a_{A_i},\sum_{i=1}^{2}\lambda_i\sigma_{A_i}\right),\\ \sqrt[p]{1-\left(1-\left(\mu_{A_1}\right)^p\right)^{\lambda_1}\left(1-\left(\mu_{A_2}\right)^p\right)^{\lambda_2}},\\ \prod_{i=1}^{2}\left(\eta_{A_i}\right)^{\lambda_i}\end{array}\right)$$

$$=\left(\begin{array}{c}\left(\sum_{i=1}^{2}\lambda_i a_{A_i},\sum_{i=1}^{2}\lambda_i\sigma_{A_i}\right),\\ \sqrt[p]{1-\prod_{i=1}^{2}\left(1-\left(\mu_{A_i}\right)^p\right)^{\lambda_i}},\prod_{i=1}^{2}\left(\eta_{A_i}\right)^{\lambda_i}\end{array}\right).$$

Thus, result holds when $n=2$.

Step 2. Suppose that Eq. 4.2 is holds for $n = k$, that is

$$\mathcal{WA}\left(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_k\right) = \left(\begin{array}{c} \left(\sum_{i=1}^{k} \lambda_i a_{A_i}, \sum_{i=1}^{k} \lambda_i \sigma_{A_i} \right), \\ \sqrt[p]{1 - \prod_{i=1}^{k} \left(1 - \left(\mu_{A_i}\right)^p\right)^{\lambda_i}}, \prod_{i=1}^{k} \left(\eta_{A_i}\right)^{\lambda_i} \end{array} \right).$$

Step 3. Now we have to prove that Eq. 4.2 is holds for $n = k + 1$, based on the operational laws of the p, q-QRONFNs, we can get

$$\begin{aligned} \bigoplus_{i=1}^{k+1} \lambda_i \mathcal{A}_i &= \bigoplus_{i=1}^{k} \lambda_i \mathcal{A}_i \oplus \lambda_{k+1} \mathcal{A}_{k+1} \\ &= \left(\begin{array}{c} \left(\sum_{i=1}^{k} \lambda_i a_{A_i}, \sum_{i=1}^{k} \lambda_i \sigma_{A_i} \right), \\ \sqrt[p]{1 - \prod_{i=1}^{k} \left(1 - \left(\mu_{A_i}\right)^p\right)^{\lambda_i}}, \prod_{i=1}^{k} \left(\eta_{A_i}\right)^{\lambda_i} \end{array} \right) \oplus \\ &\quad \left(\begin{array}{c} \left(\lambda_{k+1} a_{A_{k+1}}, \lambda_{k+1} \sigma_{A_{k+1}}\right), \\ \sqrt[p]{1 - \left(1 - \left(\mu_{A_{k+1}}\right)^p\right)^{\lambda_{k+1}}}, \left(\eta_{A_{k+1}}\right)^{\lambda_{k+1}} \end{array} \right) \\ &= \left(\begin{array}{c} \left(\sum_{i=1}^{k} \lambda_i a_{A_i} + \lambda_{k+1} a_{A_{k+1}}, \sum_{i=1}^{k} \lambda_i \sigma_{A_i} + \lambda_{k+1} \sigma_{A_{k+1}} \right), \\ \sqrt[p]{\left(\sqrt[p]{1 - \prod_{i=1}^{k} \left(1 - \left(\mu_{A_i}\right)^p\right)^{\lambda_i}} \right)^p + \left(\sqrt[p]{1 - \left(1 - \left(\mu_{A_{k+1}}\right)^p\right)^{\lambda_{k+1}}} \right)^p - \left(\begin{array}{c} \left(\sqrt[p]{1 - \prod_{i=1}^{k} \left(1 - \left(\mu_{A_i}\right)^p\right)^{\lambda_i}} \right)^p \\ \left(\sqrt[p]{1 - \left(1 - \left(\mu_{A_{k+1}}\right)^p\right)^{\lambda_{k+1}}} \right)^p \end{array} \right)}, \\ \prod_{i=1}^{k} \left(\eta_{A_i}\right)^{\lambda_1} \left(\eta_{A_{k+1}}\right)^{\lambda_{k+1}} \end{array} \right), \end{aligned}$$

$$= \left(\begin{array}{c} \left(\sum_{i=1}^{k+1} \lambda_i a_{A_i}, \sum_{i=1}^{k+1} \lambda_i \sigma_{A_i} \right), \\ \sqrt[p]{\begin{array}{c} 1 - \prod_{i=1}^{k} \left(1 - (\mu_{A_i})^p\right)^{\lambda_i} + 1 - \left(1 - \left(\mu_{A_{k+1}}\right)^p\right)^{\lambda_{k+1}} + \\ - \left(1 - \prod_{i=1}^{k} \left(1 - (\mu_{A_i})^p\right)^{\lambda_i}\right) \left(1 - \left(1 - \left(\mu_{A_{k+1}}\right)^p\right)^{\lambda_{k+1}}\right) \end{array}}, \\ \prod_{i=1}^{k+1} (\eta_{A_i})^{\lambda_i} \end{array} \right)$$

$$= \left(\begin{array}{c} \left(\sum_{i=1}^{k+1} \lambda_i a_{A_i}, \sum_{i=1}^{k+1} \lambda_i \sigma_{A_i} \right), \\ \sqrt[p]{1 - \prod_{i=1}^{k+1} \left(1 - (\mu_{A_i})^p\right)^{\lambda_i}}, \prod_{i=1}^{k+1} (\eta_{A_i})^{\lambda_i} \end{array} \right).$$

Therefore, the proof of Theorem 4.2 is completed. ∎

Theorem 4.3. *Let* $\Delta = \{\mathcal{A}_i = ((a_{A_i}, \sigma_{A_i}), \mu_{A_i}, \eta_{A_i}) : i = 1, 2, 3, \ldots, n\}$ *be a collection of* p, q*-QRONFNs on a fixed set* U *and* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)^T$ *be the weight vector of* Δ *where* λ_j *indicates the importance degree of* Δ*, satisfying* $0 \leq \lambda_1, \lambda_2, \ldots, \lambda_n \leq 1$ *and* $\sum_{i=1}^{n} \lambda_i = 1$*. Then their accumulated outcome utilizing the* p, q*-QRONFWA operator is again* p, q*-QRONFN.*

Proof. It's proof immediately followed by Theorem 3.5. ∎

According to theories, our suggested p, q-QRONFWA operator meets the following idempotency, boundedness, and monotonicity properties:

Theorem 4.4. *(Idempotency) Let* $\{\mathcal{A}_i = ((a_{A_i}, \sigma_{A_i}), \mu_{A_i}, \eta_{A_i}) : i = 1, 2, 3, \ldots, n\}$ *be a collection of* p, q*-QRONFNs on a fixed set* U*. If* $\mathcal{A}_i = \mathcal{A} = ((a_A, \sigma_A), \mu_A, \eta_A)$ *for every* $i = 1, 2, \ldots, n$*, then* $\mathcal{WA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \mathcal{A}$*.*

Proof. Then, since $\mathcal{A}_1 = \mathcal{A}_2 = \ldots = \mathcal{A}_n = \mathcal{A}$ by Theorem 4.2, we have

$$\mathcal{WA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \left(\begin{array}{c} \left(\sum_{i=1}^{n} \lambda_i a_{A_i}, \sum_{i=1}^{n} \lambda_i \sigma_{A_i} \right), \\ \sqrt[p]{1 - \prod_{i=1}^{n} \left(1 - (\mu_{A_i})^p\right)^{\lambda_i}}, \prod_{i=1}^{n} (\eta_{A_i})^{\lambda_i} \end{array} \right)$$

$$
\begin{aligned}
&= \left(\begin{array}{c} \left(\sum_{i=1}^{n} \lambda_i a_A, \sum_{i=1}^{n} \lambda_i \sigma_A \right), \\ \sqrt[p]{1-(1-(\mu_A)^p)^{\sum_{i=1}^{n} \lambda_i}}, (\eta_A)^{\sum_{i=1}^{n} \lambda_i} \end{array} \right) \\
&= \left(\begin{array}{c} \left(a_A \sum_{i=1}^{n} \lambda_i, \sigma_A \sum_{i=1}^{n} \lambda_i \right), \\ \sqrt[p]{1-(1-(\mu_A)^p)^{1}}, (\eta_A)^{1} \end{array} \right) \\
&= ((a_A, \sigma_A), \mu_A, \eta_A) \\
&= \mathcal{A}.
\end{aligned}
$$

Therefore we obtain that $\mathcal{WA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \mathcal{A}$. ∎

The p, q-QRONFWA operator has the following features, which are easily proved.

Proposition 4.5. *Let $\{\mathcal{A}_i = ((a_{A_i}, \sigma_{A_i}), \mu_{A_i}, \eta_{A_i}) : i = 1, 2, 3, \ldots, n\}$ be a collection of p, q-QRONFNs on a fixed set U. Then the following properties hold.*

(1) *If $\mathcal{A}_i = \emptyset$ for every $i = 1, 2, 3, \ldots, n$, then $\mathcal{WA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \emptyset$;*
(2) *If $\mathcal{A}_i = \mathbb{U}$ for every $i = 1, 2, 3, \ldots, n$, then $\mathcal{WA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \mathbb{U}$.*

The following Theorem 4.6 establishes some properties of p, q-QRONFWA operator.

Theorem 4.6. *(Monotonicity) Let $\{\mathcal{A}_i = ((a_{A_i}, \sigma_{A_i}), \mu_{A_i}, \eta_{A_i}) : i = 1, 2, 3, \ldots, n\}$ and $\{\mathcal{B}_i = ((a_{B_i}, \sigma_{B_i}), \mu_{B_i}, \eta_{B_i}) : i = 1, 2, 3, \ldots, n\}$ be any collections of p, q-QRONFNs on a fixed set U. If $\mathcal{A}_i \preceq \mathcal{B}_i$ for every $i = 1, 2, 3, \ldots, n$, then $\mathcal{WA}(\mathcal{A}_1, \ldots, \mathcal{A}_n) \preceq \mathcal{WA}(\mathcal{B}_1, \ldots, \mathcal{B}_n)$.*

Proof. Then, since $\mathcal{A}_i \preceq \mathcal{B}_i$ for every $i = 1, 2, 3, \ldots, n$, we have $\mu_{A_i} \leq \mu_{B_i}, a_{A_i} \leq a_{B_i}$ and $\eta_{A_i} \geq \eta_{B_i}, \sigma_{A_i} \geq \sigma_{B_i}$, that is,

$$
\begin{gathered}
0 \leq (\mu_{A_i})^p \leq (\mu_{B_i})^p \\
0 \leq 1 - (\mu_{A_i})^p \geq 1 - (\mu_{B_i})^p \\
(1 - (\mu_{A_i})^p)^{\lambda_i} \geq (1 - (\mu_{B_i})^p)^{\lambda_i} \geq 0 \\
\prod_{i=1}^{n} (1 - (\mu_{A_i})^p)^{\lambda_i} \geq \prod_{i=1}^{n} (1 - (\mu_{B_i})^p)^{\lambda_i} \geq 0 \\
0 \leq 1 - \prod_{i=1}^{n} (1 - (\mu_{A_i})^p)^{\lambda_i} \leq 1 - \prod_{i=1}^{n} (1 - (\mu_{B_i})^p)^{\lambda_i}
\end{gathered}
$$

$$
\sqrt[p]{1 - \prod_{i=1}^{n} (1 - (\mu_{A_i})^p)^{\lambda_i}} \leq \sqrt[p]{1 - \prod_{i=1}^{n} (1 - (\mu_{B_i})^p)^{\lambda_i}}. \tag{4.3}
$$

From these calculations, we obtain

$$
\prod_{i=1}^{n} (\eta_{A_i})^{\lambda_i} \geq \prod_{i=1}^{n} (\eta_{B_i})^{\lambda_i}, \tag{4.4}
$$

and

$$\sum_{i=1}^{n} \lambda_i a_{A_i} \leq \sum_{i=1}^{n} \lambda_i a_{B_i}. \tag{4.5}$$

Thus, by Eqs. (4.3), (4.4) and (4.5) we have $\mathcal{WA}(\mathcal{A}_1, \ldots, \mathcal{A}_n) \preceq \mathcal{WA}(\mathcal{B}_1, \ldots, \mathcal{B}_n)$. ∎

We now demonstrate the properties of the p, q-QRONFWA operator.

Theorem 4.7. *(Boundedness) Let* $\{\mathcal{A}_i = ((a_{A_i}, \sigma_{A_i}), \mu_{A_i}, \eta_{A_i}) : i = 1, 2, 3, \ldots, n\}$ *be a collection of* p, q*-QRONFNs on a fixed set* U*. If* $\mathcal{A}^+ = \left(\left(\bigvee_{i=1}^{n} a_{A_i}, \bigwedge_{i=1}^{n} \sigma_{A_i}\right), \bigvee_{i=1}^{n} \mu_{A_i}, \bigwedge_{i=1}^{n} \eta_{A_i}\right)$ *and* $\mathcal{A}^- = \left(\left(\bigwedge_{i=1}^{n} a_{A_i}, \bigvee_{i=1}^{n} \sigma_{A_i}\right), \bigwedge_{i=1}^{n} \mu_{A_i}, \bigvee_{i=1}^{n} \eta_{A_i}\right)$*, then*

$$\mathcal{A}^- \preceq \mathcal{WA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \preceq \mathcal{A}^+.$$

Proof. For the membership degree of $\mathcal{WA}(\mathcal{A}_1, \ldots, \mathcal{A}_n)$, we have

$$0 \leq \bigwedge_{i=1}^{n} (\mu_{A_i})^p \leq (\mu_{A_i})^p$$

$$1 - \bigwedge_{i=1}^{n} (\mu_{A_i})^p \geq 1 - (\mu_{A_i})^p \geq 0$$

$$\left(1 - \bigwedge_{i=1}^{n} (\mu_{A_i})^p\right)^{\lambda_i} \geq (1 - (\mu_{A_i})^p)^{\lambda_i} \geq 0$$

$$\prod_{i=1}^{n} \left(1 - \bigwedge_{i=1}^{n} (\mu_{A_i})^p\right)^{\lambda_i} \geq \prod_{i=1}^{n} (1 - (\mu_{A_i})^p)^{\lambda_i} \geq 0$$

$$0 \leq 1 - \prod_{i=1}^{n} \left(1 - \bigwedge_{i=1}^{n} (\mu_{A_i})^p\right)^{\lambda_i} \leq 1 - \prod_{i=1}^{n} (1 - (\mu_{A_i})^p)^{\lambda_i}$$

$$\sqrt[p]{1 - \prod_{i=1}^{n} \left(1 - \bigwedge_{i=1}^{n} (\mu_{A_i})^p\right)^{\lambda_i}} \leq \sqrt[p]{1 - \prod_{i=1}^{n} (1 - (\mu_{A_i})^p)^{\lambda_i}}$$

$$\sqrt[p]{1 - \left(1 - \bigwedge_{i=1}^{n} (\mu_{A_i})^p\right)^{\sum_{i=1}^{n} \lambda_i}} \leq \sqrt[p]{1 - \prod_{i=1}^{n} (1 - (\mu_{A_i})^p)^{\lambda_i}}$$

$$\sqrt[p]{1 - \left(1 - \bigwedge_{i=1}^{n} (\mu_{A_i})^p\right)} \leq \sqrt[p]{1 - \prod_{i=1}^{n} (1 - (\mu_{A_i})^p)^{\lambda_i}}$$

$$\bigwedge_{i=1}^{n} \mu_{A_i} \leq \sqrt[p]{1 - \prod_{i=1}^{n} (1 - (\mu_{A_i})^p)^{\lambda_i}} \tag{4.6}$$

and

$$\bigvee_{i=1}^{n}(\mu_{A_i})^p \geq (\mu_{A_i})^p \geq 0$$

$$0 \leq 1-\bigvee_{i=1}^{n}(\mu_{A_i})^p \leq 1-(\mu_{A_i})^p$$

$$0 \leq \left(1-\bigvee_{i=1}^{n}(\mu_{A_i})^p\right)^{\lambda_i} \leq (1-(\mu_{A_i})^p)^{\lambda_i}$$

$$0 \leq \prod_{i=1}^{n}\left(1-\bigvee_{i=1}^{n}(\mu_{A_i})^p\right)^{\lambda_i} \leq \prod_{i=1}^{n}(1-(\mu_{A_i})^p)^{\lambda_i}$$

$$1-\prod_{i=1}^{n}\left(1-\bigvee_{i=1}^{n}(\mu_{A_i})^p\right)^{\lambda_i} \geq 1-\prod_{i=1}^{n}(1-(\mu_{A_i})^p)^{\lambda_i} \geq 0$$

$$\sqrt[p]{1-\prod_{i=1}^{n}\left(1-\bigvee_{i=1}^{n}(\mu_{A_i})^p\right)^{\lambda_i}} \geq \sqrt[p]{1-\prod_{i=1}^{n}(1-(\mu_{A_i})^p)^{\lambda_i}}$$

$$\sqrt[p]{1-\left(1-\bigvee_{i=1}^{n}(\mu_{A_i})^p\right)^{\sum_{i=1}^{n}\lambda_i}} \geq \sqrt[p]{1-\prod_{i=1}^{n}(1-(\mu_{A_i})^p)^{\lambda_i}}$$

$$\sqrt[p]{1-\left(1-\bigvee_{i=1}^{n}(\mu_{A_i})^p\right)^{1}} \geq \sqrt[p]{1-\prod_{i=1}^{n}(1-(\mu_{A_i})^p)^{\lambda_i}}$$

$$\bigvee_{i=1}^{n}\mu_{A_i} \geq \sqrt[p]{1-\prod_{i=1}^{n}(1-(\mu_{A_i})^p)^{\lambda_i}}. \tag{4.7}$$

Then, by Eqs. (4.6) and (4.7), we get

$$\bigwedge_{i=1}^{n}\mu_{A_i} \leq \sqrt[p]{1-\prod_{i=1}^{n}(1-(\mu_{A_i})^p)^{\lambda_i}} \leq \bigvee_{i=1}^{n}\mu_{A_i}. \tag{4.8}$$

For the non-membership degree of $\mathcal{WA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n)$, we have

$$\bigwedge_{i=1}^{n}(\eta_{A_i})^{\lambda_i} \leq (\eta_{A_i})^{\lambda_i} \leq \bigvee_{i=1}^{n}(\eta_{A_i})^{\lambda_i}$$

$$\prod_{i=1}^{n}\bigwedge_{i=1}^{n}(\eta_{A_i})^{\lambda_i} \leq \prod_{i=1}^{n}(\eta_{A_i})^{\lambda_i} \leq \prod_{i=1}^{n}\bigvee_{i=1}^{n}(\eta_{A_i})^{\lambda_i}$$

$$\bigwedge_{i=1}^{n}(\eta_{A_i})^{\sum_{i=1}^{n}\lambda_i} \leq \prod_{i=1}^{n}(\eta_{A_i})^{\lambda_i} \leq \bigvee_{i=1}^{n}(\eta_{A_i})^{\sum_{i=1}^{n}\lambda_i}$$

$$\bigwedge_{i=1}^{n}(\eta_{A_i}) \leq \prod_{i=1}^{n}(\eta_{A_i})^{\lambda_i} \leq \bigvee_{i=1}^{n}(\eta_{A_i}). \tag{4.9}$$

It can be similarly proved that

$$\bigwedge_{i=1}^{n}(a_{A_i}) \leq \sum_{i=1}^{n}(\lambda_i a_{A_i}) \leq \bigvee_{i=1}^{n}(a_{A_i}) \tag{4.10}$$

and

$$\bigwedge_{i=1}^{n}(\sigma_{A_i}) \leq \sum_{i=1}^{n}(\lambda_i \sigma_{A_i}) \leq \bigvee_{i=1}^{n}(\sigma_{A_i}). \tag{4.11}$$

Therefore, by Eqs. (4.8), (4.9), (4.10) and (4.11), we have

$$\mathcal{A}^{-} \preceq \mathcal{WA}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \preceq \mathcal{A}^{+}.$$

■

4.2. p, q-QRONFWG OPERATORS

On the basis of the p, q-QRONFNs, we will present the principles and basic operations of the p, q quasi-rung orthopair normal fuzzy weighted geometric operator. The p, q-QRONFWG operators are defined as follows.

Definition 4.8. Let $\Delta = \{\mathcal{A}_i = ((a_{A_i}, \sigma_{A_i}), \mu_{A_i}, \eta_{A_i}) : i = 1, 2, 3, \ldots, n\}$ be a collection of p, q-QRONFNs on a fixed set U and $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)^T$ be the weight vector of Δ, where λ_j indicates the importance degree of Δ, satisfying $\lambda_1, \lambda_2, \ldots, \lambda_n \in [0, 1]$ and $\sum_{i=1}^{n}\lambda_i = 1$, and let p, q-**QRONFWG** $\mathcal{WG} : (\mathcal{QR}(U))^n \to \mathcal{QR}(U)$ if

$$\mathcal{WG}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \bigotimes_{i=1}^{n}\mathcal{A}_i^{\lambda_i}, \tag{4.12}$$

then the function p, q-QRONFWG is called the p, q-**QRONFWG operator**.

We may derive the specific form of the aggregation result from the operational laws of p, q-QRONFNs in Definition 4.8, which is given in Theorem 4.9.

Theorem 4.9. *Let $\Delta = \{\mathcal{A}_i = ((a_{A_i}, \sigma_{A_i}), \mu_{A_i}, \eta_{A_i}) : i = 1, 2, 3, \ldots, n\}$ be a collection of p, q-QRONFNs on a fixed set U. If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)^T$ is the weight vector of Δ, where λ_j indicates the importance degree of Δ, satisfying $\lambda_1, \lambda_2, \ldots, \lambda_n \in [0, 1]$ and $\sum_{i=1}^{n}\lambda_i = 1$, then*

$$\mathcal{WG}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \left(\begin{array}{c} \left(\prod_{i=1}^{n}(a_{A_i})^{\lambda_i}, \prod_{i=1}^{n}(\sigma_{A_i})^{\lambda_i}\right), \\ \prod_{i=1}^{n}(\mu_{A_i})^{\lambda_i}, \sqrt[q]{1 - \prod_{i=1}^{n}(1 - (\eta_{A_i})^q)^{\lambda_i}} \end{array} \right). \tag{4.13}$$

Proof. To prove the Theorem 4.9, we use mathematical induction on n. For this, we proceed as follows.

Step 1. Now for $n = 2$, we have

$$\bigotimes_{i=1}^{2}\mathcal{A}_i^{\lambda_i} = \begin{pmatrix} \left((a_{A_1})^{\lambda_1},(\sigma_{A_1})^{\lambda_1}\right), \\ (\mu_1)^{\lambda_{A_1}}, \sqrt[q]{1-(1-(\eta_{A_1})^q)^{\lambda_1}}, \end{pmatrix} \otimes \begin{pmatrix} \left((a_{A_2})^{\lambda_2},(\sigma_{A_2})^{\lambda_2}\right), \\ (\mu_2)^{\lambda_{A_2}}, \sqrt[q]{1-(1-(\eta_{A_2})^q)^{\lambda_2}} \end{pmatrix}$$

$$= \begin{pmatrix} \left((a_{A_1})^{\lambda_1}(a_{A_2})^{\lambda_2},(\sigma_{A_1})^{\lambda_1}(\sigma_{A_2})^{\lambda_2}\right), \\ (\mu_{A_1})^{\lambda_1}(\mu_{A_2})^{\lambda_2}, \\ \sqrt[q]{\begin{array}{c} \left(\sqrt[q]{1-(1-(\eta_{A_1})^q)^{\lambda_1}}\right)^q + \\ \left(\sqrt[q]{1-(1-(\eta_{A_2})^q)^{\lambda_2}}\right)^q + \\ -\left(\begin{array}{c} \left(\sqrt[q]{1-(1-(\eta_{A_1})^q)^{\lambda_1}}\right)^q \\ \left(\sqrt[q]{1-(1-(\eta_{A_2})^q)^{\lambda_2}}\right)^q \end{array}\right) \end{array}} \end{pmatrix}$$

$$= \begin{pmatrix} \left(\prod_{i=1}^{2}(a_{A_i})^{\lambda_i},\prod_{i=1}^{2}(\sigma_{A_i})^{\lambda_i}\right),\prod_{i=1}^{2}(\mu_{A_i})^{\lambda_i}, \\ \sqrt[q]{\begin{array}{c} 1-(1-(\eta_{A_1})^q)^{\lambda_1}+1-(1-(\eta_{A_2})^q)^{\lambda_2}+ \\ -\left(1-(1-(\eta_{A_1})^q)^{\lambda_1}\right)\left(1-(1-(\eta_{A_2})^q)^{\lambda_2}\right) \end{array}} \end{pmatrix}$$

$$= \begin{pmatrix} \left(\prod_{i=1}^{2}(a_{A_i})^{\lambda_i},\prod_{i=1}^{2}(\sigma_{A_i})^{\lambda_i}\right),\prod_{i=1}^{2}(\mu_{A_i})^{\lambda_i}, \\ \sqrt[q]{1-(1-(\eta_{A_1})^q)^{\lambda_1}(1-(\eta_{A_2})^q)^{\lambda_2}} \end{pmatrix}$$

$$= \begin{pmatrix} \left(\prod_{i=1}^{2}(a_{A_i})^{\lambda_i},\prod_{i=1}^{2}(\sigma_{A_i})^{\lambda_i}\right), \\ \prod_{i=1}^{2}(\mu_{A_i})^{\lambda_i}, \sqrt[q]{1-\prod_{i=1}^{2}(1-(\eta_{A_i})^q)^{\lambda_i}} \end{pmatrix}.$$

Thus, result holds when $n = 2$.

Step 2. Suppose that Eq. (4.13) holds for $n = k$, that is

$$\mathcal{WG}(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_k) = \begin{pmatrix} \left(\prod_{i=1}^{k}(a_{A_i})^{\lambda_i},\prod_{i=1}^{k}(\sigma_{A_i})^{\lambda_i}\right), \\ \prod_{i=1}^{k}(\mu_{A_i})^{\lambda_i}, \sqrt[q]{1-\prod_{i=1}^{k}(1-(\eta_{A_i})^q)^{\lambda_i}} \end{pmatrix}.$$

Step 3. Now we have to prove that Eq. 4.13 is holds for $n = k+1$, based on the operational laws of the p,q-QRONFNs, we can get

$$\begin{aligned}
\bigotimes_{i=1}^{k+1}\mathcal{A}_i^{\lambda_i} &= \bigotimes_{i=1}^{k}\mathcal{A}^{\lambda_i}\otimes\mathcal{A}_{k+1}^{\lambda_{k+1}}\\
&= \begin{pmatrix} \left(\prod_{i=1}^{k}(a_{A_i})^{\lambda_i}, \prod_{i=1}^{k}(\sigma_{A_i})^{\lambda_i}\right), \\ \prod_{i=1}^{k}(\mu_{A_i})^{\lambda_i}, \sqrt[q]{1-\prod_{i=1}^{k}(1-(\eta_{A_i})^q)^{\lambda_i}} \end{pmatrix}\otimes \\
&\quad \begin{pmatrix} \left((a_{A_{k+1}})^{\lambda_{k+1}}, (\sigma_{A_{k+1}})^{\lambda_{k+1}}\right), \\ (\mu_{A_{k+1}})^{\lambda_{k+1}}, \sqrt[q]{1-(1-(\eta_{A_{k+1}})^q)^{\lambda_{k+1}}} \end{pmatrix}\\
&= \begin{pmatrix} \left(\prod_{i=1}^{k}(a_{A_i})^{\lambda_1}(a_{A_{k+1}})^{\lambda_{k+1}}, \prod_{i=1}^{k}(\sigma_{A_i})^{\lambda_1}(\sigma_{A_{k+1}})^{\lambda_{k+1}}\right), \\ \prod_{i=1}^{k}(\mu_{A_i})^{\lambda_1}(\mu_{A_{k+1}})^{\lambda_{k+1}}, \\ \sqrt{\begin{array}{l}\left(\sqrt[q]{1-\prod_{i=1}^{k}(1-(\eta_{A_i})^q)^{\lambda_i}}\right)^q + \\ \left(\sqrt[q]{1-(1-(\eta_{A_{k+1}})^q)^{\lambda_{k+1}}}\right)^q + \\ -\left(\begin{array}{l}\left(\sqrt[q]{1-\prod_{i=1}^{k}(1-(\eta_{A_i})^q)^{\lambda_i}}\right)^q \\ \left(\sqrt[q]{1-(1-(\eta_{A_{k+1}})^q)^{\lambda_{k+1}}}\right)^q\end{array}\right)\end{array}}\end{pmatrix}\\
&= \begin{pmatrix} \left(\prod_{i=1}^{k+1}(a_{A_i})^{\lambda_i}, \prod_{i=1}^{k+1}(\sigma_{A_i})^{\lambda_i}\right), \\ \prod_{i=1}^{k+1}(\mu_{A_i})^{\lambda_i}, \sqrt{1-\prod_{i=1}^{k}(1-(\eta_{A_i})^q)^{\lambda_i}(1-(\eta_{A_{k+1}})^q)^{\lambda_{k+1}}} \end{pmatrix}\\
&= \begin{pmatrix} \left(\prod_{i=1}^{k+1}(a_{A_i})^{\lambda_i}, \prod_{i=1}^{k+1}(\sigma_{A_i})^{\lambda_i}\right), \\ \prod_{i=1}^{k+1}(\mu_{A_i})^{\lambda_i}, \sqrt{1-\prod_{i=1}^{k+1}(1-(\eta_{A_i})^q)^{\lambda_i}} \end{pmatrix}.
\end{aligned}$$

Therefore, the proof of Theorem 4.9 is completed. ∎

Theorem 4.10. *Let* $\Delta = \{\mathcal{A}_i = ((a_{A_i}, \sigma_{A_i}), \mu_{A_i}, \eta_{A_i}) : i = 1,2,3,\ldots,n\}$ *be a collection of* p,q*-QRONFNs on a fixed set* U *and* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)^T$ *be the weight vector of* Δ*, where* λ_j *indicates the importance degree of* Δ*, satisfying* $\lambda_1, \lambda_2, \ldots, \lambda_n \in [0,1]$ *and*

$\sum_{i=1}^{n} \lambda_i = 1$. *Then their accumulated outcome utilizing the* p, q*-QRONFWG operator is again* p, q*-QRONFN.*

Proof. Its proof immediately followed by Theorem 4.9. ∎

The p, q-QRONFWG operator has various desired qualities, which are as follows:

Theorem 4.11. *(Idempotency) Let* $\{\mathcal{A}_i = ((a_{A_i}, \sigma_{A_i}), \mu_{A_i}, \eta_{A_i}) : i = 1, 2, 3, \ldots, n\}$ *be a collection of* p, q*-QRONFNs on a fixed set* U*. If* $\mathcal{A}_i = \mathcal{A} = ((a_A, \sigma_A), \mu_A, \eta_A)$ *for every* $i = 1, 2, \ldots, n$*, then* $\mathcal{WG}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \mathcal{A}$.

Proof. Then, since $\mathcal{A}_1 = \mathcal{A}_2 = \ldots = \mathcal{A}_n = \mathcal{A}$ by Theorem 4.9, we have

$$\begin{aligned}
\mathcal{WG}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) &= \left(\begin{array}{c} \left(\prod_{i=1}^{n} (a_{A_i})^{\lambda_i}, \prod_{i=1}^{n} (\sigma_{A_i})^{\lambda_i} \right), \\ \prod_{i=1}^{n} (\mu_{A_i})^{\lambda_i}, \sqrt[q]{1 - \prod_{i=1}^{n} (1 - (\eta_{A_i})^q)^{\lambda_i}} \end{array} \right) \\
&= \left(\begin{array}{c} \left((a_A)^{\sum_{i=1}^{n} \lambda_i}, (\sigma_A)^{\sum_{i=1}^{n} \lambda_i} \right), \\ (\mu_A)^{\sum_{i=1}^{n} \lambda_i}, \sqrt[q]{1 - (1 - (\eta_A)^q)^{\sum_{i=1}^{n} \lambda_i}} \end{array} \right) \\
&= \left(\begin{array}{c} \left((a_A)^1, (\sigma_A)^1 \right), \\ (\mu_A)^1, \sqrt[q]{1 - (1 - (\eta_A)^q)^1} \end{array} \right) \\
&= ((a_A, \sigma_A), \mu_A, \eta_A) \\
&= \mathcal{A}.
\end{aligned}$$

Therefore we obtain that $\mathcal{WG}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \mathcal{A}$. ∎

It is simple to demonstrate that the p, q-QRONFWG operator satisfies the following properties.

Proposition 4.12. *Let* $\{\mathcal{A}_i = ((a_{A_i}, \sigma_{A_i}), \mu_{A_i}, \eta_{A_i}) : i = 1, 2, 3, \ldots, n\}$ *be a collection of* p, q*-QRONFNs on a fixed set* U *Then the following properties hold.*

(1) *If* $\mathcal{A}_i = \emptyset$ *for every* $i = 1, 2, 3, \ldots, n$*, then* $\mathcal{WG}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \emptyset$;
(2) *If* $\mathcal{A}_i = \mathbb{U}$ *for every* $i = 1, 2, 3, \ldots, n$*, then* $\mathcal{WG}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) = \mathbb{U}$.

The following theorem expresses the characteristics of the p, q-QRONFWG operator:

Theorem 4.13. *(Monotonicity) Let* $\{\mathcal{A}_i = ((a_{A_i}, \sigma_{A_i}), \mu_{A_i}, \eta_{A_i}) : i = 1, 2, 3, \ldots, n\}$ *and* $\{\mathcal{B}_i = ((a_{B_i}, \sigma_{B_i}), \mu_{B_i}, \eta_{B_i}) : i = 1, 2, 3, \ldots, n\}$ *be any collections of* p, q*-QRONFNs on a fixed set* U*. If* $\mathcal{A}_i \preceq \mathcal{B}_i$ *for every* $i = 1, 2, \ldots, n$*, then* $\mathcal{WG}(\mathcal{A}_1, \ldots, \mathcal{A}_n) \preceq \mathcal{WG}(\mathcal{B}_1, \ldots, \mathcal{B}_n)$.

Proof. Then, since $\mathcal{A}_i \preceq \mathcal{B}_i$ for every $i = 1, 2, 3, \ldots, n$, we have $a_{A_i} \leq a_{B_i}$, $\sigma_{A_i} \geq \sigma_{B_i}$ and $\mu_{A_i} \leq \mu_{B_i}$, $\eta_{A_i} \geq \eta_{B_i}$, that is, $(a_{A_i})^{\lambda_i} \leq (a_{B_i})^{\lambda_i}$

$$\prod_{i=1}^{n} (a_{A_i})^{\lambda_i} \leq \prod_{i=1}^{n} (a_{B_i})^{\lambda_i}. \tag{4.14}$$

It can be similarly proved that

$$\prod_{i=1}^{n}(\sigma_{A_i})^{\lambda_i} \geq \prod_{i=1}^{n}(\sigma_{B_i})^{\lambda_i} \tag{4.15}$$

and

$$\prod_{i=1}^{n}(\mu_{A_i})^{\lambda_i} \leq \prod_{i=1}^{n}(\mu_{B_i})^{\lambda_i}. \tag{4.16}$$

From these calculations, we obtain

$$\begin{gathered}
(\eta_{A_i})^q \geq (\eta_{B_i})^q \geq 0 \\
0 \leq 1-(\eta_{A_i})^q \leq 1-(\eta_{B_i})^q \\
0 \leq (1-(\eta_{A_i})^q)^{\lambda_i} \leq (1-(\eta_{B_i})^q)^{\lambda_i} \\
0 \leq \prod_{i=1}^{n}(1-(\eta_{A_i})^q)^{\lambda_i} \leq \prod_{i=1}^{n}(1-(\eta_{B_i})^q)^{\lambda_i} \\
1-\prod_{i=1}^{n}(1-(\eta_{A_i})^q)^{\lambda_i} \geq 1-\prod_{i=1}^{n}(1-(\eta_{B_i})^q)^{\lambda_i} \geq 0
\end{gathered}$$

$$\sqrt[q]{1-\prod_{i=1}^{n}(1-(\eta_{A_i})^q)^{\lambda_i}} \geq \sqrt[q]{1-\prod_{i=1}^{q}(1-(\eta_{B_i})^q)^{\lambda_i}}. \tag{4.17}$$

By Eqs. (4.14), (4.15), (4.16) and (4.17), $\mathcal{WG}(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n) \preceq \mathcal{WG}(\mathcal{B}_1,\mathcal{B}_2,\ldots,\mathcal{B}_n)$. ∎

The following Theorem 4.14 establishes some of the features of the p,q-QRONFWG operator.

Theorem 4.14. *(Boundedness) Let* $\{\mathcal{A}_i=((a_{A_i},\sigma_{A_i}),\mu_{A_i},\eta_{A_i}) : i=1,2,3,\ldots,n\}$ *be a collection of* p,q*-QRONFNs on a fixed set* U*. If* $\mathcal{A}^+ = \left(\left(\bigvee_{i=1}^{n} a_{A_i}, \bigwedge_{i=1}^{n}\sigma_{A_i}\right), \bigvee_{i=1}^{n}\mu_{A_i}, \bigwedge_{i=1}^{n}\eta_{A_i}\right)$ *and* $\mathcal{A}^- = \left(\left(\bigwedge_{i=1}^{n} a_{A_i}, \bigvee_{i=1}^{n}\sigma_{A_i}\right), \bigwedge_{i=1}^{n}\mu_{A_i}, \bigvee_{i=1}^{n}\eta_{A_i}\right)$*, then*

$$\mathcal{A}^- \preceq \mathcal{WG}(\mathcal{A}_1,\mathcal{A}_2,\ldots,\mathcal{A}_n) \preceq \mathcal{A}^+.$$

Proof. For the membership degree of $\mathcal{WG}(\mathcal{A}_1,\ldots,\mathcal{A}_n)$, we have

$$\begin{gathered}
\bigwedge_{i=1}^{n}(\mu_{A_i})^{\lambda_i} \leq (\mu_{A_i})^{\lambda_i} \leq \bigvee_{i=1}^{n}(\mu_{A_i})^{\lambda_i} \\
\prod_{i=1}^{n}\bigwedge_{i=1}^{n}(\mu_{A_i})^{\lambda_i} \leq \prod_{i=1}^{n}(\mu_{A_i})^{\lambda_i} \leq \prod_{i=1}^{n}\bigvee_{i=1}^{n}(\mu_{A_i})^{\lambda_i} \\
\bigwedge_{i=1}^{n}(\mu_{A_i})^{\sum_{i=1}^{n}\lambda_i} \leq \prod_{i=1}^{n}(\mu_{A_i})^{\lambda_i} \leq \bigvee_{i=1}^{n}(\mu_{A_i})^{\sum_{i=1}^{n}\lambda_i}
\end{gathered}$$

$$\bigwedge_{i=1}^{n}(\mu_{A_i}) \leq \prod_{i=1}^{n}(\mu_{A_i})^{\lambda_i} \leq \bigvee_{i=1}^{n}(\mu_{A_i}). \tag{4.18}$$

Similarly we can prove

$$\bigwedge_{i=1}^{n}(a_{A_i}) \leq \prod_{i=1}^{n}(a_{A_i})^{\lambda_i} \leq \bigvee_{i=1}^{n}(a_{A_i}) \tag{4.19}$$

and

$$\bigwedge_{i=1}^{n}(\sigma_{A_i}) \leq \prod_{i=1}^{n}(\sigma_{A_i})^{\lambda_i} \leq \bigvee_{i=1}^{n}(\sigma_{A_i})\,. \tag{4.20}$$

For the non-membership degree of $\mathcal{WG}(\mathcal{A}_1,\ldots,\mathcal{A}_n)$, we have

$$0 \leq \bigwedge_{i=1}^{n}(\eta_{A_i})^q \leq (\eta_{A_i})^q$$

$$1-\bigwedge_{i=1}^{n}(\eta_{A_i})^q \geq 1-(\eta_{A_i})^q \geq 0$$

$$\left(1-\bigwedge_{i=1}^{n}(\eta_{A_i})^q\right)^{\lambda_i} \geq (1-(\eta_{A_i})^q)^{\lambda_i} \geq 0$$

$$\prod_{i=1}^{n}\left(1-\bigwedge_{i=1}^{n}(\eta_{A_i})^q\right)^{\lambda_i} \geq \prod_{i=1}^{n}(1-(\eta_{A_i})^q)^{\lambda_i} \geq 0$$

$$0 \leq 1-\prod_{i=1}^{n}\left(1-\bigwedge_{i=1}^{n}(\eta_{A_i})^q\right)^{\lambda_i} \leq 1-\prod_{i=1}^{n}(1-(\eta_{A_i})^q)^{\lambda_i}$$

$$\sqrt[q]{1-\prod_{i=1}^{n}\left(1-\bigwedge_{i=1}^{n}(\eta_{A_i})^q\right)^{\lambda_i}} \leq \sqrt[q]{1-\prod_{i=1}^{n}(1-(\eta_{A_i})^q)^{\lambda_i}}$$

$$\sqrt[q]{1-\left(1-\bigwedge_{i=1}^{n}(\eta_{A_i})^q\right)^{\sum_{i=1}^{n}\lambda_i}} \leq \sqrt[q]{1-\prod_{i=1}^{n}(1-(\eta_{A_i})^q)^{\lambda_i}}$$

$$\sqrt[q]{1-\left(1-\bigwedge_{i=1}^{n}(\eta_{A_i})^q\right)} \leq \sqrt[q]{1-\prod_{i=1}^{n}(1-(\eta_{A_i})^q)^{\lambda_i}}$$

$$\bigwedge_{i=1}^{n}\eta_{A_i} \leq \sqrt[q]{1-\prod_{i=1}^{n}(1-(\eta_{A_i})^q)^{\lambda_i}} \tag{4.21}$$

and

$$\bigvee_{i=1}^{n}(\eta_{A_i})^q \geq (\eta_{A_i})^q \geq 0$$

$$0 \leq 1-\bigvee_{i=1}^{n}(\eta_{A_i})^q \leq 1-(\eta_{A_i})^q$$

$$0 \leq \left(1-\bigvee_{i=1}^{n}(\eta_{A_i})^q\right)^{\lambda_i} \leq (1-(\eta_{A_i})^q)^{\lambda_i}$$

$$0 \leq \prod_{i=1}^{n}\left(1-\bigvee_{i=1}^{n}(\eta_{A_i})^q\right)^{\lambda_i} \leq \prod_{i=1}^{n}(1-(\eta_{A_i})^q)^{\lambda_i}$$

$$1-\prod_{i=1}^{n}\left(1-\bigvee_{i=1}^{n}(\eta_{A_i})^q\right)^{\lambda_i} \geq 1-\prod_{i=1}^{n}(1-(\eta_{A_i})^q)^{\lambda_i} \geq 0$$

$$\sqrt[q]{1-\prod_{i=1}^{n}\left(1-\bigvee_{i=1}^{n}(\eta_{A_i})^q\right)^{\lambda_i}} \geq \sqrt[q]{1-\prod_{i=1}^{n}(1-(\eta_{A_i})^q)^{\lambda_i}}$$

$$\sqrt[q]{1-\left(1-\bigvee_{i=1}^{n}(\eta_{A_i})^q\right)^{\sum_{i=1}^{n}\lambda_i}} \geq \sqrt[q]{1-\prod_{i=1}^{n}(1-(\eta_{A_i})^q)^{\lambda_i}}$$

$$\sqrt[q]{1-\left(1-\bigvee_{i=1}^{n}(\eta_{A_i})^q\right)^{1}} \geq \sqrt[q]{1-\prod_{i=1}^{n}(1-(\eta_{A_i})^q)^{\lambda_i}}$$

$$\bigvee_{i=1}^{n}\eta_{A_i} \geq \sqrt[q]{1-\prod_{i=1}^{n}(1-(\eta_{A_i})^q)^{\lambda_i}}. \tag{4.22}$$

Then, by Eqs. (4.21) and (4.22), we get

$$\bigwedge_{i=1}^{n}\eta_{A_i} \leq \sqrt[q]{1-\prod_{i=1}^{n}(1-(\eta_{A_i})^q)^{\lambda_i}} \leq \bigvee_{i=1}^{n}\eta_{A_i}. \tag{4.23}$$

Therefore, by Eqs. (4.18), (4.19),(4.20) and (4.23), we have

$$\mathcal{A}^- \preceq \mathcal{WG}(\mathcal{A}_1, \mathcal{A}_2, \ldots, \mathcal{A}_n) \preceq \mathcal{A}^+.$$

∎

5. Application

In the subsequent sections, we detail the construction and application of the MADM method, demonstrating its efficacy through illustrative examples and case studies. This work not only advances the theoretical understanding of p,q-QRONFSs but also provides practical solutions for decision-makers facing complex, multi-attribute challenges in uncertain environments.

5.1. Problem and solution steps

MADM is a critical process in various fields, where decisions are made based on multiple, often conflicting, criteria. In environments characterized by uncertainty and imprecision, traditional decision-making approaches may fall short in effectively capturing the complexity of the information at hand. To address these challenges, advanced fuzzy set theories have been developed, offering a more nuanced way to model and process uncertain data. One such advancement is the p,q-QRONFS, which extends the traditional fuzzy set by introducing a flexible framework for managing a wider range of uncertainty. This

manuscript introduces a novel multi-attribute decision-making method called p,q-quasi-rung orthopair normal fuzzy multi-attribute decision-making (p,q-QRONFMADM). This method leverages the power of p,q-QRONFSs to define real number weights and aggregate fuzzy information, providing a robust and comprehensive approach to decision-making in complex environments.

In a p,q-QRONFMADM problem, decision-makers are required to evaluate a set of alternatives, denoted as $A = \{A_1, A_2, A_3, \ldots, A_m\}$, against multiple attributes $G = \{G_1, G_2, \ldots, G_n\}$. Each alternative is assessed by a group of decision-makers, who express their evaluations as p,q-QRONFNs. These evaluations are then aggregated into a p,q-quasi-rung orthopair normal fuzzy decision matrix, which serves as the basis for further analysis. Suppose that $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)^T$ is the weight vector met the criteria as $0 \leq \lambda_1, \lambda_2, \ldots, \lambda_n \leq 1$ such that $\sum_{i=1}^{n} \lambda_i = 1$. A set of decision-makers are invited to appraisal all the alternatives. The expert's decisions are stated as p,q-QRONFNs, i.e., $\left\{\mathcal{A}_{ij} = \left(\left(a_{A_{ij}}, \sigma_{A_{ij}}\right), \mu_{A_{ij}}, \eta_{A_{ij}}\right) : i = 1,2,3,\ldots,m, j = 1,2,3,\ldots,n\right\}$ such that $0 \leq \left(\mu_{A_{ij}}\right)^p + \left(\eta_{A_{ij}}\right)^q \leq 1$. Thus, the p,q-quasi-rung orthopair normal fuzzy decision matrix is given as $\mathcal{D} = \left[\left(\left(a_{A_{ij}}, \sigma_{A_{ij}}\right), \mu_{A_{ij}}, \eta_{A_{ij}}\right)\right]_{m\times n}$ and presented as follows:

$$\mathcal{D} = \begin{bmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} & \ldots & \mathcal{A}_{1n} \\ \mathcal{A}_{21} & \mathcal{A}_{22} & \ldots & \mathcal{A}_{2n} \\ \mathcal{A}_{31} & \mathcal{A}_{32} & \ldots & \mathcal{A}_{3n} \\ \vdots & & & \\ \mathcal{A}_{m1} & \mathcal{A}_{m2} & \ldots & \mathcal{A}_{mn} \end{bmatrix}. \tag{5.1}$$

The algorithm follows a p,q-QRONFMADM method to interpret a MADM problem under p,q-quasi-rung orthopair normal fuzzy information using p,q-QRONFWA operator and p,q-QRONFWG operator. This method has the following steps.

5.2. Algorithm based on MADM of p,q-QRONFNs

This endeavor encompasses a meticulously delineated sequence of steps, as detailed below:

Input: Some alternatives

Step 1: Calculate the p,q-quasi-rung orthopair normal fuzzy decision matrix.

Step 2: In order to eliminate the influence of attribute type.

We consider the following technique and obtain the standardize p,q-quasi-rung orthopair normal fuzzy matrix $\mathcal{R} = [\mathcal{R}_{ij}]_{m\times n}$, where $\left\{\mathcal{R}_{ij} = \left(\left(\widetilde{a}_{A_{ij}}, \widetilde{\sigma}_{A_{ij}}\right), \widetilde{\mu}_{A_{ij}}, \widetilde{\eta}_{A_{ij}}\right) : i = 1,2,3,\ldots,m, j = 1,2,3,\ldots,n\right\}$. For benefit oriented attributes:

$$\widetilde{a}_{A_{ij}} = \frac{a_{A_{ij}}}{\bigvee_{i=1}^{m} a_{A_{ij}}}, \widetilde{\sigma}_{A_{ij}} = \frac{\sigma_{A_{ij}}}{\bigvee_{i=1}^{m} \sigma_{A_{ij}}} \times \frac{\sigma_{A_{ij}}}{a_{A_{ij}}}, \widetilde{\mu}_{A_{ij}} = \mu_{A_{ij}}, \widetilde{\eta}_{A_{ij}} = \eta_{A_{ij}}.$$

For cost-oriented attributes:

$$\widetilde{a}_{A_{ij}} = \frac{\bigvee_{i=1}^{m} a_{A_{ij}}}{a_{A_{ij}}}, \widetilde{\sigma}_{A_{ij}} = \frac{\sigma_{A_{ij}}}{\bigvee_{i=1}^{m} \sigma_{A_{ij}}} \times \frac{\sigma_{A_{ij}}}{a_{A_{ij}}}, \widetilde{\mu}_{A_{ij}} = \mu_{A_{ij}}, \widetilde{\eta}_{A_{ij}} = \eta_{A_{ij}}.$$

Step 3: Based on the standardize p,q-quasi-rung orthopair normal fuzzy matrix.

As obtained from step 2, the overall aggregated value of the alternative A_i for all $i = 1, 2, \ldots, m$, under the different criteria G_j for all $j = 1, 2, \ldots, n$ is obtained by using either

$$\mathcal{E}_i = \mathcal{WA}\left(\mathcal{R}_{i1}, \mathcal{R}_{i2}, \ldots, \mathcal{R}_{in}\right) \tag{5.2}$$

or

$$\mathcal{E}_i = \mathcal{WG}\left(\mathcal{R}_{i1}, \mathcal{R}_{i2}, \ldots, \mathcal{R}_{in}\right) \tag{5.3}$$

and hence get the collective value $\mathcal{E}_i$ for each alternative A_i for all $i = 1, 2, \ldots, m$.

Step 4: The score values of $\mathcal{SC}\left(\mathcal{E}_i\right)$ of the overall p,q-QRONFNs of $\mathcal{E}_i$ is obtained to rank the alternatives A_i.

If the score values of $\mathcal{SC}_k\left(\mathcal{E}_i\right)$ and $\mathcal{SC}_k\left(\mathcal{E}_j\right)$ are equal for two alternatives A_i and A_j, then it is required to calculate accuracy degrees of $\mathcal{AC}_k\left(\mathcal{E}_i\right)$ and $\mathcal{AC}_k\left(\mathcal{E}_j\right)$ with respect to the overall collective p,q-QRONFNs to rank the alternatives A_i and A_j, respectively, based on the aforementioned accuracy degrees $\mathcal{AC}_k\left(\mathcal{E}_i\right)$ and $\mathcal{AC}_k\left(\mathcal{E}_j\right)$.

Step 5: We select the best alternative from the rankings of all alternatives A_i according to $\mathcal{SC}_k\left(\mathcal{E}_i\right)$.

Output: Best alternative

To illustrate the sequential progression of the algorithm's steps, we provide a visual representation through the flowchart diagram of the proposed algorithm, as depicted in Fig. 2.

5.3. An Illustrative application of p,q-QRONFSs in MADM

In the increasingly interconnected global economy, businesses face complex challenges that demand advanced strategies to remain competitive. One of the most critical strategies for maintaining competitiveness is the identification and selection of suitable global partners. The decision-making process in this context is multifaceted, influenced by various factors that can significantly impact the long-term success of partnerships. These factors include, but are not limited to, research and development (R&D) competence, business operation levels, international cooperation, and creditworthiness. The complexity of this decision-making process is further heightened by the inherent uncertainty and ambiguity in global business environments. Information relevant to evaluating potential partners is often incomplete, imprecise, or subject to rapid changes due to the dynamic nature of global markets. Traditional MADM methods have been developed to assist decision-makers in evaluating and ranking alternatives based on multiple criteria. However, these conventional approaches may not fully capture the uncertainty and ambiguity that characterize real-world decision-making, particularly in global contexts. This gap

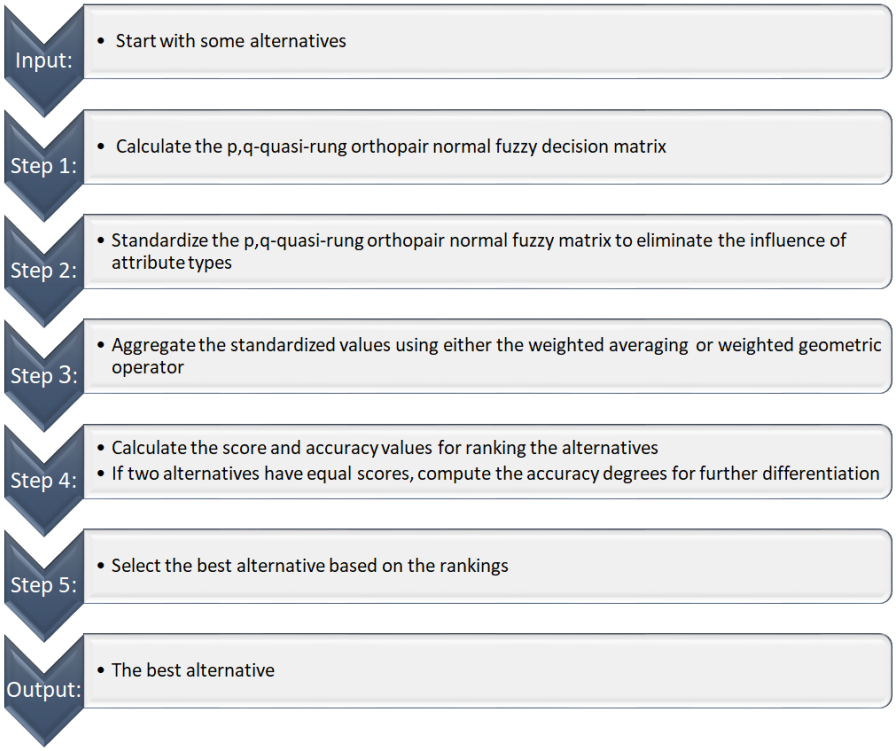

Figure 2. Flowchart diagram of proposed algorithm

necessitates the development of more sophisticated methods that can effectively handle imprecise and uncertain information. Recent advancements in fuzzy set theory, particularly the introduction of p, q-QRONFSs, offer a promising avenue for enhancing MADM techniques. These sets provide a more flexible and nuanced representation of uncertainty, making them well-suited for complex decision-making scenarios.

In this manuscript, we introduce an advanced MADM approach based on the p, q-QRONFMADM framework. This approach leverages the p, q-QRONFSs to enhance decision-making reliability in complex and uncertain environments. By incorporating p, q-QRONFSs, our proposed method allows decision-makers to better manage the ambiguity and uncertainty inherent in global business environments, leading to more informed and robust decisions. To demonstrate the effectiveness of the proposed p, q-QRONFMADM technique, we apply it to a specific decision-making problem involving the selection of a suitable global business partner. The problem scenario, derived from [29], involves an organization that has shortlisted five potential partner enterprises, denoted as $\mathbb{A} = \{\mathbb{A}_1, \mathbb{A}_2, \mathbb{A}_3, \mathbb{A}_4, \mathbb{A}_5\}$. These enterprises are evaluated based on four key attributes: R&D competence ($\mathbb{G}_1$), business operation level ($\mathbb{G}_2$), international cooperation level ($\mathbb{G}_3$), and credit level ($\mathbb{G}_4$), all of which are considered benefit-oriented. The weight vector $\lambda = \{0.2, 0.3, 0.2, 0.3\}$ is used to reflect the relative importance of each attribute. In Section 5.1, we apply the p, q-QRONFMADM model to the global partner

selection problem, demonstrating its capability in managing complex and uncertain information. The results, summarized in Table 2, underscore the model's robustness and adaptability, establishing it as a valuable decision-support tool in global business.

Table 2. Evaluations of decision makers

	$\mathbb{G}_1$	$\mathbb{G}_2$	$\mathbb{G}_3$	$\mathbb{G}_4$
$\mathbb{A}_1$	$\begin{pmatrix}(8,0.7),\\0.8,0.1\end{pmatrix}$	$\begin{pmatrix}(4,0.5),\\0.6,0.2\end{pmatrix}$	$\begin{pmatrix}(8,0.5),\\0.9,0.1\end{pmatrix}$	$\begin{pmatrix}(4,0.3),\\0.7,0.2\end{pmatrix}$
$\mathbb{A}_2$	$\begin{pmatrix}(6,0.4),\\0.6,0.4\end{pmatrix}$	$\begin{pmatrix}(5,0.6),\\0.8,0.2\end{pmatrix}$	$\begin{pmatrix}(4,0.4),\\0.5,0.5\end{pmatrix}$	$\begin{pmatrix}(8,0.5),\\0.4,0.6\end{pmatrix}$
$\mathbb{A}_3$	$\begin{pmatrix}(9,0.8),\\0.7,0.3\end{pmatrix}$	$\begin{pmatrix}(7,0.8),\\0.6,0.4\end{pmatrix}$	$\begin{pmatrix}(4,0.3),\\0.4,0.6\end{pmatrix}$	$\begin{pmatrix}(6,0.6),\\0.4,0.06\end{pmatrix}$
$\mathbb{A}_4$	$\begin{pmatrix}(7,0.6),\\0.5,0.5\end{pmatrix}$	$\begin{pmatrix}(8,0.7),\\0.4,0.6\end{pmatrix}$	$\begin{pmatrix}(5,0.4),\\0.8,0.2\end{pmatrix}$	$\begin{pmatrix}(6,0.4),\\0.6,0.3\end{pmatrix}$
$\mathbb{A}_5$	$\begin{pmatrix}(5,0.3),\\0.6,0.4\end{pmatrix}$	$\begin{pmatrix}(7,0.5),\\0.8,0.2\end{pmatrix}$	$\begin{pmatrix}(5,0.3),\\0.5,0.5\end{pmatrix}$	$\begin{pmatrix}(8,0.6),\\0.4,0.6\end{pmatrix}$

5.4. The MADM based on the p,q-QRONFWA and p,q-QRONFWG operators

Based on the given information, the steps of developed approach are executed as follows:

Step 1: Develop a p,q-quasi-rung orthopair normal fuzzy decision matrix

To evaluate the five alternatives $\mathbb{A}_1$, $\mathbb{A}_2$, $\mathbb{A}_3$, $\mathbb{A}_4$, and $\mathbb{A}_5$, we first construct a p,q-quasi-rung orthopair normal fuzzy decision matrix. This matrix is denoted by $\mathcal{D} = [\mathcal{A}ij]_{5\times 4}$, where each element $\mathcal{A}_{ij}$ represents the fuzzy rating of alternative $\mathbb{A}_i$ with respect to attribute $\mathbb{G}_j$. The matrix $\mathcal{D}$ is detailed as follows:

$$\mathcal{D} = \begin{bmatrix} \begin{pmatrix}(8,0.7),\\0.8,0.1\end{pmatrix} & \begin{pmatrix}(4,0.5),\\0.6,0.2\end{pmatrix} & \begin{pmatrix}(8,0.5),\\0.9,0.1\end{pmatrix} & \begin{pmatrix}(4,0.3),\\0.7,0.2\end{pmatrix} \\ \begin{pmatrix}(6,0.4),\\0.6,0.4\end{pmatrix} & \begin{pmatrix}(5,0.6),\\0.8,0.2\end{pmatrix} & \begin{pmatrix}(4,0.4),\\0.5,0.5\end{pmatrix} & \begin{pmatrix}(8,0.5),\\0.4,0.6\end{pmatrix} \\ \begin{pmatrix}(9,0.8),\\0.7,0.3\end{pmatrix} & \begin{pmatrix}(7,0.8),\\0.6,0.4\end{pmatrix} & \begin{pmatrix}(4,0.3),\\0.4,0.6\end{pmatrix} & \begin{pmatrix}(6,0.6),\\0.4,0.06\end{pmatrix} \\ \begin{pmatrix}(7,0.6),\\0.5,0.5\end{pmatrix} & \begin{pmatrix}(8,0.7),\\0.4,0.6\end{pmatrix} & \begin{pmatrix}(5,0.4),\\0.8,0.2\end{pmatrix} & \begin{pmatrix}(6,0.4),\\0.6,0.3\end{pmatrix} \\ \begin{pmatrix}(5,0.3),\\0.6,0.4\end{pmatrix} & \begin{pmatrix}(7,0.5),\\0.8,0.2\end{pmatrix} & \begin{pmatrix}(5,0.3),\\0.5,0.5\end{pmatrix} & \begin{pmatrix}(8,0.6),\\0.4,0.6\end{pmatrix} \end{bmatrix}. \tag{5.4}$$

Step 2: Normalize the decision matrix

Since all attributes $\mathbb{G}_1$, $\mathbb{G}_2$, $\mathbb{G}_3$, and $\mathbb{G}_4$ are benefit-type, the decision matrix needs to be normalized.

$$\begin{bmatrix} \begin{pmatrix} (0.9,0.1), \\ 0.8,0.1 \end{pmatrix} & \begin{pmatrix} (0.5,0.1), \\ 0.6,0.2 \end{pmatrix} & \begin{pmatrix} (1,0.1), \\ 0.9,0.1 \end{pmatrix} & \begin{pmatrix} (0.5,0.0), \\ 0.7,0.2 \end{pmatrix} \\ \begin{pmatrix} (0.7,0.0), \\ 0.6,0.4 \end{pmatrix} & \begin{pmatrix} (0.6,0.1), \\ 0.8,0.2 \end{pmatrix} & \begin{pmatrix} (0.5,0.1), \\ 0.5,0.5 \end{pmatrix} & \begin{pmatrix} (1,0.1), \\ 0.4,0.6 \end{pmatrix} \\ \begin{pmatrix} (1,0.1), \\ 0.7,0.3 \end{pmatrix} & \begin{pmatrix} (0.9,0.1), \\ 0.6,0.4 \end{pmatrix} & \begin{pmatrix} (0.5,0.0), \\ 0.4,0.6 \end{pmatrix} & \begin{pmatrix} (0.8,0.1), \\ 0.4,0.06 \end{pmatrix} \\ \begin{pmatrix} (0.8,0.1), \\ 0.5,0.5 \end{pmatrix} & \begin{pmatrix} (1,0.1), \\ 0.4,0.6 \end{pmatrix} & \begin{pmatrix} (0.6,0.1), \\ 0.8,0.2 \end{pmatrix} & \begin{pmatrix} (0.8,0.0), \\ 0.6,0.3 \end{pmatrix} \\ \begin{pmatrix} (0.6,0.0), \\ 0.6,0.4 \end{pmatrix} & \begin{pmatrix} (0.9,0.0), \\ 0.8,0.2 \end{pmatrix} & \begin{pmatrix} (0.6,0.0), \\ 0.5,0.5 \end{pmatrix} & \begin{pmatrix} (1.0,0.1), \\ 0.4,0.6 \end{pmatrix} \end{bmatrix}. \tag{5.5}$$

Step 3: Apply p,q-QRONFWA/WG operator

We derive the overall rating values of each option $\mathbb{A}_i$ by following the p,q-QRONFWA and p,q-QRONFWG operators described in Eqs. (5.2) and (5.3), which are illustrated in Table 3.

Table 3. Calculation of the aggregated values for the alternatives

	5, 10-**QRONFWA operator**	5, 10-**QRONFWG operator**
$\mathcal{E}_1$	$\begin{pmatrix} (0.6778,0.0625), \\ 0.7788,0.1516 \end{pmatrix}$	$\begin{pmatrix} (0.6444,0.0597), \\ 0.7218,0.1901 \end{pmatrix}$
$\mathcal{E}_2$	$\begin{pmatrix} (0.7208,0.0653), \\ 0.6697,0.3837 \end{pmatrix}$	$\begin{pmatrix} (0.6972,0.0612), \\ 0.5584,0.5381 \end{pmatrix}$
$\mathcal{E}_3$	$\begin{pmatrix} (0.7875,0.0911), \\ 0.5772,0.2318 \end{pmatrix}$	$\begin{pmatrix} (0.7672,0.0967), \\ 0.5052,0.5123 \end{pmatrix}$
$\mathcal{E}_4$	$\begin{pmatrix} (0.8056,0.0620), \\ 0.6395,0.3772 \end{pmatrix}$	$\begin{pmatrix} (0.7914,0.0606), \\ 0.5426,0.5376 \end{pmatrix}$
$\mathcal{E}_5$	$\begin{pmatrix} (0.7986,0.0476), \\ 0.6697,0.3837 \end{pmatrix}$	$\begin{pmatrix} (0.7775,0.0436), \\ 0.5584,0.5381 \end{pmatrix}$

Step 4: Compute score and accuracy function

Calculate the scores function of $\mathcal{E}_i$, which is shown in Table 4.

Table 4. Computation of score function of aggregated information

	5, 10-**QRONFWA operator**	5, 10-**QRONFWG operator**
$\mathcal{SC}_1(\mathcal{E}_1)$	0.1941	0.1263
$\mathcal{SC}_1(\mathcal{E}_2)$	0.0971	0.0378
$\mathcal{SC}_1(\mathcal{E}_3)$	0.0505	0.0252
$\mathcal{SC}_1(\mathcal{E}_4)$	0.0862	0.0373
$\mathcal{SC}_1(\mathcal{E}_5)$	0.1076	0.0422

Table 5. Ranking of the alternative

	Ranking ordered
$5, 10$-QRONFWA operator	$\mathbb{A}_1 \geq \mathbb{A}_5 \geq \mathbb{A}_2 \geq \mathbb{A}_4 \geq \mathbb{A}_3$
$5, 10$-QRONFWG operator	$\mathbb{A}_1 \geq \mathbb{A}_5 \geq \mathbb{A}_2 \geq \mathbb{A}_4 \geq \mathbb{A}_3$

Step 5: Rank alternatives

Calculate the ranking order of the alternatives, which is shown in Table 5.

5.5. Influence of p, q-QRONFWA and p, q-QRONFWG operators

To analyze the sensitivity and adaptability of the proposed p, q-QRONFWA/WG operators, various parameter combinations of p and q were examined. Tables 6, 7, and Figures 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, present the ranking results of five alternatives $\mathbb{A}_1$–$\mathbb{A}_5$ under different (p, q) settings.

For the p, q-QRONFWA operator (Table 6 and Figures 3, 4, 5, 6, 7, 8), the ranking pattern exhibits stability for lower values of p and q, such as $(1, 1)$ and $(1, 2)$, where the order is $\mathbb{A}_1 \geq \mathbb{A}_5 \geq \mathbb{A}_4 \geq \mathbb{A}_3 \geq \mathbb{A}_2$. However, as p and q increase (e.g., $(2, 3)$, $(3, 1)$, and higher combinations), a shift occurs, where $\mathbb{A}_2$ surpasses $\mathbb{A}_4$. From $p \geq 3$, the ranking pattern stabilizes as $\mathbb{A}_1 \geq \mathbb{A}_5 \geq \mathbb{A}_2 \geq \mathbb{A}_4 \geq \mathbb{A}_3$. This demonstrates that larger p values, which increase the degree of nonlinearity in aggregation, enhance the impact of higher membership degrees while diminishing the influence of lower ones. Consequently, the averaging operator becomes more optimistic as p increases.

In contrast, the p, q-QRONFWG operator (Table 7 and Figures 9, 10, 11, 12, 13, 14) displays greater ranking consistency across all examined (p, q) pairs. For nearly all parameter combinations, the ranking remains $\mathbb{A}_1 \geq \mathbb{A}_5 \geq \mathbb{A}_4 \geq \mathbb{A}_2 \geq \mathbb{A}_3$, with only minor shifts observed when p and q are large. This reflects the geometric operators inherent conservatism and its tendency to reduce the influence of extreme values, yielding more stable and balanced aggregation outcomes.

Comparative analysis indicates that both operators consistently identify $\mathbb{A}_1$ as the best-performing alternative and $\mathbb{A}_3$ as the least preferred across all parameter values, confirming the robustness of the proposed framework. Nevertheless, the p, q-QRONFWA operator provides greater flexibility in reflecting decision-makers risk attitudes: lower (p, q) values yield balanced decisions, while higher values emphasize favorable criteria more strongly. Conversely, the p, q-QRONFWG operator maintains stable rankings and is better suited for risk-averse scenarios. Overall, the sensitivity analysis highlights that the parameters p and q effectively control the degree of optimism or conservatism in decision aggregation. The results confirm that the proposed p, q-QRONFWA and p, q-QRONFWG operators are adaptable, reliable, and capable of capturing nuanced decision preferences under uncertainty.

TABLE 6. Ranking of the alternative options using the p, q-QRONFWA for various p, q values

p, q	Ranking ordered	p, q	Ranking ordered
$1, 1$	$\mathbb{A}_1 \geq \mathbb{A}_5 \geq \mathbb{A}_4 \geq \mathbb{A}_3 \geq \mathbb{A}_2$	$2, 1$	$\mathbb{A}_1 \geq \mathbb{A}_5 \geq \mathbb{A}_4 \geq \mathbb{A}_2 \geq \mathbb{A}_3$
$1, 2$	$\mathbb{A}_1 \geq \mathbb{A}_5 \geq \mathbb{A}_4 \geq \mathbb{A}_3 \geq \mathbb{A}_2$	$2, 2$	$\mathbb{A}_1 \geq \mathbb{A}_5 \geq \mathbb{A}_4 \geq \mathbb{A}_2 \geq \mathbb{A}_3$
$1, 3$	$\mathbb{A}_1 \geq \mathbb{A}_5 \geq \mathbb{A}_4 \geq \mathbb{A}_2 \geq \mathbb{A}_3$	$2, 3$	$\mathbb{A}_1 \geq \mathbb{A}_5 \geq \mathbb{A}_4 \geq \mathbb{A}_2 \geq \mathbb{A}_3$
$1, 4$	$\mathbb{A}_1 \geq \mathbb{A}_5 \geq \mathbb{A}_4 \geq \mathbb{A}_2 \geq \mathbb{A}_3$	$2, 4$	$\mathbb{A}_1 \geq \mathbb{A}_5 \geq \mathbb{A}_4 \geq \mathbb{A}_2 \geq \mathbb{A}_3$
$1, 5$	$\mathbb{A}_1 \geq \mathbb{A}_5 \geq \mathbb{A}_4 \geq \mathbb{A}_2 \geq \mathbb{A}_3$	$2, 5$	$\mathbb{A}_1 \geq \mathbb{A}_5 \geq \mathbb{A}_4 \geq \mathbb{A}_2 \geq \mathbb{A}_3$
$1, 10$	$\mathbb{A}_1 \geq \mathbb{A}_5 \geq \mathbb{A}_4 \geq \mathbb{A}_2 \geq \mathbb{A}_3$	$2, 10$	$\mathbb{A}_1 \geq \mathbb{A}_5 \geq \mathbb{A}_4 \geq \mathbb{A}_2 \geq \mathbb{A}_3$
$3, 1$	$\mathbb{A}_1 \geq \mathbb{A}_5 \geq \mathbb{A}_2 \geq \mathbb{A}_4 \geq \mathbb{A}_3$	$4, 1$	$\mathbb{A}_1 \geq \mathbb{A}_5 \geq \mathbb{A}_2 \geq \mathbb{A}_4 \geq \mathbb{A}_3$
$3, 2$	$\mathbb{A}_1 \geq \mathbb{A}_5 \geq \mathbb{A}_2 \geq \mathbb{A}_4 \geq \mathbb{A}_3$	$4, 2$	$\mathbb{A}_1 \geq \mathbb{A}_5 \geq \mathbb{A}_2 \geq \mathbb{A}_4 \geq \mathbb{A}_3$
$3, 3$	$\mathbb{A}_1 \geq \mathbb{A}_5 \geq \mathbb{A}_2 \geq \mathbb{A}_4 \geq \mathbb{A}_3$	$4, 3$	$\mathbb{A}_1 \geq \mathbb{A}_5 \geq \mathbb{A}_2 \geq \mathbb{A}_4 \geq \mathbb{A}_3$
$3, 4$	$\mathbb{A}_1 \geq \mathbb{A}_5 \geq \mathbb{A}_2 \geq \mathbb{A}_4 \geq \mathbb{A}_3$	$4, 4$	$\mathbb{A}_1 \geq \mathbb{A}_5 \geq \mathbb{A}_2 \geq \mathbb{A}_4 \geq \mathbb{A}_3$
$3, 5$	$\mathbb{A}_1 \geq \mathbb{A}_5 \geq \mathbb{A}_2 \geq \mathbb{A}_4 \geq \mathbb{A}_3$	$4, 5$	$\mathbb{A}_1 \geq \mathbb{A}_5 \geq \mathbb{A}_2 \geq \mathbb{A}_4 \geq \mathbb{A}_3$
$3, 10$	$\mathbb{A}_1 \geq \mathbb{A}_5 \geq \mathbb{A}_2 \geq \mathbb{A}_4 \geq \mathbb{A}_3$	$4, 10$	$\mathbb{A}_1 \geq \mathbb{A}_5 \geq \mathbb{A}_2 \geq \mathbb{A}_4 \geq \mathbb{A}_3$
$5, 1$	$\mathbb{A}_1 \geq \mathbb{A}_5 \geq \mathbb{A}_2 \geq \mathbb{A}_4 \geq \mathbb{A}_3$	$10, 1$	$\mathbb{A}_1 \geq \mathbb{A}_5 \geq \mathbb{A}_2 \geq \mathbb{A}_4 \geq \mathbb{A}_3$
$5, 2$	$\mathbb{A}_1 \geq \mathbb{A}_5 \geq \mathbb{A}_2 \geq \mathbb{A}_4 \geq \mathbb{A}_3$	$10, 2$	$\mathbb{A}_1 \geq \mathbb{A}_5 \geq \mathbb{A}_2 \geq \mathbb{A}_4 \geq \mathbb{A}_3$
$5, 3$	$\mathbb{A}_1 \geq \mathbb{A}_5 \geq \mathbb{A}_2 \geq \mathbb{A}_4 \geq \mathbb{A}_3$	$10, 3$	$\mathbb{A}_1 \geq \mathbb{A}_5 \geq \mathbb{A}_2 \geq \mathbb{A}_4 \geq \mathbb{A}_3$
$5, 4$	$\mathbb{A}_1 \geq \mathbb{A}_5 \geq \mathbb{A}_2 \geq \mathbb{A}_4 \geq \mathbb{A}_3$	$10, 4$	$\mathbb{A}_1 \geq \mathbb{A}_5 \geq \mathbb{A}_2 \geq \mathbb{A}_4 \geq \mathbb{A}_3$
$5, 5$	$\mathbb{A}_1 \geq \mathbb{A}_5 \geq \mathbb{A}_2 \geq \mathbb{A}_4 \geq \mathbb{A}_3$	$10, 5$	$\mathbb{A}_1 \geq \mathbb{A}_5 \geq \mathbb{A}_2 \geq \mathbb{A}_4 \geq \mathbb{A}_3$
$5, 10$	$\mathbb{A}_1 \geq \mathbb{A}_5 \geq \mathbb{A}_2 \geq \mathbb{A}_4 \geq \mathbb{A}_3$	$10, 10$	$\mathbb{A}_1 \geq \mathbb{A}_5 \geq \mathbb{A}_2 \geq \mathbb{A}_4 \geq \mathbb{A}_3$

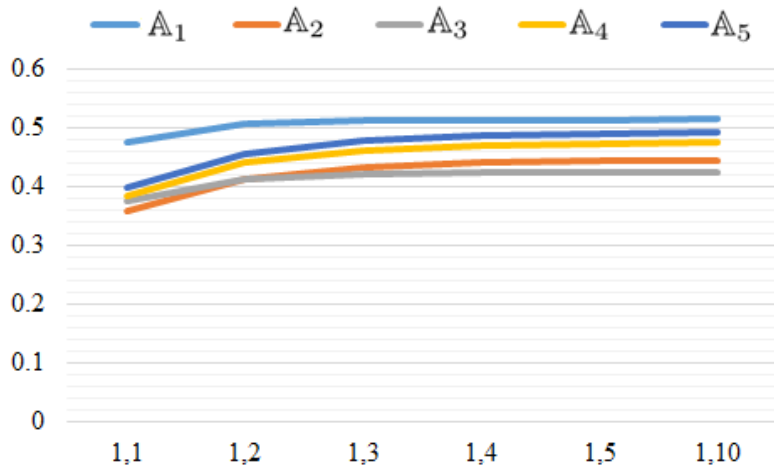

Figure 3. Ranking of the alternative options using the $1, q$-QRONFWA operator for various q value

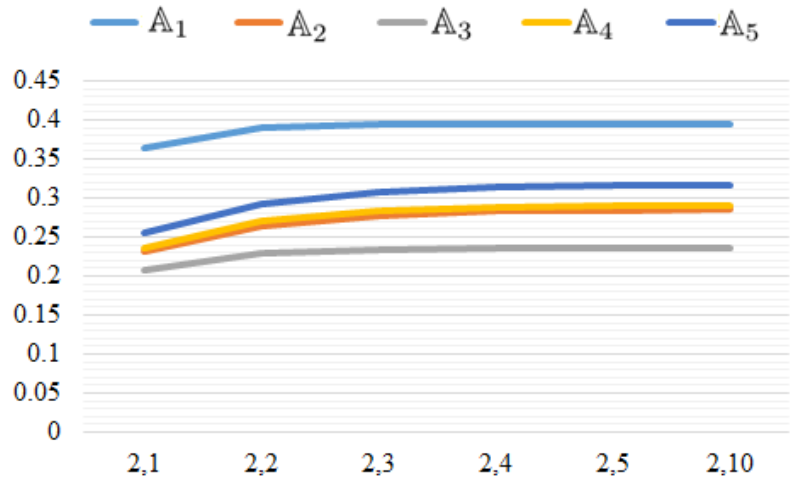

Figure 4. Ranking of the alternative options using the $2, q$-QRONFWA operator for various p value

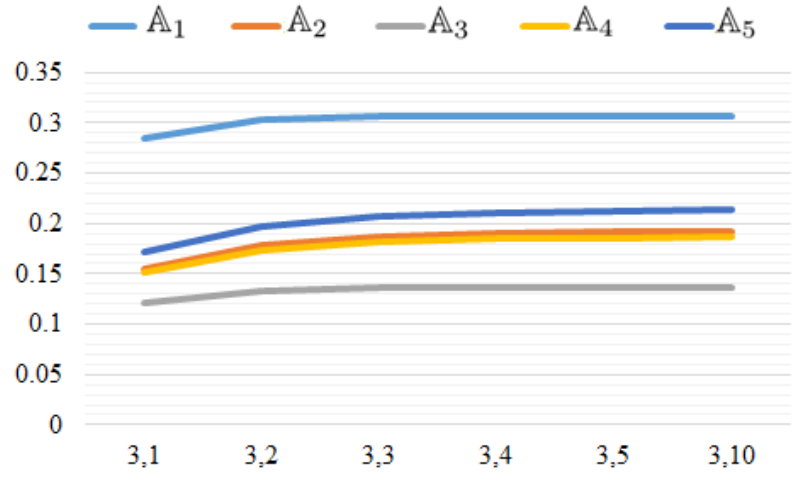

Figure 5. Ranking of the alternative options using the $3, q$-QRONFWA operator for various p value

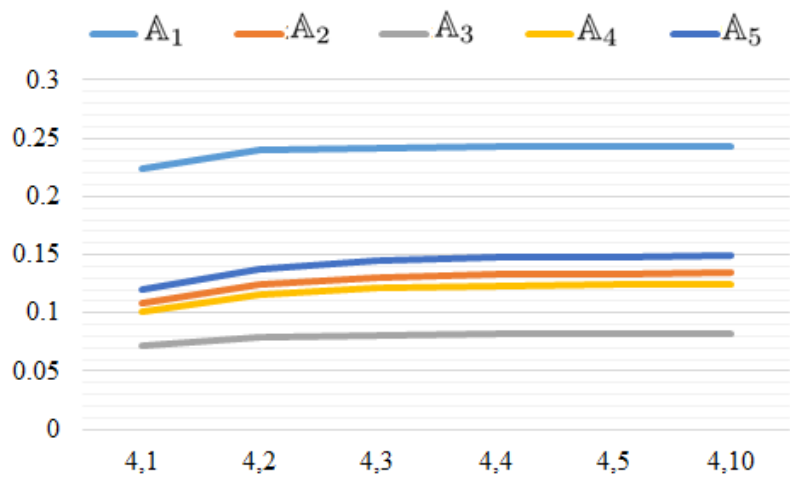

Figure 6. Ranking of the alternative options using the $4, q$-QRONFWA operator for various p value

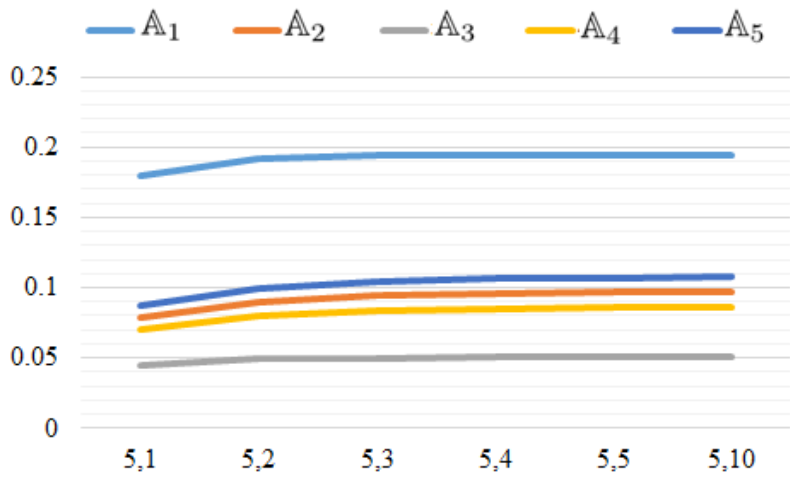

Figure 7. Ranking of the alternative options using the $5, q$-QRONFWA operator for various p value

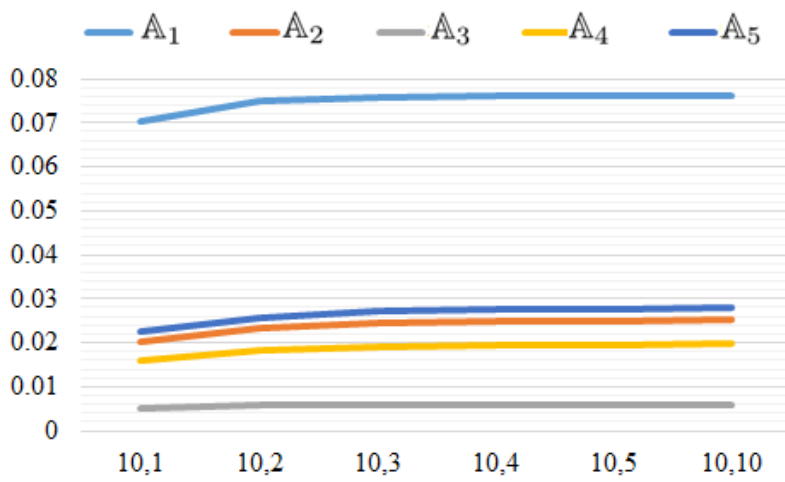

Figure 8. Ranking of the alternative options using the $10, q$-QRONFWA operator for various p value

TABLE 7. Ranking of the alternative options using the p, q-QRONFWG for various p, q values

p, q	Ranking ordered	p, q	Ranking ordered
$1, 1$	$\mathbb{A}_1 \geq \mathbb{A}_5 \geq \mathbb{A}_4 \geq \mathbb{A}_2 \geq \mathbb{A}_3$	$2, 1$	$\mathbb{A}_1 \geq \mathbb{A}_5 \geq \mathbb{A}_4 \geq \mathbb{A}_2 \geq \mathbb{A}_3$
$1, 2$	$\mathbb{A}_1 \geq \mathbb{A}_5 \geq \mathbb{A}_4 \geq \mathbb{A}_2 \geq \mathbb{A}_3$	$2, 2$	$\mathbb{A}_1 \geq \mathbb{A}_5 \geq \mathbb{A}_4 \geq \mathbb{A}_2 \geq \mathbb{A}_3$
$1, 3$	$\mathbb{A}_1 \geq \mathbb{A}_5 \geq \mathbb{A}_4 \geq \mathbb{A}_2 \geq \mathbb{A}_3$	$2, 3$	$\mathbb{A}_1 \geq \mathbb{A}_5 \geq \mathbb{A}_4 \geq \mathbb{A}_2 \geq \mathbb{A}_3$
$1, 4$	$\mathbb{A}_1 \geq \mathbb{A}_5 \geq \mathbb{A}_4 \geq \mathbb{A}_2 \geq \mathbb{A}_3$	$2, 4$	$\mathbb{A}_1 \geq \mathbb{A}_5 \geq \mathbb{A}_4 \geq \mathbb{A}_2 \geq \mathbb{A}_3$
$1, 5$	$\mathbb{A}_1 \geq \mathbb{A}_5 \geq \mathbb{A}_4 \geq \mathbb{A}_2 \geq \mathbb{A}_3$	$2, 5$	$\mathbb{A}_1 \geq \mathbb{A}_5 \geq \mathbb{A}_4 \geq \mathbb{A}_2 \geq \mathbb{A}_3$
$1, 10$	$\mathbb{A}_1 \geq \mathbb{A}_5 \geq \mathbb{A}_4 \geq \mathbb{A}_2 \geq \mathbb{A}_3$	$2, 10$	$\mathbb{A}_1 \geq \mathbb{A}_5 \geq \mathbb{A}_4 \geq \mathbb{A}_2 \geq \mathbb{A}_3$
$3, 1$	$\mathbb{A}_1 \geq \mathbb{A}_5 \geq \mathbb{A}_4 \geq \mathbb{A}_2 \geq \mathbb{A}_3$	$4, 1$	$\mathbb{A}_1 \geq \mathbb{A}_5 \geq \mathbb{A}_4 \geq \mathbb{A}_2 \geq \mathbb{A}_3$
$3, 2$	$\mathbb{A}_1 \geq \mathbb{A}_5 \geq \mathbb{A}_4 \geq \mathbb{A}_2 \geq \mathbb{A}_3$	$4, 2$	$\mathbb{A}_1 \geq \mathbb{A}_5 \geq \mathbb{A}_4 \geq \mathbb{A}_2 \geq \mathbb{A}_3$
$3, 3$	$\mathbb{A}_1 \geq \mathbb{A}_5 \geq \mathbb{A}_4 \geq \mathbb{A}_2 \geq \mathbb{A}_3$	$4, 3$	$\mathbb{A}_1 \geq \mathbb{A}_5 \geq \mathbb{A}_4 \geq \mathbb{A}_2 \geq \mathbb{A}_3$
$3, 4$	$\mathbb{A}_1 \geq \mathbb{A}_5 \geq \mathbb{A}_4 \geq \mathbb{A}_2 \geq \mathbb{A}_3$	$4, 4$	$\mathbb{A}_1 \geq \mathbb{A}_5 \geq \mathbb{A}_4 \geq \mathbb{A}_2 \geq \mathbb{A}_3$
$3, 5$	$\mathbb{A}_1 \geq \mathbb{A}_5 \geq \mathbb{A}_4 \geq \mathbb{A}_2 \geq \mathbb{A}_3$	$4, 5$	$\mathbb{A}_1 \geq \mathbb{A}_5 \geq \mathbb{A}_4 \geq \mathbb{A}_2 \geq \mathbb{A}_3$
$3, 10$	$\mathbb{A}_1 \geq \mathbb{A}_5 \geq \mathbb{A}_4 \geq \mathbb{A}_2 \geq \mathbb{A}_3$	$4, 10$	$\mathbb{A}_1 \geq \mathbb{A}_5 \geq \mathbb{A}_4 \geq \mathbb{A}_2 \geq \mathbb{A}_3$
$5, 1$	$\mathbb{A}_1 \geq \mathbb{A}_5 \geq \mathbb{A}_2 \geq \mathbb{A}_4 \geq \mathbb{A}_3$	$10, 1$	$\mathbb{A}_1 \geq \mathbb{A}_5 \geq \mathbb{A}_2 \geq \mathbb{A}_4 \geq \mathbb{A}_3$
$5, 2$	$\mathbb{A}_1 \geq \mathbb{A}_5 \geq \mathbb{A}_2 \geq \mathbb{A}_4 \geq \mathbb{A}_3$	$10, 2$	$\mathbb{A}_1 \geq \mathbb{A}_5 \geq \mathbb{A}_2 \geq \mathbb{A}_4 \geq \mathbb{A}_3$
$5, 3$	$\mathbb{A}_1 \geq \mathbb{A}_5 \geq \mathbb{A}_2 \geq \mathbb{A}_4 \geq \mathbb{A}_3$	$10, 3$	$\mathbb{A}_1 \geq \mathbb{A}_5 \geq \mathbb{A}_2 \geq \mathbb{A}_4 \geq \mathbb{A}_3$
$5, 4$	$\mathbb{A}_1 \geq \mathbb{A}_5 \geq \mathbb{A}_2 \geq \mathbb{A}_4 \geq \mathbb{A}_3$	$10, 4$	$\mathbb{A}_1 \geq \mathbb{A}_5 \geq \mathbb{A}_2 \geq \mathbb{A}_4 \geq \mathbb{A}_3$
$5, 5$	$\mathbb{A}_1 \geq \mathbb{A}_5 \geq \mathbb{A}_2 \geq \mathbb{A}_4 \geq \mathbb{A}_3$	$10, 5$	$\mathbb{A}_1 \geq \mathbb{A}_5 \geq \mathbb{A}_2 \geq \mathbb{A}_4 \geq \mathbb{A}_3$
$5, 10$	$\mathbb{A}_1 \geq \mathbb{A}_5 \geq \mathbb{A}_2 \geq \mathbb{A}_4 \geq \mathbb{A}_3$	$10, 10$	$\mathbb{A}_1 \geq \mathbb{A}_5 \geq \mathbb{A}_2 \geq \mathbb{A}_4 \geq \mathbb{A}_3$

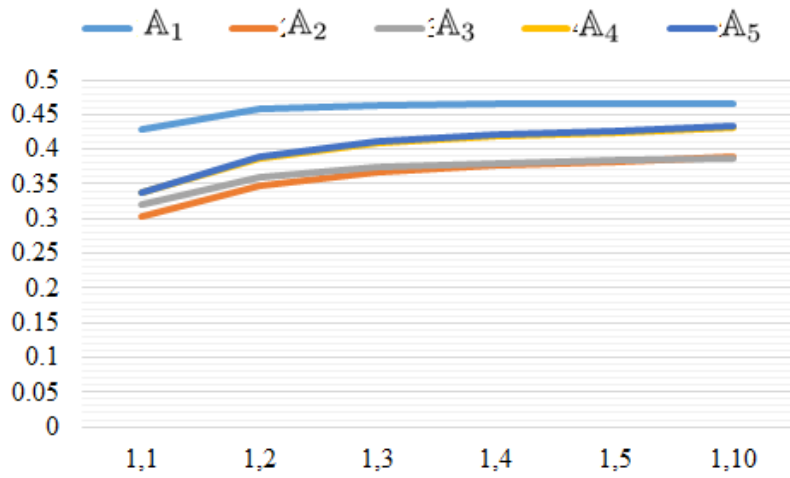

Figure 9. Ranking of the alternative options using the $1, q$-QRONFWG operator for various q value

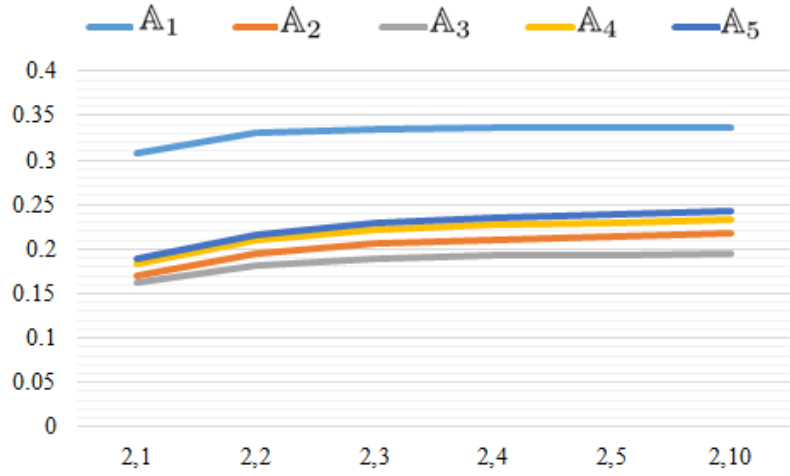

Figure 10. Ranking of the alternative options using the $2, q$-QRONFWG operator for various q value

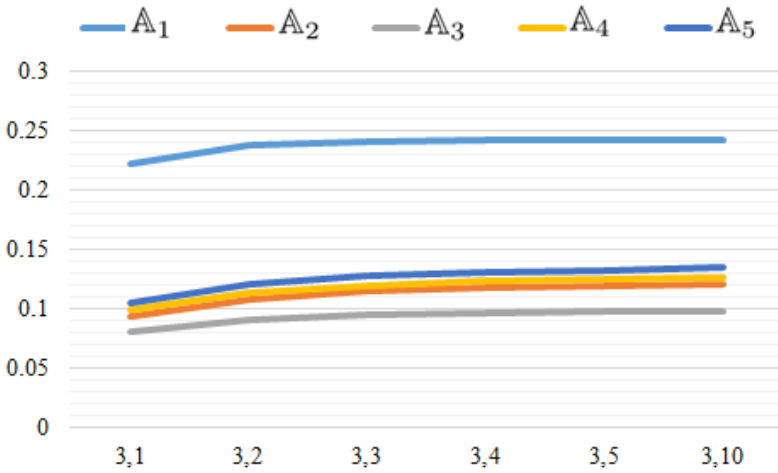

Figure 11. Ranking of the alternative options using the $3, q$-QRONFWG operator for various q value

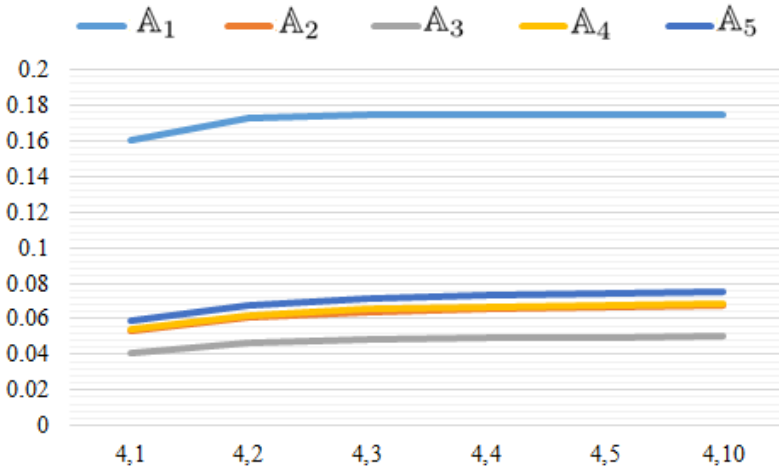

Figure 12. Ranking of the alternative options using the $4, q$-QRONFWG operator for various q value

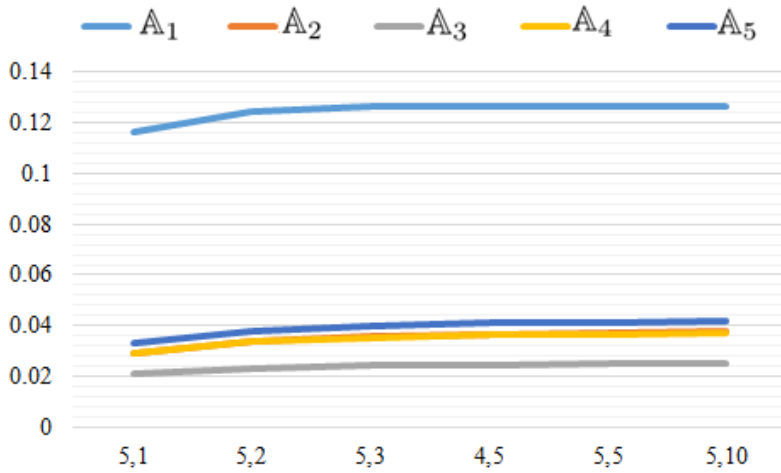

Figure 13. Ranking of the alternative options using the $5, q$-QRONFWG operator for various q value

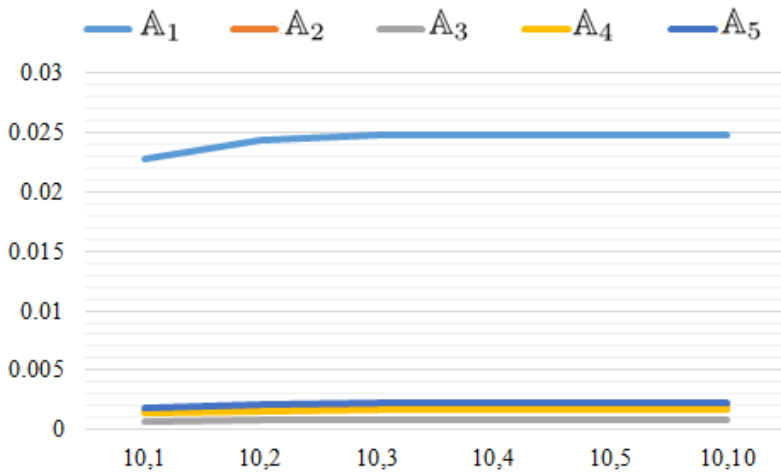

Figure 14. Ranking of the alternative options using the $10, q$-QRONFWG operator for various q value

5.6. Comparative analysis

In this section, we present a comparative analysis between the proposed p, q-QRONFWA and p, q-QRONFWG operators and several well-established operators, including $(2, 1)$-FWA[17], $(2, 1)$-FWG [17], $3, 4$-QFWA [13], $3, 4$-QFWG [13], p, q-QOFWA [14], p, q-QOFWG [14], q-RONFWA [29] and q-RONFWG [29] operators.

TABLE 8. Comparison between existing work with the proposed work

Operators	**Parameters** p, q	**Ranking**
$(2,1)$-FWA [17]	$p, q = 2, 1$	Cannot be calculated
$(2,1)$-FWG [17]	$p, q = 2, 1$	Cannot be calculated
$3,4$-QFWA [13]	$p, q = 3, 4$	Cannot be calculated
$3,4$-QFWG [13]	$p, q = 3, 4$	Cannot be calculated
p, q-QOFWA [14]	$p, q = 2, 10$	Cannot be calculated
p, q-QOFWG [14]	$p, q = 2, 10$	Cannot be calculated
q-RONFWA [29]	$p, q = 2, 2$	$\mathbb{A}_1 \geq \mathbb{A}_5 \geq \mathbb{A}_3 \geq \mathbb{A}_2 \geq \mathbb{A}_4$
q-RONFWG [29]	$p, q = 2, 2$	$\mathbb{A}_1 \geq \mathbb{A}_3 \geq \mathbb{A}_5 \geq \mathbb{A}_4 \geq \mathbb{A}_2$
The propounded methodology		
p, q-QRONFWA	$p, q = 5, 10$	$\mathbb{A}_1 \geq \mathbb{A}_5 \geq \mathbb{A}_2 \geq \mathbb{A}_4 \geq \mathbb{A}_3$
p, q-QRONFWG	$p, q = 5, 10$	$\mathbb{A}_1 \geq \mathbb{A}_5 \geq \mathbb{A}_2 \geq \mathbb{A}_4 \geq \mathbb{A}_3$

TABLE 9. Comparison between existing work with the proposed work

Operators	**Settings**	**ranking**
$(2,1)$-FWA [17]	$p, q = 2, 1$, NFN $= (1, 1)$	$\mathbb{A}_1 \geq \mathbb{A}_5 \geq \mathbb{A}_3 \geq \mathbb{A}_2 \geq \mathbb{A}_4$
$(2,1)$-FWG [17]	$p, q = 2, 1$, NFN $= (1, 1)$	$\mathbb{A}_1 \geq \mathbb{A}_5 \geq \mathbb{A}_3 \geq \mathbb{A}_2 \geq \mathbb{A}_4$
$3,4$-QFWA [13]	$p, q = 3, 4$, NFN $= (1, 1)$	$\mathbb{A}_1 \geq \mathbb{A}_5 \geq \mathbb{A}_2 \geq \mathbb{A}_4 \geq \mathbb{A}_3$
$3,4$-QFWG [13]	$p, q = 3, 4$, NFN $= (1, 1)$	$\mathbb{A}_1 \geq \mathbb{A}_2 \geq \mathbb{A}_5 \geq \mathbb{A}_4 \geq \mathbb{A}_3$
p, q-QOFWA [14]	$p, q = 5, 10$, NFN $= (1, 1)$	$\mathbb{A}_1 \geq \mathbb{A}_2 \geq \mathbb{A}_5 \geq \mathbb{A}_4 \geq \mathbb{A}_3$
p, q-QOFWG [14]	$p, q = 5, 10$, NFN $= (1, 1)$	$\mathbb{A}_1 \geq \mathbb{A}_3 \geq \mathbb{A}_2 \geq \mathbb{A}_4 \geq \mathbb{A}_3$
q-RONFWA [29]	$p, q = 2, 2$	$\mathbb{A}_1 \geq \mathbb{A}_5 \geq \mathbb{A}_3 \geq \mathbb{A}_2 \geq \mathbb{A}_4$
q-RONFWG [29]	$p, q = 2, 2$	$\mathbb{A}_1 \geq \mathbb{A}_3 \geq \mathbb{A}_5 \geq \mathbb{A}_4 \geq \mathbb{A}_2$
The propounded methodology		
p, q-QRONFWA	$p, q = 2, 10$	$\mathbb{A}_1 \geq \mathbb{A}_5 \geq \mathbb{A}_2 \geq \mathbb{A}_4 \geq \mathbb{A}_3$
p, q-QRONFWG	$p, q = 2, 10$	$\mathbb{A}_1 \geq \mathbb{A}_5 \geq \mathbb{A}_2 \geq \mathbb{A}_4 \geq \mathbb{A}_3$

TABLE 10. The characteristic comparisons of different methods

Operators	**MG**	**NMG**	**NFN**	p	q
$(2,1)$-FWA [17]	✓	✓	×	×	×
$(2,1)$-FWG [17]	✓	✓	×	×	×
$3,4$-QFWA [13]	✓	✓	×	×	×
$3,4$-QFWG [13]	✓	✓	×	×	×
p, q-QOFWA [14]	✓	✓	×	✓	✓
p, q-QOFWG [14]	✓	✓	×	✓	✓
q-RONFWA [29]	✓	✓	✓	✓	×
q-RONFWG [29]	✓	✓	✓	✓	×
The propounded methodology					
p, q-QRONFWA	✓	✓	✓	✓	✓
p, q-QRONFWG	✓	✓	✓	✓	✓

The comparative evaluation presented in Tables 8–10 demonstrates the effectiveness, flexibility, and broader applicability of the proposed p, q-QRONFMADM framework in relation to existing models. As observed in Table 8, several conventional operators such as the $(2, 1)$-FWA and $(2, 1)$-FWG [17], $3, 4$-QFWA and $3, 4$-QFWG [13], and p, q-QOFWA and p, q-QOFWG [14] are unable to yield feasible outcomes under the current decision environment. This limitation primarily arises from their structural inability to accommodate the normalization and orthopair properties intrinsic to the proposed p, q-QRONFNs framework. In contrast, the q-RONFWA and q-RONFWG operators [29] successfully provide calculable rankings but exhibit restricted parameter flexibility, as they rely solely on a single parameter (q).

The proposed p, q-QRONFWA and p, q-QRONFWG operators overcome these shortcomings by incorporating two adjustable parameters, p and q, which enhance control over the balance between MG and NMG. This dual-parameter structure allows the decision model to adaptively capture varying levels of uncertainty and decision-maker preference, thus yielding more nuanced and realistic evaluations. As reported in Table 9, both proposed operators produce consistent ranking orders ($\mathbb{A}_1 \geq \mathbb{A}_5 \geq \mathbb{A}_2 \geq \mathbb{A}_4 \geq \mathbb{A}_3$) across multiple parameter settings, reflecting high stability and robustness. Moreover, the similarity of ranking outcomes under both the averaging and geometric operators further validates the internal coherence of the proposed model.

Table 10 provides a characteristic comparison that highlights the methodological distinctions between existing approaches and the proposed framework. Earlier operators such as FWA, FWG, and QFWA models generally support MG and NMG but lack the ability to handle NFNs. The introduction of the q-RONFWA and q-RONFWG operators partially addressed this limitation by incorporating normalization; however, they remain constrained to a single control parameter. In contrast, the proposed p, q-QRONFWA and p, q-QRONFWG models simultaneously integrate MG, NMG, and NFN components under the full control of two tunable parameters, thereby offering enhanced precision and interpretability in complex decision-making environments.

In summary, the comparative analysis confirms that the proposed p, q-QRONFMADM approach generalizes and extends existing fuzzy aggregation frameworks. It not only ensures broader operational feasibility but also introduces greater modeling flexibility, robustness, and adaptability, making it particularly suitable for decision problems characterized by uncertainty, interdependence, and heterogeneous information structures.

6. Conclusion

This study developed an advanced MADM framework grounded in the p, q-QRONFS to address uncertainty, ambiguity, and incomplete information in complex decision environments. By integrating the interpretive capacity of NFNs with the structural flexibility of p, q-QROFSs, the framework provides a comprehensive mechanism for representing heterogeneous and uncertain information. Building upon this foundation, two novel aggregation operatorsthe p, q-QRONFWA and p, q-QRONFWG were introduced, incorporating dual tunable parameters (p, q) to allow precise control over the balance between membership and non-membership grades. This dual-parameter design enhances robustness, adaptability, and interpretability, overcoming the limitations of existing single-parameter methods. The resulting p, q-QRONFMADM model consistently produces stable and coherent ranking outcomes across different parameter settings, demonstrating high computational reliability and decision-making accuracy. In contrast, the proposed methodology

provides feasible, consistent, and interpretable rankings while accommodating diverse decision-maker preferences and varying levels of uncertainty. This approach is particularly applicable to strategic decision problems, including global business partner selection, where complex criteria and interdependencies prevail. Future work may explore automated parameter optimization, extensions to group decision-making contexts, and hybrid integration with predictive analytics, further enhancing the practical applicability of the framework.

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